International Journal of Applied Engineering Research ISSN 0973-4562 Volume 10, Number 24 (2015) pp 44112-44117

Quantitative Analysis of Equilibrium Solution and Stability for Non-linear Differential Equation Governing Pendulum Clock

M.C. Agarana,

Lecture/Researcher, Department of Mathematics, Covenant University, Ota, Ogun State, Nigeria
michael.agarana@covenantuniversity.edu.ng

S.A. Bishop,

Lecture/Researcher, Department of Mathematics Covenant University, Ota Ogun State, Nigeria
Sheila.bishop@covenantuniversity.edu.ng

Abstract

The problem of existence of equilibrium solution and stability for certain non-linear differential equations is one of the most fundamental areas of research in dynamical systems, usually governed by the non-linear differential equations. In this paper we implore Quantitative study approach with the use of eigenvalue and eigenvector, to obtain differential equation properties such as equilibrium and stability. Also, important characteristics of the solutions of the differential equations are deduced without actually solving them. Mathematical model for pendulum clock are considered in analysing the equilibrium points and the stability of an equilibrium point, The analysis show that the equilibrium points reduce to the only point at the origin and suggests that the solutions of the nonlinear system circle and dye at the origin.

Keywords: Quantitative study, pendulum clock, Equilibrium Solution, Stability, Non-linear Differential Equation

Introduction

The quantitative study of different equations is concerned with how to deduce important characteristic of the solutions of differential equations without actually solving them [17, 18]. In this study we implored this quantitative study approach; which is used extremely for obtaining, from the differential equation, such properties as equilibrium and stability. We have seen how a lot of information about the solutions of a differential equation is available without solving it explicitly. In particular, there are oftentimes constant solutions which we called equilibrium solutions or fixed points of the differential equation [17, 19, 20]. These are important because they represent behaviour which persists in time. The classical pendulum problem shows how this approach may be used to reveal all the main features of the solutions of a particular differential equation [12, 18, 20]. Here is an analogy for stability and equilibrium: consider a rigid rod which is firmly attached to the wall on one end and allowed to pivot as a pendulum. If we work very hard, we could balance it so that it points straight up. However, just the slightest touch will cause it to move and eventually point downwards. This is an unstable equilibrium. Another equilibrium is when the pendulum points downwards. If we move it to some nearby place, it will swing until it eventually returns to the downward position. This is a stable equilibrium [3, 20]. In practice, a stable equilibrium is important because it represents behaviour which cannot easily be changed, it represents a fundamental feature of the system. For example, stable equilibria can be useful for making predictions because lots of solutions eventually settle down near the stable equilibria (like in the pendulum clock)[8, 10, 11, 15]. A pendulum clock is a simple non-linear system. Figure 1 shows the main features of the pendulum clock. The "escape wheel" is a toothed wheel, which drives the hand of the clock through a succession of gears. It has a spindle around which is around a wire with weight at its free end. The escape wheel is arrested by the "anchor" which has two teeth. The anchor is attached to the shaft of the pendulum and rocks with it, controlling the rotation of the escape wheel. Its teeth are so designed that as the pendulum reaches its maximum amplitude on one side one teeth of the escape wheel is released, and the escape wheel is then stopped again by the other tooth on the anchor.[16] Every time this happens the anchor receives a small impulse or pressure. It is this impulse that maintains the oscillation of the pendulum, which would otherwise die away.[5, 6] The loss of potential energy due to the weight's descent is therefore fed periodically into the pendulum via the anchor mechanism. It can be shown that the system will settle into steady oscillation of fixed amplitude independently of sporadic disturbance and of initial condition.[10,] If the pendulum is swinging with two great amplitude its loss of energy per cycle due to friction is large, and the impulse supplied by the escapement is insufficient to offset this.[1, 4, 13] The amplitude consequently decreases, if the amplitude is too small, and the frictional loss is small, the impulse will over compensate and the amplitude will build up. A balanced state is therefore approached, which appears in the θ, $\dot{\theta}$ plane.[18] The motion of nonlinear pendulum is determined by Newton's law. Nonlinear dynamics has profoundly changed how scientist view the world generally.

