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On Warped Products

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Abstract
The aim of this paper is to find some properties on warped products by studying their Ricci curvatures and scalar curvatures.

Keywords: Warped product, Ricci curvature, scalar curvature.

1. Introduction

Suppose B and F are semi-Riemannian manifolds, and let $f > 0$ be a smooth function on B. The metric tensors of B and F are denoted by $g_B = \langle,\rangle$ and $g_F = (.)$, respectively. Let π and σ be the projections of $B \times F$ onto B and F, respectively.

The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with metric tensor

$g = \langle,\rangle = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$.

Explicitly, if x is tangent to $B \times F$ at (p, q), then

$\langle x,x\rangle = \langle d\pi(x), d\pi(x)\rangle + f^2(p)(d\sigma(x), d\sigma(x))$.

Then g is in fact a metric tensor.

The aim of this paper is to find some properties on warped products by studying their Ricci curvatures and scalar curvatures.

Section 2 of this paper is devoted to the notations and the propositions which are needed subsequently in this paper.

Section 3 of this paper is devoted to study the Ricci curvatures of warped products.

Section 4 of this paper is devoted to study the scalar curvatures of warped products.

2. Preliminaries

Let M be a semi-Riemannian manifold with a metric $\langle,\rangle$ and Levi-Civita connection D. The curvature tensor R assigns to each $p \in M$ a trilinear map of $M_p \times M_p \times M_p \to M_p$. If x, y, z are elements of M_p, we extend to vector fields X, Y, Z and define

$R(x,y)z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$.

Let $e_1,\dots, e_{n+1}$ is any frame with $\varepsilon_i = \langle e_i, e_i\rangle$ on M at p. As usual, the Ricci curvature Ric of M is

$$Ric(X,Y) = \sum_{i=1}^{n+1} \varepsilon_i \langle R(e_i,X)Y, e_i\rangle$$

and the scalar curvature S of M at p is

$$S = \sum_{i=1}^{n+1} \varepsilon_i Ric(e_i, e_i).$$

The divergence, div(V) of a smooth vector field V is

$$div(V) = \sum_{i=1}^{n+1} \varepsilon_i \langle D_{e_i} V, e_i\rangle$$

and the gradient, grad f of a smooth function f is the vector field metrically equivalent to the differential df. Thus

$\langle grad\, f, X\rangle = df(X) = Xf$ for all smooth vector field X.

The Laplacian Δf of a smooth function f is the divergence of its gradient.

Let N be a submanifold of a semi-Riemannian manifold M. Recall that any vector field Y on M splits into the sum $Y^\top + Y^\perp$ of its components tangent to N and perpendicular to N, respectively. The mean curvature vector field H of N in M is defined by

$$H_p = \sum_{i=1}^{k} \varepsilon_i (D_{e_i} e_i)^\perp,$$

where $e_1,\dots, e_k$ is any frame on N at p.

From now, let B and F be semi-Riemannian manifolds with dim F = n > 1 and let f > 0 be a smooth function on B. Let $M = B \times_f F$ be the warped product. The notion of lift of a vector field on B or F to $B \times F$, the set of all such lifts being denoted as usual by L(B) and L(F), respectively. Typically we use the same notation for a vector field and for its lift.

If X, Y ∈ L(B) and V, W ∈ L(F), then we have the following propositions which are from [2].

Proposition 2.1.

(1) $D_X Y \in L(B)$ is the lift of $D_X Y$ on B.

(2) $D_X V = D_V X = \frac{Xf}{f} V$.

(3) $(D_V W)^\perp = -\frac{\langle V,W\rangle}{f} grad\, f$.

(4) $(D_V W)^\top \in L(F)$ is the lift of $\nabla_V W$ on F.

Proposition 2.2. Let ${}^B Ric$ and ${}^F Ric$ be the lifts of the Ricci curvatures of B and F, respectively. Then

(1) $\mathrm{Ric}(X,Y) = {}^{B}\mathrm{Ric}(X,Y) - \frac{n}{f}H^f(X,Y)$.
(2) $\mathrm{Ric}(X,V) = 0$.
(3) $\mathrm{Ric}(V,W) = {}^{F}\mathrm{Ric}(V,W)$
$- \langle V,W\rangle\left\{\frac{\nabla f}{f} + (n-1)\frac{\langle \mathrm{grad}\, f, \mathrm{grad}\, f\rangle}{f^2}\right\}$,
where H^f is the Hessian of f.
It is well-known that if k and K are the scalar curvatures of B and F, respectively, then the scalar curvature of M = B $\times_f$ F is

(2.1) $S = k + \frac{K}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \mathrm{grad}\, f, \mathrm{grad}\, f\rangle}{f^2}$.

