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Solving Third Order Ordinary Differential Equations Using Hybrid Block Method of Order Five

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Abstract
In this work, interpolation and collocation approach is adopted in deriving hybrid block method of steplength $k = 1$ for solving third order initial value problems of ordinary differential equations. In developing this method, power series of order 7 is interpolated at the first three points while its third derivative is collocated at all points in the selected interval. In addition, some properties of the new method which include zero stability, consistency, convergence, error constant, order, and region of absolute stability are also established. The new method performs better than the existing methods in term of accuracy when solving the same problems.

Keywords: Hybrid method, Block method, Third order differential equation, Single step, Three off step points.

Introduction

This paper considers the solution to general third order initial value problem of the form

$$y''' = f(x, y, y', y''),\ y(a) = \eta_0,\ y'(a) = \eta_1,\ y''(a) = \eta_2,\ x \in [a, b]. \tag{1}$$

Equation (1) can be solved by converting to its equivalent of three first order ODEs and thereafter suitable numerical methods for first order ODEs are employed. This approach, therefore, enlarges Equation (1) and thus requires more computation to be solved. To overcome this drawback, numerous scholars have developed numerical methods for solving initial value problems of third order ODEs directly. Among these researchers are [1], [2], [5] and [6]. The implementation of direct method can be done by using two approaches namely block method and predictor-corrector method. However, subroutines to supply the starting values are needed in predictor-corrector method which leads to inefficiency of the method in terms of error [4] and [9]. Conversely, the development of separate predictors in block method is not needed and thus requires less computational burden and human effort which resulted in high accuracy of the method [3]. In general, using off-step points (hybrid method) was introduced to overcome the zero stability barrier. This barrier implies that the highest order of zero stability of linear multistep method of step length k is k+2 when k is even and k+1 when k is odd [8].

Some hybrid block methods to present direct solution of equation (1) have been proposed, for example [7], [10], [12], [13] and [11]. In the same topic, [2] developed accurate scheme by block method having an order seven for solving (1) directly but the accuracy of the method can be improved. This paper examines single step hybrid block method with uniform order 5 for solving (1) directly. This method is an improvement of existing method mentioned above.

Derivation of the Method

In this section, a hybrid single step block method with three off-step points $x_{n+\frac{1}{10}}$, $x_{n+\frac{2}{5}}$ and $x_{n+\frac{9}{10}}$ for solving (1) is derived.

Let the power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i. \quad x \in [x_n, x_{n+1}] \tag{2}$$

be an approximate solution to (1), where $n = 0, 1, 2, ..., N-1$, v denotes of the number of interpolation points, m represents the number of collocation points which is equal to the order of differential equation and $h = x_n - x_{n-1}$ is constant step size of partition of interval [a, b] which is given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$. The Third derivative of (2) is given by (3)

$$\begin{aligned} y'''(x) &= f(x, y, y', y'') \\ &= \sum_{i=3}^{v+m-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3}. \end{aligned} \tag{3}$$

In this case, v = 3 and m = 4.

Equation (2) is interpolated at x_n, $x_{n+\frac{1}{10}}$ and $x_{n+\frac{2}{5}}$, while equation (3) is collocated at all points i.e. x_n, $x_{n+\frac{1}{10}}$, $x_{n+\frac{2}{5}}$, $x_{n+\frac{9}{10}}$ and x_{n+1} in the selected interval. This yields the following equations which can be written in matrix form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{10} & \frac{1}{100} & \frac{1}{1000} & \frac{1}{10000} & \frac{1}{100000} & \frac{1}{1000000} & \frac{1}{10000000} \\ 1 & \frac{2}{5} & \frac{4}{25} & \frac{8}{125} & \frac{16}{625} & \frac{32}{3125} & \frac{64}{15625} & \frac{128}{78125} \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{12}{5h^3} & \frac{3}{5h^3} & \frac{3}{25h^3} & \frac{21}{1000h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{5h^3} & \frac{48}{5h^3} & \frac{192}{25h^3} & \frac{672}{125h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{108}{5h^3} & \frac{243}{5h^3} & \frac{2187}{25h^3} & \frac{137781}{1000h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{10}} \\ y_{n+\frac{2}{5}} \\ f_n \\ f_{n+\frac{1}{10}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{9}{10}} \\ f_{n+1} \end{pmatrix} \tag{4}$$