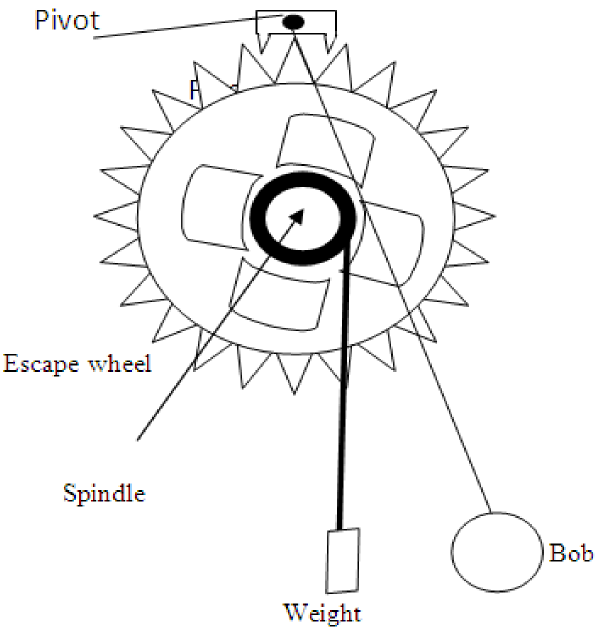

Figure 1. Weight driven clock mechanism

2. Preliminaries

2.1 Basic Definitions

With reference to the articles [17, 18, 20] we introduce in this section the following basic definitions and theorems:

2.1.1Definition 1

A physical system is said to be autonomous if its differential equation does not contain the independent variable (time t, say) explicitly. Hence if this differential equation is of second order, it is of the form

$$F(y, y', y'') = 0 \tag{1}$$

where

$$y' = \frac{dy}{dt} = v \text{, is the velocity} \tag{2}$$

By chain rule

$$v' = \frac{dv}{dt} = \frac{dv}{dy}.\frac{dy}{dt} = \frac{dv}{dy}v \tag{3}$$

2.1.2.Definition 2

An equilibrium solution of the system

$$y' = Ay \tag{4}$$

is a point (y_1, y_2) where

$$y' = 0 \tag{5}$$

that is

$$y'_1 = 0 = y'_2 \tag{6}$$

An equilibrium solution is a constant solution of the system, and is usually called a critical point.

2.1.3.Definition 3

A plane autonomous system is a pair of simultaneous first-order differential equations $x' = f(x, y), y' = g(x, y)$. This system has an equilibrium point (or fixed point or critical point or singular point) (x_0, y_0) when $f(x_0, y_0) = g(x_0, y_0) = 0$

2.1.4.Definition 4

The equilibrium point q is said to be stable if given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\phi(t, p) - q\| < \varepsilon$ for all $t > 0$ and for all p such that $\|p - q\| < \delta$ if δ can be chosen not only so that the solution q is stable but also so that $\phi(t, p) \to q$ as $t \to \infty$, then q is said to be asymptotically stable. If q is not stable it is said to be unstable.

2.2.Basic Theorem

2.2.1. Stability theorem:

Let $\dot{\boldsymbol{\theta}} = f(\boldsymbol{\theta})$ be an autonomous differential equation. Suppose $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_e$ is an equilibrium, that is, $f'(\boldsymbol{\theta}_e) = 0$, then if $f'(\boldsymbol{\theta}_e) < 0$, the equilibrium $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_e$ is stable and if $f'(\boldsymbol{\theta}_e) > 0$, then equilibrium $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_e$ is unstable.

3. Governing Equation and Analysis

3. Governing Equation and Analysis

The equation that governs the motion of a pendulum clock can be approximated as follows [7, 17, 18]

$$I\ddot{\theta} + K\dot{\theta} + C\,\theta = f(\theta) \tag{7}$$

Which can be written as

$$I\dot{\theta}\frac{d\dot{\theta}}{dt} + K\,\dot{\theta} + C\,\theta = f(\theta) \tag{8}$$

where,

I is the moment of the inertia of the pendulum, K is a small damping constant, C is another constant determined by gravity, θ is the angular displacement, and f(θ) is the moment Equation (8) relates $\dot{\theta}$ and θ instead of θ and t. It is an autonomous system. By integrating it with respect to θ;

$$I\int \dot{\theta}\,\frac{d\dot{\theta}}{d\theta}\,d\theta + K\int \dot{\theta}\,d\theta + C\int \theta\,d\,\theta = \int f(\theta)d\theta \tag{9}$$

which becomes

$$\frac{1}{2}I\,\dot{\theta}^2 + K\,\theta + \frac{1}{2}C\,\theta^2 = \int f(\theta)d\theta \tag{10}$$

