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Distance Two Labeling of Some Special Graphs

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Abstract
An L(2,1) labeling (or) distance two labeling of a graph G is a function f from the vertex set V(G) to the set of all non-negative integers such that $|f(x) - f(y)| \geq 2$ if d(x,y)=1 and $|f(x) - f(y)| \geq 1$ if d(x,y)=2. The L(2, 1) labeling number $\lambda(G)$ of G is the smallest number k such that G has an L(2, 1) labeling with max{f(v), $v \in V(G)$} = k. In this paper the L(2, 1) labeling number $\lambda(G)$ for some special graphs like (m, n) kite graph, Jelly fish graph J(m, n) and coconut tree graph CT(m, n) have been determined.

Keywords: L(2, 1) labeling, λ number, cycle, path.

1 Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions.Labeled graphs serve as useful models for broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication networks and data base management and models for constraint programming over finite domain. The concept of graph labeling was introduced by Rosa in 1967. Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, total cordial labeling, k-graceful labeling and odd graceful labeling etc., have been studied in over1100 papers.

Hale [6] introduced the graph theory model of assignment of channels in 1980 in which a graph was considered whose vertices represented the stations and the edges represented the stations. A channel assignment problem was designed in such a way that the vertices of distance two are considered to be close and vertices which are adjacent, are considered to be very close which paved way for distance two labeling of graphs. The graphs considered here are all finite, undirected and simple. V(G) and E(G) denote the vertex set and the edge set of G. Labeling with a condition of distance two was introduced by J.R. Griggs and R.K.Yeh[5] who proved that every graph with maximum degree k has an L(2, 1)-labeling with span at most k+2k and proved the conjecture for 2-regular graphs. G.J. Chang and D. Kuo [1] improved this upper bound to k^2+k. Chang et al. [2] generalized this to obtain $k^2+(d-1)k$ as an upper bound on the minimum span of an L(d,1)-labeling. Z. Fredi and J.H. Kang [3] proved it for 3-regular Hamiltonian graphs and for the incidence graphs of projective planes. Tight bounds on the maximum span have been obtained for special classes of graphs like paths, cycles, wheels, complete k-partite graphs and graphs with diameter 2 [1], trees [1,2], etc. Bounds have also been obtained for various other graph families like chordal graphs and unit interval graphs [9], hyper cubes [5, 9, 10], and planar graphs [7]. The pair sum labeling of (m, n) kite, Jelly fish J(m, n) and coconut tree CT(m, n) is studied in [8] by K.Manimekalai and K. Thirusangu.In this paper, we study the L(2, 1) labeling of (m, n) kite, Jelly fish J(m, n) and coconut tree CT(m, n).

2. Preliminaries

Definition 2.1.
Graph labeling is an assignment of integers to vertices or edges or both under certain conditions

Definition 2.2.
An L(2,1) labeling of a graph is a labeling of the vertices with non-negative integers such that the labels on adjacent vertices differ by at least 2 and the labels on vertices at distance 2 differ by at least 1.

Definition 2.3.
An (m, n) kite is obtained by attaching a cycle of length m with a path of length n. Some kites are shown in Figure 1.

Definition 2.4.
The jelly fish graph J(m, n) is obtained by attaching pendent edges to all the vertices of degree two in $K_4-\{e\}$ graph. Since there are two vertices of degree two, let m pendent edges (the head part) be attached to one vertex and n pendent edges (the tail part) be attached to the other vertex. The jelly fish graph J(7, 5) is shown in Figure 2.

Definition 2.5.
A coconut tree CT(m, n) is the graph obtained by attaching n pendent edges at one end vertex of a path of length m. A coconut tree CT(7, 9) is shown in Figure 3.

Definition 2.6.
Any function $f : V(G) \to N \cup \{0\}$is said to be a valid L(2,1) labeling if and only if it satisfies the condition $d(u, v) + |f(u) - f(v)| \geq 3$.

3. Main Results

Theorem 3.1.
The L(2, 1) labeling number $\lambda(G)$ of an (m, n) kite is 4.
Proof.
Denote the vertices of an (m,n) kite as follows:

$$V(C_i) = \{v_i / 1 \leq i \leq m\}$$
$$V(P_j) = \{u_j / 1 \leq j \leq n\}$$

where v_i are vertices on the cycle and u_j are the vertices on the path such that$v_m$is attached to u_1. The number of vertices on the cycle and the path are chosen in such away that both m and n are greater than two.