3. Ricci curvature of a warped product

In the sections 3, 4, whenever B = [0, ∞), $g_B = dt^2$.

Theorem 3.1. Let $M = [0,\infty) \times_f F$, $c(t) = (t,p)$ and H be the mean curvature vector field of 0×F in M. For some non-positive constant r, suppose $\mathrm{Ric}(c', c') \geq r$ and $\langle H, c'(0)\rangle \geq \sqrt{-nr}$. Then
(1) $f(t) = f(0)e^{-\sqrt{-r/n}\,t}$ and $\mathrm{Ric}(c', c') = r$.
(2) If F is compact and A(t) is the volume of $F_t = \{(t,p) | p \in F\}$ then A(t) is decreasing.

Proof. (1) From Proposition 2.1 and Proposition 2.2,

$\mathrm{Ric}(c', c') = -\frac{n}{f}H^f(c', c') = -n\frac{f''}{f}$

and

$\langle H, c'(0)\rangle = \sum_{i=1}^{n} \varepsilon_i \langle (D_{e_i} e_i)^{\perp}, c'(0)\rangle$

$= -\sum_{i=1}^{n} \frac{1}{f(0)}\langle \mathrm{grad}\, f, c'(0)\rangle = -n\frac{f'(0)}{f(0)}$.

From our assumptions,

(3.1) $-n\frac{f''}{f} \geq r, -n\frac{f'(0)}{f(0)} \geq \sqrt{-nr}$.

Hence, (3.1) leads to the differential inequality

(3.2) $f'' + \frac{r}{n}f \leq 0, f'(0) \leq -\sqrt{\frac{-r}{n}}f(0)$.

The solution of the differential equation $s'' + (r/n)s = 0$, with the initial conditions $s(0) = f(0), s'(0) = -\sqrt{\frac{-r}{n}}f(0)$ is

$s(t) = f(0)e^{-\sqrt{\frac{-r}{n}}t}$.

By the below Lemma 3.2, on [0, ∞)
$f(t) \leq s(t)$.
Moreover $f - s$ is monotonically decreasing on [0, ∞) because $(f-s)'(0) \leq 0$ and $(f-s)'' \leq -(r/n)(f-s) \leq 0$. But since $0 \leq f \leq s$ and $s(t) \to 0$ as $t \to \infty$, it follows that $f - s \to 0$ as $t \to \infty$. Since $(f-s)(0) = 0$, we have $f \equiv s$ on $[0,\infty)$ and

$\mathrm{Ric}(c', c') = -n\frac{f''}{f} = -n\left(\frac{-r}{n}\right) = r$.

(2) The map $\sigma|F_t: F_t \to F$ is a positive homothety onto F. Let $(\sigma|F_t)^{-1} = \phi$. Then
the volume of F_t = the volume of F with the induced metric $\phi^*\langle,\rangle$.
Let $E_1, \cdots, E_n$ be a frame field on F and let $Y_1, \cdots, Y_n$ be the lifts to $[0,\infty) \times F$ of, $E_1, \cdots, E_n$ respectively.
Let $\langle d\phi(E_i), d\phi(E_j)\rangle = \langle Y_i(t), Y_j(t)\rangle = g_{ij}(t)$ and let $g(t) = \sqrt{\det(g_{ij}(t))}$. Then

the volume of $F_t = \int_F g(t)dV$,

where dV denotes the volume element of the metric g_F. Since
$g_{ij}(t) = \langle Y_i(t), Y_j(t)\rangle = f^2(t)\left(d\sigma(Y_i), d\sigma(Y_j)\right)$
$= f^2(t)g_F(E_i,E_j) = f^2(t)\delta_{ij}$,
we have

$g(t) = \sqrt{\det(g_{ij}(t))} = f^n(t)$.

Hence,
the volume of $F_t = f^n(t) \times$ (the volume of F).
Since $f^n(t)$ is decreasing, A(t) is also decreasing.

Lemma 3.2. Suppose $k: \mathbb{R} \to \mathbb{R}$ is smooth. Let f, s be smooth functions such that f is a solution of the differential inequality $f'' + kf \leq 0$, s is a solution of the differential equality $s'' + ks = 0, f(0) = s(0), f'(0) \leq s'(0)$ and f and s are both positive in some interval $(0,a)$. Let α and β be, respectively, the first positive zeros of f and s. Then
(i) α ≤ β
(ii) f ≤ s on [0, β] and equality at a point c implies equality on the interval $[0,c]$.