Gaussian elimination method is applied to find the coefficients $a_i = 0(1)7$.

$$a_0 = y_n$$
$$a_1 = \frac{40}{3}y_{n+\frac{1}{10}} - \frac{25}{2}y_n - \frac{5}{6}y_{n+\frac{2}{5}} + \frac{h^3}{7560}f_n + \frac{19h^3}{189000}f_{n+1} + \frac{8839h^3}{1512000}f_{n+\frac{1}{10}} + \frac{2h^3}{2625}f_{n+\frac{2}{5}} - \frac{263h^3}{1512000}f_{n+\frac{9}{10}}$$
$$a_2 = 25y_n - \frac{100}{3}y_{n+\frac{1}{10}} + \frac{25}{3}y_{n+\frac{2}{5}} - \frac{229h^3}{18000}f_n - \frac{139h^3}{162000}f_{n+1} - \frac{16643h^3}{259200}f_{n+\frac{1}{10}} - \frac{19h^3}{2700}f_{n+\frac{2}{5}} + \frac{43h^3}{28800}f_{n+\frac{9}{10}}$$
$$a_3 = \frac{h^3}{6}f_n$$
$$a_4 = -h^3\left(\frac{263}{432}f_n + \frac{1}{36}f_{n+1} - \frac{25}{36}f_{n+\frac{1}{10}} + \frac{5}{48}f_{n+\frac{2}{5}} - \frac{5}{108}f_{n+\frac{9}{10}}\right)$$
$$a_5 = h^3\left(\frac{7}{8}f_n + \frac{49}{324}f_{n+1} - \frac{415}{324}f_{n+\frac{1}{10}} + \frac{109}{216}f_{n+\frac{2}{5}} - \frac{1}{4}f_{n+\frac{9}{10}}\right)$$
$$a_6 - h^3\left(\frac{5}{9}f_n + \frac{35}{162}f_{n+1} - \frac{575}{648}f_{n+\frac{1}{10}} + \frac{25}{54}f_{n+\frac{2}{5}} - \frac{25}{72}f_{n+\frac{9}{10}}\right)$$
$$a_7 = h^3\left(\frac{25}{189}f_n + \frac{50}{567}f_{n+1} - \frac{125}{567}f_{n+\frac{1}{10}} + \frac{25}{189}f_{n+\frac{2}{5}} - \frac{25}{189}f_{n+\frac{9}{10}}\right)$$

The values of $a_i = 0(1)7$ are then substituted into equation (2) to give a continuous implicit scheme of the form

$$y(x) = \sum_{i=0,\frac{1}{10},\frac{2}{5},\frac{9}{10}}^{1} \alpha_i y_{n+i} + \sum_{i=\frac{1}{10},\frac{2}{5},\frac{9}{10}}^{1} \beta_i f_{n+i} \qquad (5)$$

The first and second derivative of equation (5) are given in the following equations (6) and (7) respectively.

$$y'(x) = \sum_{i=\frac{1}{10},\frac{2}{5},\frac{9}{10}}^{1} \frac{d}{dx}\alpha_i(x) y_{n+i} + \sum_{i=\frac{1}{10},\frac{2}{5},\frac{9}{10}}^{1} \frac{d}{dx}\beta_i(x) f_{n+i} \qquad (6)$$

$$y''(x) = \sum_{i=\frac{1}{10},\frac{2}{5},\frac{9}{10}}^{1} \frac{d}{dx^2}\alpha_{s_i}(x) y_{n+s_i} + \sum_{i=\frac{1}{10},\frac{2}{5},\frac{9}{10}}^{1} \frac{d}{dx^2}\beta_i(x) f_{n+i} \qquad (7)$$