Case (i):
If $m \equiv 0 \pmod 3$, define a mapping$f : V(G) \to N \cup \{0\}$by

$$f(v_{3i-2}) = 0; \quad 1 \leq i \leq \frac{m}{3}$$
$$f(v_{3i-1}) = 2; \quad 1 \leq i \leq \frac{m}{3}$$
$$f(v_{3i}) = 4; \quad 1 \leq i \leq \frac{m}{3}$$
$$f(u_1) = 1; \quad f(u_2) = 3$$
$$f(u_{3j}) = 0; \quad 1 \leq j \leq \left\lceil \frac{n-1}{3} \right\rceil$$
$$f(u_{3j+1}) = 2; \quad 1 \leq j \leq \left\lceil \frac{n-2}{3} \right\rceil$$
$$f(u_{3j+2}) = 4; \quad 1 \leq j \leq \left\lceil \frac{n-3}{3} \right\rceil$$

Claim:
If $m \equiv 0 \pmod 3$, then the L(2,1) labeling number $\lambda(G)$ of an (m,n) kite is 4.Let u, vbe any two vertices in$V(G)$.

Subcase (i):
Let u, vbe any two adjacent vertices on the cycle, such that $u = v_{3i-1}$and $v = v_{3i-2}$.
Then$f(u) = 2$, $f(v) = 0$ and $d(u, v) = 1$.
Therefore $d(u, v) + |f(u) - f(v)| = 1 + |2-0| \geq 3$.

Subcase (ii):
Let u, vbe any two vertices on the cycle at a distance two, such that $u = v_{3i}$and $v = v_{3i-2}$.
Then$f(u) = 4$ and $f(v) = 0$ and $d(u, v) = 2$.
Therefore$d(u, v) + |f(u) - f(v)| = 2 + |4-0| = 6 \geq 3$.

Subcase (iii):
Let u, vbe any two adjacent vertices on the path, such that $u = u_{3j}$and$v = u_{3j+1}$.
Then$f(u) = 0$ and $f(v) = 2$ and $d(u, v) = 1$.
Therefore $d(u, v) + |f(u) - f(v)| = 1 + |2-0| \geq 3$

Subcase (iv):
Let u, vbe any two vertices on the path at a distance two, such that $u = u_{3j}$and $v = u_{3j+2}$.
Then$f(u) = 0$and $f(v) = 4$ and $d(u, v) = 2$.
Therefore$d(u, v) + |f(u) - f(v)| = 2 + |0-4| = 6 \geq 3$.

Subcase (v):
Let u, v be any two vertices, on the cycle and the path respectively at a distance two such that $u = v_{3i}$ and $v = u_{3j}$.
Then $f(u) = 4$ and $f(v) = 0$ and $d(u, v) = 2$.
Therefore $d(u, v) + |f(u) - f(v)| = 2 + |4-0| = 6 \geq 3$.

Subcase (vi):
Let u, vbe any two adjacent vertices on the cycle and the path respectively, such that $u = v_{3i}$ and $v = u_1$.
Then$f(u) = 4$ and $f(v) = 1$, then $d(u, v) = 1$.
Therefore $d(u, v) + |f(u) - f(v)| = 1 + |4-1| = 4 \geq 3$.

Case (ii):
If $m \equiv 1 \pmod 3$, define a mapping$f : V(G) \to N \cup \{0\}$ by

$$f(v_{3i+1}) = 0; \quad 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil$$
$$f(v_{3i+2}) = 2; \quad 1 \leq i \leq \left\lceil \frac{m-3}{3} \right\rceil$$
$$f(v_{3i+3}) = 4; \quad 1 \leq i \leq \left\lceil \frac{m-3}{3} \right\rceil$$
$$f(v_1) = 3, \quad f(v_2) = 1, \quad f(v_3) = 4$$
$$f(u_{3j-2}) = 0; \quad 1 \leq j \leq \left\lceil \frac{n+1}{3} \right\rceil$$
$$f(u_{3j-1}) = 2; \quad 1 \leq j \leq \left\lceil \frac{n}{3} \right\rceil$$
$$f(u_{3j}) = 4; \quad 1 \leq j \leq \left\lceil \frac{n-1}{3} \right\rceil$$

Claim:

If m ≡ 1 (mod 3), then the L(2,1) labeling number λ(G) of an (m,n) kite is 4.
Let u, vbe any two vertices inV(G).