Theorem 3.3. Let $M = [0,\infty) \times_f F$, and $c(t) = (t,p)$. Let H be the mean curvature vector field of $F_t = \{(t,p) | p \in F\}$ in M.
If $\mathrm{Ric}(c', c') > 0$, then $h(t) = \langle H, c'(t)\rangle < 0$ for all t.
Proof. From Proposition 2.1 and Proposition 2.2, we know

$\mathrm{Ric}(c', c') = -n\frac{f''}{f}, h(t) = -n\frac{f'(t)}{f(t)}$.

The given condition $\mathrm{Ric}(c', c') > 0$ implies
$f''(t) < 0$.
Thus the graph of f, except at for some t_0, is bellow that of its tangent line at t_0. This is the graph of
$y = f(t_0) + f'(t_0)(t - t_0)$.
Since $f'' < 0$, either f' is always positive on $[0,\infty)$ or f has a maximum point after which $f' < 0$. In the latter case, $f(t_1) = 0$ for some t_1. It contradicts to the hypothesis that $f > 0$ on $[0,\infty)$. Hence f' is always positive on $[0,\infty)$ and so $h(t)$ is always negative on $[0,\infty)$.

4. Scalar curvature of a warped product

Lemma 4.1. Let k and K be the scalar curvatures of B and F, respectively, and let S be the scalar curvature of $M = B \times_f F$. Then

(4.1) $\frac{4n}{n+1}\Delta u + Su - ku - K u^{\frac{n-3}{n+1}} = 0$,

where $u = f^{(n+1)/2}$.
Proof. If $\alpha = 2/(n+1)$, then $f = u^{\alpha}$, and

(4.2) $\Delta f = \sum_{i=1}^{n} \varepsilon_i\{e_i e_i - D_{e_i} e_i\}(u^{\alpha}) = \alpha u^{\alpha-1}\Delta u + \alpha(\alpha-1)u^{\alpha-2}\sum \varepsilon_i (e_i(u))^2$.

Moreover, since $\langle \mathrm{grad}\, f, e_i\rangle = e_i(f)$, $\mathrm{grad}\, f = \sum \varepsilon_i e_i(f)e_i$ and so

(4.3) $\frac{\langle \mathrm{grad}\, f, \mathrm{grad}\, f\rangle}{f^2} = \alpha^2 u^{-2}\sum \varepsilon_i (e_i(u))^2$.

From (4.2) and (4.3), we have

$$(4.4)\ 2n\frac{\Delta f}{f} + n(n-1)\frac{\langle \mathrm{grad}\, f, \mathrm{grad}\, f\rangle}{f^2} 2n\alpha u^{-1}\Delta u,$$

since $\alpha = 2/(n+1)$. By substituting (4.4) to (2.1), we get (4.1).

Lemma 4.2. Let $B = [0,\infty)$. Then, (4.1) becomes

$$(4.5)\ \frac{4n}{n+1}u''(t) + S\cdot u(t) - K(x)\cdot u^{\frac{n-3}{n+1}}(t) = 0,$$

for $t \in [0,\infty)$ and $x \in F$.

Proof. Since

$$\Delta u = \frac{d}{dt}\frac{d}{dt}u(t) - D_{\frac{d}{dt}}\frac{d}{dt}u(t) = u''(t)$$

(4.1) becomes (4.5).

Theorem 4.3. For $n \geq 3$, let $M = [0,\infty) \times_f F$ be the Riemannian warped product (n+1)−manifold with F a n−manifold. Suppose F admits a metric of zero scalar curvature. Then there is a metric of constant scalar curvature on M.

Proof. From our hypothesis, F admits a metric of zero scalar curvature. Hence, the equation (4.5) becomes

$$\tfrac{4n}{n+1}u''(t) + S\cdot u(t) = 0,$$

and so

$$S = -\frac{4n}{n+1}\cdot\frac{u''(t)}{u(t)}.$$

To be S constant, $u''(t)/u(t)$ must be constant. Let $u''(t)/u(t) = \pm a^2$. A Solution of the differential equation is $u(t) = e^{at}$, or $u(t) = \sin at$.

Then $S = \pm\frac{4n}{n+1}a^2$ and so it is constant.

Theorem 4.4. Let $M = [0,\infty) \times_f F$ be the semi-Riemannian warped product manifold with F a 3−manifold. Suppose F admits a metric of constant scalar curvature K. Then there is a metric of constant scalar curvature on M.