where,

$$\alpha_0 = \frac{((h-10x+10x_n)(2h-5x+5x_n))}{(2h^2)}$$
$$\alpha_{\frac{1}{10}} = \frac{(20(x-x_n)(2h-5x+5x_n))}{(3h^2)}$$
$$\alpha_{\frac{2}{5}} = \frac{-(5(x-x_n)(h-10x+10x_n))}{(6h^2)}$$
$$\beta_0 = \frac{(x-x_n)(2h-5x+5x_n)(h-10x+10x_n)}{(378000h^4)}(25h^4 - 2092h^3x + 2092\,h^3x_n + 4725\,h^2x^2 - 9450\,h^2xx_n + 4725\,h^2x_n^2 - 3700hx^3 + 11100hx^2x_n - 11100hxx_n^2 + 3700hx_n^3 + 1000x^4 - 4000x^3x_n + 6000x^2x_n^2 - 4000xx_n^3 + 1000x_n^4)$$
$$\beta_{\frac{1}{10}} = \frac{(x-x_n)(2h-5x+5x_n)(h-10x+10x_n)}{(9072000h^4)}(26517h^4 + 40210h^3 - 40210h^3x_n - 160300h^2x^2 + 320600h^2xx_n - 160300h^2x_n^2 + 141000hx - 423000hx^2x_n + 423000hxx_n^2 - 141000hx_n^3 - 40000x^4 + 160000x^3x - 240000x^2x_n^2 + 160000xx_n^3 - 40000x_n^4)$$
$$\beta_{\frac{2}{5}} = \frac{(x-x_n)(2h-5x+5x_n)(h-10x+10x_n)}{(378000h^4)}(144h^4 + 470h^3x + 2275\,h^2x^2 - 4550\,h^2xx_n + 2275\,h^2x_n^2 - 3000hx^3 + 9000hx^2x_n - 470h^3x_n - 9000hxx_n^2 + 3000hx_n^3 + 1000x^4 - 4000x^3x_n + 1000x_n^4 + 6000x^2x_n^2 - 4000xx_n^3)$$
$$\beta_{\frac{9}{10}} = -\frac{(x-x_n)(2h-5x+5x_n)(h-10x+10x_n)}{(3024000h^4)}(1030\,h^3x - 1030h^3x_n + 6300h^2x^2 - 12600h^2xx_n + 6300h^2x_n^2 - 17000hx^3 + 51000hx^2x_n - 51000hxx_n^2 + 17000hx_n^3 + 8000x^4 - 32000x^3x_n + 48000x^2x_n^2 - 32000xx_n^3 + 8000x_n^4 + 263h^4)$$
$$\beta_1 = \frac{(x-x_n)(2h-5x+5x_n)(h-10x+10x_n)}{(1134000h^4)}(57h^4 + 226h^3x + 1400h^2x^2 - 2800h^2xx_n + 1400h^2x_n^2 - 3900hx^3 + 11700hx^2x_n - 226h^3x_n - 11700hxx_n^2 + 3900hx_n^3 + 2000x^4 - 8000x^3x_n + 12000x^2x_n^2 - 8000xx_n^3 + 2000x_n^4)$$

Evaluating Equation (5) at the non-interpolating point i.e. $x_{n+\frac{9}{10}}$ and x_{n+1}. Next, evaluating (6) and (7) at all points produce the discrete schemes and its derivatives. This can be written as

$$A^{[3]_3}Y_m^{[3]_3} = B^{[3]_3}R_1^{[3]_3} + h^3\left[D^{[3]_3}R_2^{[3]_3} + E^{[3]_3}R_3^{[3]_3}\right] \qquad (8)$$

where,

$$A^{[3]_3} = \begin{pmatrix} 20 & \frac{-15}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-40}{3h} & \frac{5}{6h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-20}{3h} & \frac{-5}{6h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{40}{3h} & \frac{-35}{6h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{140}{3h} & \frac{-85}{6h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{160}{3h} & \frac{-95}{6h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{200}{3h^2} & \frac{-50}{3h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{200}{3h^2} & \frac{-50}{3h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{200}{3h^2} & \frac{-50}{3h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{200}{3h^2} & \frac{-50}{3h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{200}{3h^2} & \frac{-50}{3h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Y_m^{[3]_3} = \begin{pmatrix} y_{n+\frac{1}{10}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{9}{10}} \\ y_{n+1} \\ y'_{n+\frac{1}{10}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{9}{10}} \\ y'_{n+1} \\ y''_{n+\frac{1}{10}} \\ y''_{n+\frac{2}{5}} \\ y''_{n+\frac{9}{10}} \\ y''_{n+1} \end{pmatrix}$$