Subcase (i):

Let u, v be any two adjacent vertices on the cycle, such that u = $v_{3i+1}$and v = v_{3i+2}.
Thenf(u) = 0, f(v) = 2 and d(u, v) = 1.
Therefored(u, v) + |f(u) – f(v)| = 1 + |0–2| ≥ 3.

Subcase (ii):

Let u, vbe any two vertices on the cycle at a distance two, such that u = $v_1$and u = v_3.
Thenf(u) = 3, f(v) = 4 and d(u, v) = 2.
Therefore d(u, v) + |f(u) – f(v)| = 2 + |3–4| ≥ 3.

Subcase (iii):

Let u, vbe any two adjacent vertices on the path, such thatu = u_{3j-2}andv = u_{3j-1}.
Thenf(u) = 0, f(v) = 2 and d(u, v) = 1.
Therefore d(u, v) + |f(u) – f(v)| = 1 + |0–2| ≥ 3.

Subcase (iv):

Let u, vbe any two vertices on the path at a distance two, such thatu = $u_{3j-2}$and v = u_{3j}.
Thenf(u) = 0, f(v) = 4 and d(u, v) = 2.
Therefore d(u, v) + |f(u) – f(v)| = 2 + |0–4| ≥ 3.

Subcase (v):

Let u, vbe any two adjacent vertices on the cycle and the path respectively, such that u = $v_{3i+3}$and v = u_{3j-2}.
Thenf(u) = 4, f(v) = 0 and d(u, v) = 1.
Therefore d(u, v) + |f(u) – f(v)| = 1 + |4–0| ≥ 3.

Subcase (vi):

Let u, vbe any two vertices on the cycle and the path respectively at a distance two, such that u = $v_{3i+3}$and v = u_{3j-1}.
Thenf(u) = 4, f(v) = 2 and d(u, v) = 2.
Therefore d(u, v) + |f(u) – f(v)| = 2 + |4–2| ≥ 3.

Case (iii):

If m ≡ 2 (mod 3), define a mappingf : V(G) → N ∪ {0}by

$$f(v_{3i}) = 0; \quad 1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$$

$$f(v_{3i+1}) = 2; \quad 1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$$

$$f(v_{3i+2}) = 4; \quad 1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$$

$$f(v_1) = 1, \quad f(v_2) = 3$$

$$f(u_{3j-2}) = 0; \quad 1 \le j \le \left\lceil \frac{n+1}{3} \right\rceil$$

$$f(u_{3j-1}) = 2; \quad 1 \le j \le \left\lceil \frac{n}{3} \right\rceil$$

$$f(u_{3j}) = 4; \quad 1 \le j \le \left\lceil \frac{n-1}{3} \right\rceil$$

Claim:

If m ≡ 2 (mod 3), then the L(2, 1) labeling number λ(G)of an (m,n) kite is 4.Let u, vbe any two vertices in V(G).

Subcase (i):

Let u, vbe any two adjacent vertices on the cycle, such that u = v_{3i}andv = u_{3i}.
Thenf(u) = 0, f(v) = 2 and d(u, v) = 1.
Therefore d(u, v) + |f(u) – f(v)| = 1 + |0–2| ≥ 3.

Subcase (ii):

Let u, vbe any two vertices on the cycle at a distance two, such that u = $v_{3i}$and v = v_1.
Thenf(u) = 0, f(v) = 1 and d(u, v) = 2.
Therefore d(u, v) + |f(u) – f(v)| = 2 + |0–1| ≥ 3.

Subcase (iii):

Let u, vbe any two adjacent vertices on the path, such that u = u_{3j-2}andv = u_{3j-1}.
Thenf(u) = 0, f(v) = 2 and d(u, v) = 1.
Therefore d(u, v) + |f(u) – f(v)| = 1 + |0–2| ≥ 3.

Subcase (iv):

Let u, vbe any two vertices on the path at a distance two, such that u = $u_{3j-2}$and v = u_{3j}.
Thenf(u) = 0, f(v) = 4 and d(u, v) = 2.
Therefore d(u, v) + |f(u) – f(v)| = 2 + |0–4| ≥ 3.