A given pair of values $(\theta, \dot{\theta})$ is called a state of the dynamical system. A state evolves as time progresses; we know that a given state determines all subsequent states since it serves as initial conditions for the subsequent motion.[2, 9, 18] Equation (10) is a differential equation for θ in terms of t, but it cannot be solved in terms of elementary functions. It is therefore not easy to obtain a useful representation of θ as a function of time. We shall show how it is possible by working directly with equation (10) to reveal the main characteristics of the solution. Note that when the pendulum hangs without swinging, θ = 0, $\dot{\theta} = 0$. The corresponding function θ(t) = 0. This is a perfectly legitimate constant solution of equation (10). If the suspension consists of light rod there is a second position of equilibrium, where it is balanced vertically on end. This is the state θ = π, $\dot{\theta} = 0$, another constant solution of equation (10). Substituting three particular states into equation (10) we obtain the value of I, K and C by solving the resulting system of equations.

Taking $\int f(\theta)\ to\ be\ 0$ which its value is at (0, 0) and considering the following states of the system:

i) $(\theta, \dot{\theta}) = (\pi, 0)$
ii) $(\theta, \dot{\theta}) = (-\pi, 0)$
iii) $(\theta, \dot{\theta}) = (2\pi, 0)$

By substituting these states into the following equation

$$\frac{1}{2}\mathrm{I}\,\dot{\theta}^2 + K\,\theta + \frac{1}{2}\mathrm{C}\,\theta^2 = 0 \quad (11)$$

we have respectively:

$$\frac{1}{2}\mathrm{I}\,(0^2) + \mathrm{K}\,\pi + {}^1\!/_2\,\mathrm{C}\,\pi^2 = 0 \quad (12)$$

$$\frac{1}{2}\mathrm{I}\,(0^2) - \mathrm{K}\,\pi + {}^1\!/_2\,\mathrm{C}\,(-\pi)^2 = 0 \quad (13)$$

$$\frac{1}{2}\mathrm{I}\,(0^2) + 2\mathrm{K}\,\pi + {}^1\!/_2\,\mathrm{C}\,(2\pi)^2 = 0 \quad (14)$$

Equation (12), (13) and (14) become respectively:

$$\mathrm{K}\,\pi + \frac{1}{2}\pi^2 C = 0 \quad (15)$$

$$-\mathrm{K}\,\pi + \frac{1}{2}\pi^2\,\mathrm{C} = 0 \quad (16)$$

$$2\mathrm{K}\pi + 2\pi^2\ \mathrm{C} = 0 \quad (17)$$

Subtracting equation (16) from (15) we have

$$2\mathrm{K}\pi = 0 \quad (18)$$

Substitute equation (18) into equation (17) we have

$$2\pi^2\,\mathrm{C} = 0 \quad (19)$$

Equations (18) and (19) give K = 0 and C = 0 Also substituting K = 0, C = 0 into equation (11), we have I =0. Therefore, for any state of the system of the form (nπ, 0), n = 1, 2, 3, …, the constants I, K and C in equation (9) are always zero if $\int f(\theta)$ is taking to be zero. $[\int f(\theta) = 0\ for\ state\ (0,0)]$

Now for $\int f(\theta) \neq 0$, and $f(\theta)$ a periodic function in θ, Fourier series of f(θ) on –L ≤ θ ≤ L is given as [21, 22]

$$f(\theta) = \sum_{n=0}^{\infty} An\ \ Cos\,(\frac{n\pi\theta}{L}) + \sum_{n-1}^{\infty} Bn\ Sin(\frac{n\pi\theta}{L}) \quad (20)$$

Determining formulae for the coefficients of An and Bn, we take advantage of the fact that $[Cos\,(\frac{n\pi\theta}{L})]_{n=0}^{\infty}$ and $[Sin\,(\frac{n\pi\theta}{L})]_{n-1}^{\infty}$ are naturally orthogonal on-L ≤ θ ≤ L.