$$B^{[3]_3} = \begin{pmatrix} \frac{27}{2} & 0 & 0 \\ 10 & 0 & 0 \\ \frac{-25}{2h} & -1 & 0 \\ \frac{-15}{2h} & 0 & 0 \\ \frac{15}{2h} & 0 & 0 \\ \frac{65}{2h} & 0 & 0 \\ \frac{75}{2h} & 0 & 0 \\ \frac{50}{h^2} & 0 & -1 \\ \frac{50}{h^2} & 0 & 0 \\ \frac{50}{h^2} & 0 & 0 \\ \frac{50}{h^2} & 0 & 0 \\ \frac{50}{h^2} & 0 & 0 \end{pmatrix}, \quad R_1^{[3]_3} = \begin{pmatrix} y_n \\ y'_n \\ y''_n \end{pmatrix}, \quad D^{[3]_3} = \begin{pmatrix} -\frac{3}{1000} \\ -\frac{41}{12000} \\ \frac{1}{7560h} \\ \frac{937}{1680000h} \\ -\frac{89}{21000h} \\ \frac{18065}{15120000h} \\ \frac{149}{21000h} \\ -\frac{229}{9000h^2} \\ \frac{313}{18000h^2} \\ -\frac{397}{9000h^2} \\ \frac{1081}{18000h^2} \\ \frac{37512}{648000h^2} \end{pmatrix}, \quad R_2^{[3]_3} = (\ f_n\ ),$$

$$E^{[3]_2} = \begin{pmatrix} \frac{1061}{48000} & \frac{127}{2000} & \frac{201}{16000} & -\frac{31}{6000} \\ \frac{7}{375} & \frac{837}{20000} & \frac{19}{3750} & -\frac{13}{6000)} \\ \frac{8839}{1512000h} & \frac{2}{2625h} & -\frac{263}{1512000h} & \frac{19}{189000h} \\ -\frac{1817}{378000h} & -\frac{4217}{5040000h} & \frac{43}{210000h} & -\frac{899}{7560000h} \\ \frac{24953}{1512000h} & \frac{331}{39375h} & -\frac{1333}{840000h} & \frac{173}{189000h} \\ \frac{14017}{378000h} & \frac{321901}{1680000h)} & \frac{104897}{1890000h)} & -\frac{35863}{1512000h)} \\ \frac{12001}{378000h} & \frac{15203}{63000h} & \frac{4007}{42000h} & -\frac{3361}{94500h} \\ -\frac{16643}{129600h^2} & -\frac{19}{1350h^2} & \frac{43}{14400h^2} & -\frac{139}{81000h^2} \\ -\frac{883}{12960h^2} & -\frac{481}{27000h^2} & \frac{163}{36000h^2} & -\frac{427}{162000h^2} \\ \frac{19709}{129600h^2)} & \frac{899}{6750h^2} & -\frac{1321}{72000h^2} & \frac{853}{81000h^2} \\ -\frac{3583}{64800h^2} & \frac{13471}{27000h^2} & \frac{13027}{36000h^2} & -\frac{21419*h}{162000h^2} \\ -\frac{6643}{129600h^2} & \frac{1337}{2700h^2} & \frac{6043}{14400h^2} & -\frac{7139}{81000h^2} \end{pmatrix}, \quad R_3^{[3]_3} = \begin{pmatrix} f_{n+\frac{1}{10}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{9}{10}} \\ f_{n+1} \end{pmatrix}$$

Multiplying equation (8) by inverse of $A^{[3]_3}$ gives a hybrid block method as below

$$A^{[3]_3}Y_m^{[3]_3} = B^{[3]_3}R_1^{[3]_3} + h^3\left[D^{[3]_3}R_2^{[3]_3} + E^{[3]_3}R_3^{[3]_3}\right] \qquad (8)$$

where,

$$\bar{B}^{[3]_3} = \begin{pmatrix} 1 & \frac{h}{10} & \frac{h^2}{200} \\ 1 & \frac{2h)}{5} & \frac{2h^2)}{25} \\ 1 & \frac{9h)}{10} & \frac{81h^2)}{200} \\ 1 & h & \frac{h^2}{2} \\ 0 & 1 & \frac{h}{10} \\ 0 & 1 & \frac{2h)}{5} \\ 0 & 1 & \frac{9h)}{10} \\ 0 & 1 & h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{D}^{[3]_3} = \begin{pmatrix} \frac{4309}{37800000} \\ \frac{1171}{590625} \\ \frac{9477}{1400000} \\ \frac{29}{3024} \\ \frac{1283}{432000h} \\ \frac{98}{16875h} \\ \frac{12646582742679557}{5277655813324800000h} \\ \frac{7}{216h} \\ \frac{257}{6000h^2} \\ -\frac{7}{375h^2} \\ \frac{171}{2000h^2} \\ \frac{7388718138654718 3}{886646176638566400h^2} \end{pmatrix}$$