Subcase (v):

Let u, vbe any two adjacent vertices on the cycle and the path respectively, such that u = $v_{3i+2}$and v = u_{3j-2}.
Thenf(u) = 4, f(v) = 0 and d(u, v) = 1.
Therefore d(u, v) + |f(u) – f(v)| = 1 + |4–0| ≥ 3.

Subcase (vi):

Let u, vbe any two vertices on the cycle and the path respectively at a distance two, such that u = $v_1$and v = u_{3j-2}.
Thenf(u) = 1, f(v) = 0 and d(u, v) = 2.
Therefore d(u, v) + |f(u) – f(v)| = 2 + |1–0| ≥ 3.

Similarly for all the other possibilities of u and v d(u, v) + |f(u) – f(v)| ≥ 3. Therefore the L(2, 1) labeling of an (m, n) kite is $\lambda(G) = 4$.

Example 3.1.

L(2,1) labeling of (12,5)Kite, (7,3)Kite and (11,6)Kite are shown in Figure 1.

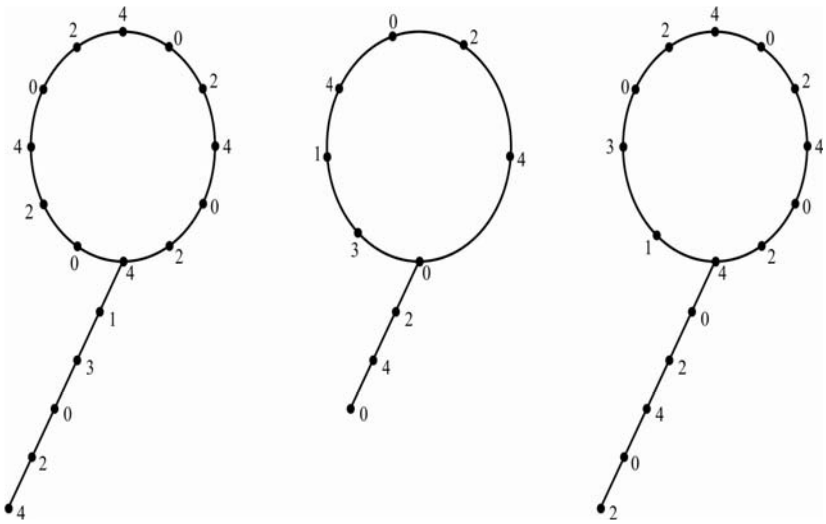

Figure 1.(12, 5) Kite (7, 3) Kite (11, 6) Kite

Theorem 3.2.

The L(2, 1) labeling number $\lambda(G)$ of Jelly fish graph J(m, n) is max(m, n)+4.

Proof.

Denote the vertices of the J(m, n) as follows

$V(C_i) = \{u_i / 1 \leq i \leq 4\}$

$V(H_j) = \{v_j / 1 \leq j \leq m\}$

$V(T_k) = \{w_k / 1 \leq k \leq n\}$

where C_i is K_4–{e}graph (body of jelly fish) in which $\{e\} = u_2 u_4$.

H_j denotes the mpendent edges attached to the vertex u_4 of the K_4–{e} (the head part) and T_k denotes the n pendent edges attached to the vertex u_2 of the K_4–{e}(the tail part).

Case (i):

Define a mapping $f : V(G) \to N \cup \{0\}$ such that

$$f(u_1) = 0, \quad f(u_2) = 2, \quad f(u_3) = 4, \quad f(u_4) = 6$$

$$f(v_1) = 1, \quad f(v_2) = 2, \quad f(v_3) = 3$$

$$f(v_{3+j}) = 7 + j, \quad 1 \leq j \leq m - 3$$

$$f(w_k) = 4 + k, \quad 1 \leq k \leq n$$

Claim.

The L(2,1) labeling number $\lambda(G)$ of J(m,n) is max (m,n) + 4.

Let u, vbe any two vertices in V(G).

Subcase (i).

Let u, vbe any two adjacent vertices K_4–{e}, such that $u = u_1$and $v = u_2$.

Thenf(u) = 0, f(v) = 2 and d(u, v) = 1.

Therefore $d(u, v) + |f(u) - f(v)| = 1 + |0–2| \geq 3$.

Subcase (ii).

Let u, v be any two vertices at a distance two on H_j, such that $u = v_1$ and $v = v_2$.

Then f(u) = 1, f(v) = 2 and d(u, v) = 2.

Therefore $d(u, v) + |f(u) - f(v)| = 2 + |1–2| \geq 3$.