Let us consider the following formula

$$\int_{-L}^{L} Cos\left(\frac{n\pi\theta}{L}\right) Cos\left(\frac{m\pi\theta}{L}\right) d\theta = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases} \quad (21)$$

$$\int_{-L}^{L} Sin\,(\frac{n\pi\theta}{L})\ Sin\left(\frac{m\pi\theta}{L}\right) d\theta = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \quad (22)$$

$$\int_{-L}^{L} Sin\,(\frac{n\pi\theta}{L})\ Cos\left(\frac{m\pi\theta}{L}\right) d\theta = 0 \quad (23)$$

Now, multiplying both sides of equation (20) by $Cos\,(\frac{m\pi\theta}{L})$ *and* integrating from –L to L, we have

$$\int_{-L}^{L} f(\theta) Cos\left(\frac{m\pi\theta}{\theta}\right) d\theta = \int_{-L}^{L}\sum_{n-0}^{\infty} An \int_{-L}^{L} Cos\left(\frac{n\pi\theta}{L}\right) Cos\left(\frac{m\pi\theta}{L}\right) d\theta + \int_{-L}^{L}\sum_{n=1}^{\infty} Bn\,Sin\left(\frac{n\pi\theta}{L}\right) Cos\left(\frac{m\pi\theta}{L}\right) d\theta \quad (24)$$

Interchange the integral and the summation we have

$$\int_{-L}^{L} f(\theta) Cos\left(\frac{m\pi\theta}{L}\right) d\theta = \sum_{n=0}^{\infty} An \int_{-L}^{L} Cos\left(\frac{n\pi\theta}{L}\right) Cos\left(\frac{m\pi\theta}{L}\right) d\theta + \sum_{n=1}^{\infty} Bn \int_{-L}^{L} Sin\left(\frac{n\pi\theta}{L}\right) Cos\left(\frac{m\pi\theta}{L}\right) d\theta \quad (25)$$

Taking advantage of the fact that the trigonometric ratios, sine and cosine are naturally orthogonal; the integral in equation (25) will always be zero and in the first series the integral will be zero if n ≠ m and so this reduces to,

$$\int_{-L}^{L} f(\theta) Cos\left(\frac{m\pi\theta}{L}\right) d\theta = \begin{cases} \mathrm{Am}\,(2L) & \text{if } n = m = 0 \\ \mathrm{Am}\,(L) & \text{if } n = m \neq 0 \end{cases} \quad (26)$$

Solving for Am gives

$$\mathrm{Ao} = \frac{1}{2L}\int_{-L}^{L} f(\theta) d\theta,\ m = 0 \quad (27)$$

$$\mathrm{Am} = \frac{1}{L}\int_{-L}^{L} f(\theta) Cos\left(\frac{m\pi\theta}{L}\right) d\theta,\ m = 1, 2, 3 \ldots \quad (28)$$

Similarly, multiply both side of equation (20) by $Sin(\frac{m\pi\theta}{L})$ and integrating both side from –L to L and interchanging the integral and summation gives

$$\int_{-L}^{L} f(\theta) Sin\left(\frac{m\pi\theta}{L}\right) d\theta = \sum_{n=0}^{\infty} An \int_{-L}^{L} Cos\left(\frac{n\pi\theta}{L}\right) Sin\left(\frac{m\pi\theta}{L}\right) d\theta + \sum_{n=1}^{\infty} Bn \int_{-L}^{L} Sin\left(\frac{n\pi\theta}{L}\right) Sin\left(\frac{m\pi\theta}{L}\right) d\theta \quad (29)$$

In this case the integral in the first series will always be zero and the second will be zero if n ≠ m and so we have

$$\int_{-L}^{L} f(\theta) Sin\left(\frac{m\pi\theta}{L}\right) d\theta = Bn(L)\ \text{(for n = m)} \quad (30)$$

Finally, solving for Bn gives

$$\mathrm{Bn} = \frac{1}{L}\int_{-L}^{L} f(\theta)\ Sin\left(\frac{m\pi\theta}{L}\right) d\theta,\ m = 1, 2, 3 \ldots \quad (31)$$

Now from equation (28), for n = m ≠ 0

$$\mathrm{An} = \frac{1}{L}\int_{-L}^{L} f(\theta) Cos\left(\frac{m\pi\theta}{L}\right) d\theta,\ m = 1, 2, 3 \ldots \quad (32)$$