$$\bar{E}^{[3]_2} = \begin{pmatrix} \frac{734157108005569223}{1276770494359535616 0000} & \frac{-11}{1890000} & \frac{149}{60480000} & \frac{-167}{113400000} \\ \frac{633206107608291808 7}{797981558974709760000} & \frac{97}{118125} & \frac{-4}{23625} & \frac{172}{1771875} \\ \frac{1473008816515561281}{2251799813685248000 0} & \frac{6561}{140000} & \frac{8991}{2240000} & \frac{-9972489539872523867 3}{638385247179767808000 00} \\ \frac{1027383664993894471 1}{127677049435953561600} & \frac{211}{3024} & \frac{17}{1512} & \frac{-5}{1134} \\ \frac{139745289062423008 7}{638385247179767808000h} & \frac{-413}{2160000h} & \frac{173}{2160000h} & \frac{-31}{648000h} \\ \frac{989891198096035091 9}{159596311794941952000h} & \frac{224}{16875h} & \frac{-44}{16875h} & \frac{76}{50625h} \\ \frac{1468125}{100000000} & \frac{2035125}{100000000h} & \frac{529875}{100000000h} & \frac{-22275}{10000000h} \\ \frac{985162418487296071 1}{638385247179767808000h} & \frac{55}{216h} & \frac{5}{54h} & \frac{-11}{324h} \\ \frac{769707397564124507}{127677049435953561 60h^2} & \frac{-101}{27000h^2} & \frac{37}{24000h^2} & \frac{-149}{162000h^2} \\ \frac{682145223559051 25}{243194379878006784h^2} & \frac{497}{3375h^2} & \frac{-8}{375h^2} & \frac{124}{10125h^2} \\ \frac{518688013333561423}{709316941310853120 0h^2} & \frac{513}{1000h^2} & \frac{2871}{8000h^2} & \frac{-261}{2000h^2} \\ \frac{492581209243648071 1}{638385247179767808 00h^2} & \frac{55}{108h^2} & \frac{5}{12h^2} & \frac{-7}{81h^2} \end{pmatrix}$$

In finding the order of the main block in (8). The vectors

$$\begin{pmatrix} y_{n+\frac{1}{10}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{9}{10}} \\ y_{n+1} \end{pmatrix}, \begin{pmatrix} f_{n+\frac{1}{10}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{9}{10}} \\ f_{n+1} \end{pmatrix}$$