Subcase (iii).

Let u, v be any two vertices at a distance two on $H_j$and K_4–{e} such that $u = v_1$ and $v = u_1$.

Then f(u) = 1, f(v) = 0 and d(u, v) = 2.

Therefore $d(u, v) + |f(u) - f(v)| = 2 + |1–0| \geq 3$.

Subcase (iv).

Let u, v be any two vertices at a distance two on T_k, such that $u = w_k$ and $v = w_{k+1}$.

Then f(u) = 4+k, f(v) = 5+k and d(u, v) = 2.

Therefore $d(u, v) + |f(u) - f(v)| = 2 + |4+k–5–k| \geq 3$.

Subcase (v).

Let u, v be any two vertices at a distance two on T_k and K_4–{e} respectively such that $u = w_1$ and $v = u_3$.

Then f(u) = 5, f(v) = 4 and d(u, v) = 2.

Therefore $d(u, v) + |f(u) - f(v)| = 2 + |5–4| \geq 3$.

Example 3.2.

L(2,1) labeling of Jelly fish J(7, 5) is shown in Figure 2.

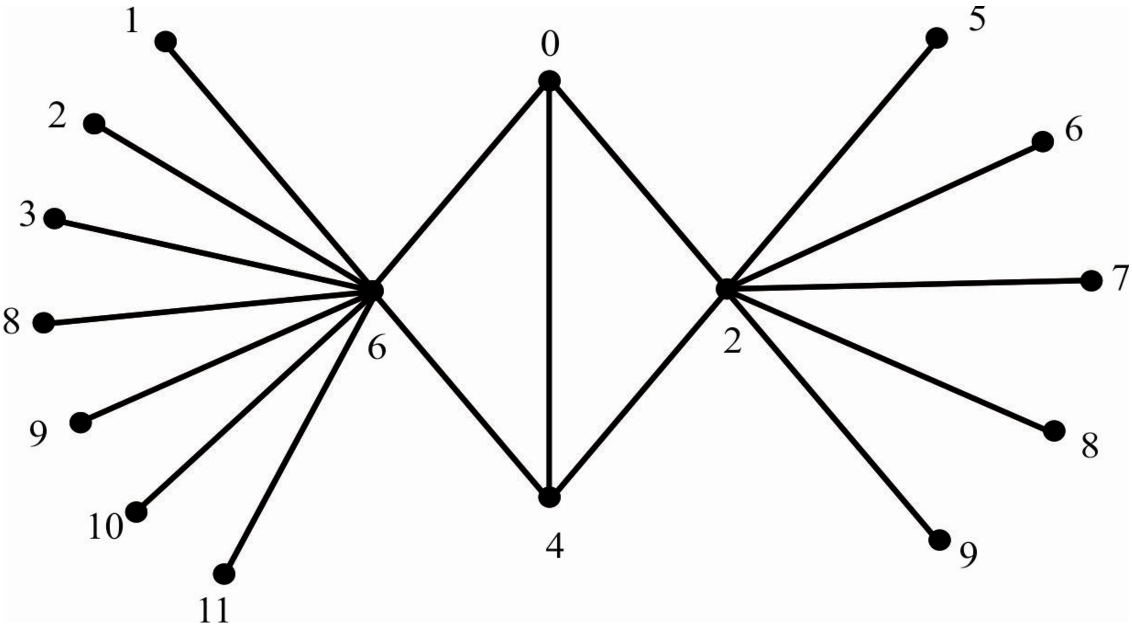

Figure 2

Hence the labeling number $\lambda(G)$ of a J(m, n) jelly fish graph ismax(m, n)+4.

Theorem 3.3.

The L(2, 1) labeling number $\lambda(G)$ of a coconut tree graph CT(m,n) is n + 2.

Proof.

Denote the vertices of the coconut tree graph CT(m,n) as follows

$V(P_i) = \{u_i / 1 \le i \le m\}$

$V(L_j) = \{v_j / 1 \le j \le n\}$

where $P_i$denotes a path of length m (stem of the coconut tree) and $L_j$denote the n pendent edges (leaves) attached to the end vertex of P_i.