So equation (20) becomes

$$f(\theta) = \sum_{n=0}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\theta) Cos\left(\frac{m\pi\theta}{L}\right) Cos\left(\frac{n\pi\theta}{L}\right) d\theta + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\theta) Sin\left(\frac{m\pi\theta}{L}\right) Sin\left(\frac{n\pi\theta}{L}\right) d\theta \quad (33)$$

For our problem n = m, therefore

$$f(\theta) = \sum_{n=0}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\theta) Cos\left(\frac{n\pi\theta}{L}\right) Cos\left(\frac{n\pi\theta}{L}\right) d\theta + \sum_{n-1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\theta) Sin\left(\frac{n\pi\theta}{L}\right) Sin\left(\frac{n\pi\theta}{L}\right) d\theta \quad (34)$$

Generally, equation (34) can be written as

$$f(\theta) = \frac{1}{L}\sum_{n=0}^{\infty}\int_{-L}^{L} f(\theta)[Cos^2\left(\frac{n\pi\theta}{L}\right) + Sin^2\left(\frac{n\pi\theta}{L}\right)]\ d\theta \quad (35)$$

$$f(\theta) = \frac{1}{L}\sum_{n=0}^{\infty}\int_{-L}^{L} f(\theta)\ d\theta \quad (36)$$

$$f(\theta) = \frac{1}{L}\int_{-L}^{L} f(\theta)\ d\theta \quad (37)$$

At equilibrium, $f(\theta) = 0$, equation (37) therefore becomes:

$$0 = \int_{-L}^{L} f(\theta) d\theta, \quad (38)$$

Therefore at equilibrium when angular velocity θ = 0, the motion of the pendulum clock can be approximated as equation (11).

Generally at any state (nπ, L) equation (11) becomes

$$\frac{1}{2}I\,(n\pi)^2 + KL + \frac{1}{2}CL^2 = \frac{1}{L}\sum_{n=0}^{\infty}\int_{-L}^{L} f(\theta)[(-1)^{2n} + (-1)^n(0)]d\theta\ |\theta = L \quad (39)$$

Since $Cos\,(n\pi) = (-1)^n$ and $Sin(n\pi) = 0$

$$\frac{1}{2}\ I\,(n\pi)^2 + KL + \frac{1}{2}CL^2 = \frac{1}{L}\sum_{n=0}^{\infty}\int_{-L}^{L} f(\theta) d\theta \quad (40)$$

$$\frac{1}{2}\ L\,[2I(n\pi)^2 + 2LK + CL^2)] = \int_{-L}^{L} f(\theta) d\theta \quad (41)$$

For this work n is always taken to be an integer.

4. Further Analysis

4.1 Converting the 2nd order ODE to 1st order system

Equation (7) governing the motion of pendulum clock can be written as

$\ddot{\theta}$ = f(θ) – (K/I)$\dot{\theta}$ – (C/I) θ (42)

Equation (42) can then be converted to a first order system as follows: Let X_1 = θ (the angular displacement of the pendulum clock) and X_2 =$\dot{\theta}$ (the angular velocity of the pendulum clock) then

$X'_1 = \dot{\theta} = X_2$ (43)

$X'_2 = \ddot{\theta}$ = f(θ) – (K/I)$\dot{\theta}$ – (C/I) θ (44)

This implies

$X'_1 = X_2$ (45)

$X'_2 = f(X_1) - (K/I)X2 - (C/I) X_1$ (46)

Without solving these equations, we will examine the behaviour of their solutions. The equilibrium points and stability of these points can be shown also.

4.2 The Equilibrium Solution

The system considered above is autonomous. Equations (45) and (46) possess equilibrium point θ^* That is

$X'_1 = 0$ (47)

$f(X_1) - (K/I)X2 - (C/I) X_1 = 0$ (48)

Equation (70) implies

$f(X_1) = 0$ (49)

which implies X(t) = X_1 is an equilibrium solutionInitial condition; X(0) = X_1 implies solution X(t) = X_1 for all time t. Now If initial condition is X(0) = $X_{0,}$ is close to X_1 (X_0 not equal to X_1), then there are different possibilities: Unstable equilibrium: X(t) moves away from X_1 as time t increases. Stable equilibrium: X(t) moves toward the equilibrium solution X_1 (or at least does not get further away). Substituting equation () into equation (), we have:

$-(K/I)X2 - (C/I) X_1 = 0$ (50)

$X_2 = -(C/K)X_1$ (51)

Differentiating both sides of equation (51) we have $X'_2 = K_1X'_1$ for some constant K_1 (52)

This implies

$X'_2 = K_1X_2$ (53)

For equilibrium $X'_2 = 0$, which implies $X_2 = 0$ Point (0, 0) is the equilibrium solution..