are expanded in Taylor series and its terms are collected in powers of h to give

$$\left[\begin{array}{l}
\sum_{j=0}^{\infty} \frac{(\frac{1}{10})^j h^j}{j!} y_n^j - y_n - \frac{h}{10} y_n' - \frac{h^2}{200} y_n'' - \frac{4309h^3}{37800000} y_n''' \\
\quad - \frac{734157108005569223}{12767704943595356160000} \sum_{j=0}^{\infty} \frac{(\frac{1}{10})^j h^{j+3}}{j!} y_n^{j+3} \\
\quad + \frac{11}{1890000} \sum_{j=0}^{\infty} \frac{(\frac{2}{5})^j h^{j+3}}{j!} y_n^{j+3} \\
\quad - \frac{149}{60480000} \sum_{j=0}^{\infty} \frac{(\frac{9}{10})^j h^{j+3}}{j!} y_n^{j+3} \\
\quad + \frac{167}{113400000} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{(\frac{2}{5})^j h^j}{j!} y_n^j - y_n - \frac{2h)}{5} y_n' - \frac{2h^2)}{25} y_n'' - \frac{1171h^3}{590625} y_n''' \\
\quad - \frac{633206107608291808 7}{797981558974709760000} \sum_{j=0}^{\infty} \frac{(\frac{1}{10})^j h^{j+3}}{j!} y_n^{j+3} \\
\quad - \frac{97}{118125} \sum_{j=0}^{\infty} \frac{(\frac{2}{5})^j h^{j+3}}{j!} y_n^{j+2} \\
\quad + \frac{4}{23625} \sum_{j=0}^{\infty} \frac{(\frac{9}{10})^j h^{j+3}}{j!} y_n^{j+2} \\
\quad - \frac{172}{1771875} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{(\frac{9}{10})^j h^j}{j!} y_n^j - y_n - \frac{9h)}{10} y_n' - \frac{81h^2)}{200} y_n'' - \frac{9477h^3}{1400000} y_n''' \\
\quad - \frac{1473008816515561281}{22517998136852480000} \sum_{j=0}^{\infty} \frac{(\frac{1}{10})^j h^{j+3}}{j!} y_n^{j+3} \\
\quad - \frac{6561}{140000} \sum_{j=0}^{\infty} \frac{(\frac{2}{5})^j h^{j+3}}{j!} y_n^{j+2} \\
\quad - \frac{8991}{2240000} \sum_{j=0}^{\infty} \frac{(\frac{9}{10})^j h^{j+3}}{j!} y_n^{j+2} \\
\quad + \frac{99724895398725238673}{63838524717976780800000} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - h y_n' - \frac{h^2}{2} y_n'' - \frac{29h^3}{3024} y_n''' \\
\quad - \frac{10273836649938944711}{127677049435953561600} \sum_{j=0}^{\infty} \frac{(\frac{1}{10})^j h^{j+3}}{j!} y_n^{j+3} \\
\quad - \frac{211}{3024} \sum_{j=0}^{\infty} \frac{(\frac{2}{5})^j h^{j+3}}{j!} y_n^{j+2} \\
\quad - \frac{17}{1512} \sum_{j=0}^{\infty} \frac{(\frac{9}{10})^j h^{j+3}}{j!} y_n^{j+2} \\
\quad + \frac{5}{1134} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3}
\end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Comparing the coefficients of h^j and y^j. This gives the order of the method to be $[5, 5, 5, 5]^T$ with error constant

$$\begin{bmatrix} 6.414187E^{-10} \\ -3.027302E^{-8} \\ -6.150938E^{-8} \\ 2.579365E^{-7} \end{bmatrix}$$

Zero Stability

Definition 1:

The hybrid block method is said to be zero stable if the first characteristic polynomial π (r) having roots such that $|r_z| \leq 1$, and if $|r_z| = 1$ then the multiplicity of r_z must not greater than two. In finding the zero-stability, we only put into consideration the coefficients of *y*-function according to Definition 1. That is

$$\Pi(r) = |r\,\widehat{I}^{[3]_3} - \widehat{B}_1^{[3]_3}| = 0.$$

where $\widehat{I}^{[3]_3}$ and $\widehat{B}_1^{[3]_3}$ are the coefficients of y_{n+i}, $i = \frac{1}{10}, \frac{2}{5}, \frac{9}{10}$ and y_n in (8). This yields,

$$|r\,\widehat{I}^{[3]_3} - \widehat{B}_1^{[3]_3}| = \left| z\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = r^3(r-1)$$

This implies *r* = 0, 0, 0, 1. Hence, our method is zero stable

Consistency

Definition 2:

The new one step hybrid block method in (8) is said to be consistent if order of the method is greater than or equal one i.e. P≥1.

This mean, our method is consistent, because it satisfies the conditions listed in **Definition 2**

Convergence

Theorem [Henrici, 1962]

Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.

Therefore, since our method is consistent and zero stable, it implies the main method is convergent.

Region of Absolute Stability

The hybrid block method in (8) is said to be absolutely stable if for a given *h*, all roots of the characteristic polynomial

$\pi(z, \bar{h}) = \rho(z) - \bar{h}\,\sigma(z)$, satisfies $|z_t| < 1$.

In this article, the boundary locus method was adopted to determine the region of absolute stability. The test equation $y''' = \lambda^3 y$ is substituted in the main method in (8) where $\bar{h} = \lambda^3 h^3$ and $\lambda = \frac{df}{dy}$. Substituting $r = \cos\theta - i \sin\theta$ and considering real part yields

$$\bar{h}(\theta, h) = \frac{(7000000000000(\cos(\theta) - 1))}{(9\cos(\theta) - 958)} \qquad (10)$$

Hence, the interval of stability gives $(0, 1.4478e^{10})$.

This is demonstrated in the Figure below.