Case (i):

Define a mapping $f : V(G) \to N \cup \{0\}$ such that

$$f(u_{3i-2}) = 0, \quad 1 \le i \le \left\lceil \frac{m+1}{3} \right\rceil$$

$$f(u_{3i-1}) = 2, \quad 1 \le i \le \left\lceil \frac{m}{3} \right\rceil$$

$$f(u_{3i}) = 4, \quad 1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$$

$$f(v_j) = 2 + j, \quad 1 \le j \le n$$

Claim:

The L(2,1) labeling number $\lambda(G)$ of CT(m,n) is n+2.

Let u, vbe any two vertices in$V(G)$.

Subcase (i).

Let u, vbe any two adjacent vertices on the path P_i, such that $u = u_{3i-2}$and $v = u_{3i-1}$.

f(u) = 0, f(v) = 2 and d(u, v) = 1.

Therefore $d(u, v) + |f(u) - f(v)| = 1 + |0-2| \ge 3$.

Subcase (ii).

Let u, vbe any two adjacent vertices on the path P_i and leaves L_j respectively, at a distance two, such that $u = v_j$and $v = v_{j+1}$.

f(u) = 2+j, f(v) = 3+j and d(u, v) = 2.

Therefore $d(u, v) + |f(u) - f(v)| = 2 + |2+j-3-j| \ge 3$.

Subcase (iii).

Let u, vbe any two adjacent vertices on the leaves Lj and the path P_i respectively, such that $u = v_1$and $v = u_{3i-2}$.

f(u) = 3, f(v) = 0 and d(u, v) = 1.

Therefore $d(u, v) + |f(u) - f(v)| = 1 + |3-0| \ge 3$.

Hence the labeling number of the coconut tree graph $\lambda(CT(m, n))$ is n + 2.

i.e., Number of leaves + 2.

Example 3.3.

L(2, 1) labeling of CT(7, 9) is shown in Figure 3.

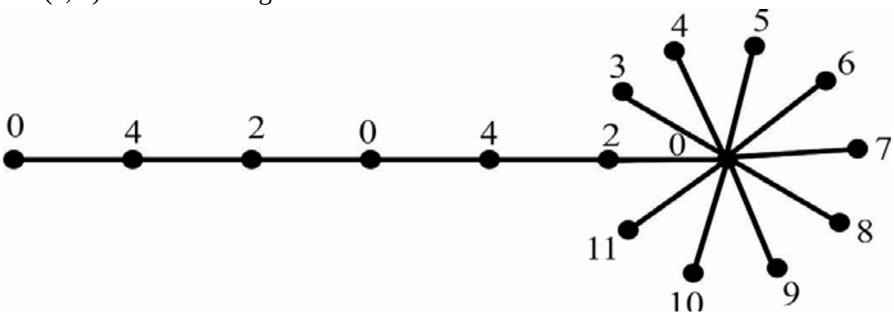

Figure 3.

4. Conclusion

In this paper the labeling number for some special graphs like (m,n) kite graph, Jelly fish graphJ(m, n) and coconut tree graph CT(m, n) graph have been determined.

References

1. G.J. Chang, D. Kuo, The L(2, 1) labeling problem on graphs,*SIAMJ Discrete Math.*,**9** (1996), 309-316.
2. G.J. Chang, D.Kuo, On L labeling of graphs,*SIAMJ Discrete Math.*,**220** (2000), 57 - 66.
3. Z. Fredi,J.H. Kang,L(2,1)-labeling of 3-regular Hamiltonian graphs.
4. J.A. Gallian, A Dynamic survey of Graph labeling, *The Electronic Journal of Combinatorics*, **16** (2009), 251 - 256.
5. J.R. Griggs, R.K.Yeh, Labeling with a condition of distance two,*SIAM Discrete Math.*,**5** (1992), 586 - 595.
6. W.K. Hale, The Frequency Assignment: theory and application,*Proc IEEE.*, **68**, 1497- 1514.
7. K. Jonas, Graph coloring analogues with a condition at distance two: L(2, 1)-labelling and list λ-labelings,*Ph.D. thesis*, University of South Carolina, 1993.
8. K. Manimekalai, K. Thirusangu, Pair sum labeling of some special graphs,*International Journal of Computer Applications*,**69** (2013), 34-38.
9. D. Sakai, Labeling chordal graphs: distance two conditions,*SIAM J. Disc. Math.*, **7** (1994), 133-140.
10. M.A. Whittlesey,J.P. Georges,D.W. Mauro,On the λ number of Q_n and related graphs,*SIAM J. Disc. Math.*,**8** (1995), 499 - 506.