4.3 Using Eigenvalues and eigenvector

The above linear system of linear equations usually has exactly one solution, located at the origin, if determinant of A is not equal to zero. Here, A is the matrix formed from the system of equations shown below. There are usually infinitely many solutions if the determinant of A is zero. Now writing equations (51) and (52) in matrix form we have

$$\frac{d}{dx}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} 0x_1 + x_2 \\ \frac{-c}{I}x_1 + f(\mathrm{x}_1) - \frac{k}{I}x_2 \end{matrix} \quad (54)$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-c}{I} & \frac{-k}{I} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, f(\mathrm{x}_1) = 0 \quad (55)$$

$$X' = AX \quad (56)$$

The above system usually has exactly one solution, located at the origin, if determinant of A is not equal to zero. If the determinant of A is zero, then there are many solutions. Now we chose values of C, I and K that will make the determinant of A non-zero: C=2, I=1, K=-3.

$$A = \begin{pmatrix} 0 & 1 \\ \frac{-c}{I} & \frac{-k}{I} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (57)$$

detA = 2 (58)

Finding eigenvalue (λ) of A: $\det(A - \lambda U) = 0,$ where U is a unit matrix

$$A - \lambda U = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad (59)$$

$$A - \lambda U = \begin{pmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix} \quad (60)$$

$$\det(A - \lambda U) = \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} \quad (61)$$

$$0 = \lambda^2 - 3\lambda + 2 \quad (62)$$

$\lambda =$ 2 or 1 (63)

Finding the eigenvector by using Gaussian elimination: For each eigenvalue we have $(A - \lambda U)\mathrm{x} = 0$, where x is the eigenvector associated with eigenvalue (λ)

Case 1: When $\lambda = 1$ We find vector x which satisfies $(A - 1U)\mathrm{x} = 0$

$$A - U = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (64)$$

$$= \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \quad (65)$$

$$(A - U)\mathrm{x} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (66)$$

Implies $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (67)

Case 2: When $\lambda = 2$, We find x such that

$(A - 2U)\mathrm{x} = 0$ (68)

$$A - 2U = \begin{pmatrix} 0 & 2 \\ -2 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (69)$$

$$= \begin{pmatrix} -1 & 2 \\ -2 & 5 \end{pmatrix} \quad (70)$$

$$(A - 2U)\mathrm{x} = \begin{pmatrix} -1 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (71)$$

Implies $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (72)

We can see that in both cases we have the same zero eigenvector. Which indicates exactly one solution, located at the origin. This is the equilibrium solution

4.4 Stability of the Equilibrium Solution

The equilibrium points reduce to the only point (0, 0). Let us find the nullclines and the direction of the velocity vectors along them. The x_1-nullcline is given by

$$\frac{dx_1}{dt} = x_2 = 0 \tag{73}$$

Hence the x_1-nullcline is the x-axis. The x_2 nullcline is given by

$$\frac{\partial^2 \dot{\theta}}{\partial t^2} = f(\theta) - \frac{K}{I}\dot{\theta} - \frac{C}{I}\theta = 0 \tag{74}$$

Which implies

$$\frac{\partial^2 x_2}{\partial t^2} = f(x_1) - 3x_2 - 2x_1 = 0 \tag{75}$$

$$f(x_1) - 3x_2 - 2x_1 = 0 \tag{76}$$

$$\therefore \; x_2 = \frac{1}{3}[-2x_1 + f(x_1)], \text{ for K=3, C=2 and I=1} \tag{77}$$

Hence the x_2-nullcline is the curve.

Note that for some value of $f(x_1)$, the arrangement of the curves will show that the solutions ’’circles" around the origin. But it is not clear whether the solutions circle and dye at the origin, circle away from the origin, or keep on circling periodically. A very rough approach to this problem suggests that when $f(x_1)$ is close to 0, the curve approaches $-\frac{2}{3}x_1$.