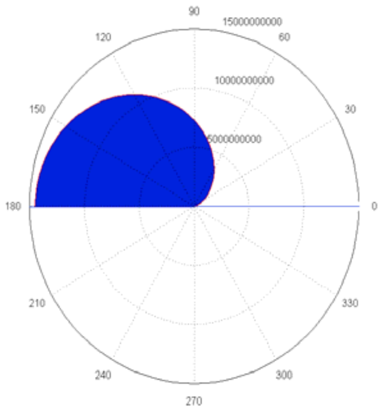

Figure 1 : Region stability of new method

Numerical Example

In order to find the accuracy of our methods, the following third order ODEs are tested. The new one step hybrid block method solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 1: $y''' + y = 0,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1.$
Exact solution: $y(x) = e^{-x}$ with $h = 0.1$.

Problem 2: $y''' = -e^{x},\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 3.$
Exact solution: $y(x) = 2 + 2x^2 - e^{x}$ with $h = 0.1$.

Table 1: Comparison of the new method with Omar[3] for solving Problem 1

x	exact solution	computed solution in our method	error in our method, $P = 5$	errors in Omar[3], $P = 7$
0.1	0.90483741803595952	0.90483741803595719	$2.331468E^{-15}$	$1.617595E^{-13}$
0.2	0.81873075307798182	0.81873075307798382	$1.998401E^{-15}$	$2.386620E^{-13}$
0.3	0.74081822068171777	0.74081822068177328	$5.551115E^{-14}$	$3.698153E^{-13}$
0.4	0.67032004603563933	0.67032004603583584	$1.965095E^{-13}$	$4.252154E^{-13}$
0.5	0.60653065971263342	0.60653065971309350	$4.600764E^{-13}$	$4.534151E^{-13}$
0.6	0.54881163609402650	0.54881163609490369	$8.771872E^{-13}$	$7.458478E^{-12}$
0.7	0.49658530379140953	0.49658530379288529	$1.475764E^{-12}$	$6.777634E^{-12}$
0.8	0.44932896411722162	0.44932896411950196	$2.280343E^{-12}$	$3.362421E^{-12}$
0.9	0.40656965974059917	0.40656965974391163	$3.312461E^{-12}$	$5.546397E^{-12}$
1.0	0.36787944117144233	0.36787944117603283	$4.590495E^{-12}$	$1.048206E^{-11}$

Table 2: Comparison of the new method with Omar[3] for solving Problem 2

x	exact solution	computed solution in our method	error in our method, $P = 5$	errors in Omar[3], $P = 7$
0.1	0.91482908192435231O	0.914829081924355080	$2.775558E^{-15}$	$2.885470E^{-13}$
0.2	0.858597241839830220	0.858597241839958230	$1.280087E^{-13}$	$1.837197E^{-12}$
0.3	0.830141192423996980	0.830141192424547090	$5.501155E^{-13}$	$4.572231E^{-12}$
0.4	0.828175302358729710	0.828175302360190880	$1.461165E^{-12}$	$8.562928E^{-12}$
0.5	0.851278729299871810	0.851278729302945130	$3.073319E^{-12}$	$1.374012E^{-11}$
0.6	0.897881199609491090	0.897881199615112260	$5.621170E^{-12}$	$2.017653E^{-11}$
0.7	0.966247292529523350	0.966247292538888190	$9.364842E^{-12}$	$2.736000E^{-11}$
0.8	1.054459071507532400	1.054459071522122700	$1.459033E^{-11}$	$3.675371E^{-11}$
0.9	1.160396888843050300	1.160396888864665200	$2.161493E^{-11}$	$4.822165E^{-11}$
1.0	1.281718171540954500	1.281718171571744100	$3.078959E^{-11}$	$6.189138E^{-11}$
1.1	1.415833976053566500	1.415833976096066100	$4.249956E^{-11}$	$7.763479E^{-11}$
1.2	1.559883077263452700	1.559883077320625800	$5.717316E^{-11}$	$9.558598E^{-11}$

Conclusion

An accurate one step hybrid block method for solving third order initial value problems directly has been developed in this work. Numerical properties for the new method which includes, consistency, order, zero stability, error constant and convergence are established.

The results generated when the new method was applied to some third order initial values problems are demonstrated in **Table 1** and **Table 2** above.

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