Hence a close system to the original nonlinear system is

$$\frac{dx_1}{dt} = x_2 \tag{78}$$

$$\frac{dx_2}{dt} = -\frac{2}{3}x_1 \tag{79}$$

which happens to be a linear system.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{3} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{80}$$

The eigenvalues are

$$\lambda = \pm\sqrt{-\frac{2}{3}} \tag{81}$$

and the eigenvector is zero:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we suggest that the solutions of nonlinear system circle and dye at the origin.

5.Conclusion

Analysis of equilibrium solution for non-linear equation governing pendulum clock was carried out using quantitative study approach. This autonomous physical system was studied using Eigenvalue and Eigenvector to analyse its equilibrium solution after converting the governing second order differential equation to a set of first order differential equations. The stability of the equilibrium solution was analysed. From the analysis we suggest that the solutions to linear system close to the original nonlinear system ‘circle and die’ at the origin.

References

[1] R.B. Kidd and S.L. Fogg (2002), A simple formula for the large-angle pendulum period, phys Teach., 40, 81-83

[2] Thomton and J. Marion, (2003), Classical Dynamics of particles and systems, 5th edition Belmont, CA: Brooks/Cole

[3] *Marrison, Warren (1948).* "The Evolution of the Quartz Crystal Clock". *Bell System Technical Journal 27: 510–588.* doi:10.1002/j.1538-7305.1948.tb01343.x.

[4] Henrique M. Oliveira and Luis V. Melo (2015), Huygens synchronization of two clocks, scientific reports 5, 11548

[5]" Huygens' Clocks". *Stories. Science Museum, London, UK.* Retrieved 2007-11-14.

[6] Sarafian (2011), A study of super-nonlinear motion of simple pendulum, Mathematica journal. dx.doi.org/doi:10.3888/tmj. 13-14

[7] M.C. Agarana and O.O. Agboola (2015), Dynamic Analysis of Damped Driven Pendulum using laplace Transform method, International journal of Mathematics and computation, 26(3), 98-109

[8]" Pendulum Clock". *The Galileo Project. Rice Univ.* Retrieved 2007-12-03.

[9] L.P. Fulcher and B.F Davis, (1976), Theoretical and experimental study of the motion of the simple pendulum, Am J. Phys 44, 51-55.

[10] A modern reconstruction can be seen at "Pendulum clock designed by Galileo, Item #1883-29". *Time Measurement. Science Museum, London, UK.* Retrieved 2007-11-14.

[11] Rockwood and Kurt Wiessenfeld, (2002), Huygen’s clocks, proceedings: Mathematical, Physical and Engineering Sciences, 458(2019), 563-579

[12] Peter J. Olver (2013) Nonlinear Ordinary, Differential equations, University of Minnesota

[13] Baker Gregory L and Blackburn James A (2005), The Pendulum: A physics case study. Oxford University Press.

[14] *Arnstein, Walt.* "The Gravity Pendulum and its Horological Quirks". *Community Articles. Timezone.com website.* Retrieved 2011-04-01.

[15*]* Bennet, Matthew; et al. (2002). *"Huygens' Clocks"* (PDF). Georgia Institute of Technology. Archived from *the original* (PDF) on 2008-04-10. Retrieved 2007-12-04., p.3, also published in Proceedings of the Royal Society of London, A 458, 563–579

[16] Headrick, Michael (2002*).* "Origin and Evolution of the Anchor Clock Escapement". Control Systems magazine, (Inst. of Electrical and Electronic

Engineers) 22 (2). Archived from *the original* on 2009-10-25. Retrieved 2007-06-06.

[17] D.W. Jordan and P. Smith (2007), Nonlinear Ordinary Differential Equations – An introduction for scientists and Engineers, Oxford University Press, New York.

[18] Daba Meshesha Gusu and O. Chandra Sekhara Reddy, (2015), Solutions of non linear ordinary differential equations determined by phase plane methods, International journal of Basic and applied sciences, 4(1), 22-27.

[19] Mathematics, *http://math.stackexchange.com/questions*?

[20] Zachary S. Tseng, (2008), Autonomous equation/stability of equilibrium solutions. 1-20

[21] Brad Osgood, (2007), the Fourier transform and its applications – a lecture note (EE261), Electrical Engineering department, Stanford University.

[22] Lecture note on Fourier Series (pdf file).www.ima.umn.edu/-miller/fourierseries.pdf