Proof. From (4.5), since $n = 3$ we have

$$(4.6)\ 3\cdot u''(t) + S\cdot u(t) - K = 0.$$

(1) In the case $K \leq 0$. Let $u(t) = e^{at} - \frac{K}{3a^2}$. Then the above differential equation (4.6) becomes

$$3a^2e^{at} + S\cdot\left(e^{at} - \frac{K}{3a^2}\right) - K = 0$$

and so

$$(S+3a^2)\cdot\left(e^{at} - \frac{K}{3a^2}\right) = 0.$$

Since $e^{at} - K/3a^2 \neq 0$, we have a constant scalar curvature $S = -3a^2$.

(2) In the case $K > 0$. Let $u(t) = \sin at + \frac{K}{3a^2}$, $(0 < a^2 < K/3)$. Then the above differential equation (4.6) becomes

$$-3a^2\sin at + S\cdot\left(\sin at + \frac{K}{3a^2}\right) - K = 0$$

and so

$$(S - 3a^2)\left(\sin at + \frac{K}{3a^2}\right) = 0.$$

Since $\sin at + K/3a^2 > 0$, we have a constant scalar curvature $S = 3a^2$.

Let F be a connected three-dimensional Riemannian manifold of constant curvature $c = -1, 0$, or 1. Let $f(s) > 0$ be a smooth function on an open interval I in $\mathbb{R}^1_1$. Then the warped product

$M(c,f) = I \times_f F$

is called a Robertson-Walker spacetime [3]. $M(c,f)$ is the manifold $I \times F$ with line element $-dt^2 + f^2(t)d\sigma^2$, where $d\sigma^2$ is the line element of F. The standard choices for F are the complete simply connected ones: H^3, R^3, S^3, with curvatures −1, 0, +1, respectively. From (2.1), The scalar curvature of $M(c,f)$ is

$$(4.7)\ S = 6\left\{\left(\frac{f'}{f}\right)^2 + \frac{c}{f^2} + \frac{f''}{f}\right\}.$$

Example 1. There is a semi-Riemannian metric of constant scalar curvature on a Robertson-Walker spacetime.

Proof. Since $\Delta u = -u''$, (4.1) becomes

$$(4.8)\ -3u'' + Su - 6c = 0.$$

(1) In the case $c = 0$. Let $u(t) = e^{at}$. Then the above differential equation (4.8) becomes

$$-3a^2e^{at} + S\cdot e^{at} = 0$$

and we have a constant scalar curvature $S = 3a^2$. In fact, $f = u^{1/2} = e^{at/2}$ and so from (4.7) we have $S = 3a^2$.

Here, if $a = 0$, then $u = 1$,and so $f = 1$. Hence $M(0,1) = I \times S$ is an open submanifold of R^4_1, we already know that $S = 0$.

(2) In the case $c = 1$. Let $u(t) = e^{at} + \frac{2}{a^2}, (a \neq 0)$. Then the above differential equation (4.8) becomes

$$-3a^2e^{at} + S\cdot\left(e^{at} + \frac{2}{a^2}\right) - 6 = 0$$

and so

$$(S - 3a^2)\left(e^{at} + \frac{2}{a^2}\right) = 0.$$

Hence, we have a constant scalar curvature $S = 3a^2$. In fact,

$$f = u^{1/2} = \left(e^{at} + \frac{2}{a^2}\right)^{1/2}$$

and

$$f' = \frac{ae^{at}}{2}u^{-1/2}, f'' = \frac{a^2e^{at}}{2}u^{-1/2} - \frac{a^2e^{2at}}{4}u^{-3/2}.$$

And so, from (4.7) we have

$$S = 6\left\{\left(\frac{f'}{f}\right)^2 - \frac{1}{f^2} + \frac{f''}{f}\right\}$$

$$= 6\left\{\frac{a^2e^{2at}}{4}\frac{1}{u^2} + \frac{1}{u} + \frac{a^2e^{at}}{2}\frac{1}{u} - \frac{a^2e^{2at}}{4}\frac{1}{u^2}\right\}$$

$$= 3a^2$$

(3) In the case $c = -1$. Let $u(t) = \sin at + \frac{2}{a^2}$, $(0 < a^2 < 2)$. Then the above differential equation (4.8) becomes

$$3a^2\sin at + S\cdot\left(\sin at + \frac{2}{a^2}\right) + 6 = 0$$

and so

$$(S + 3a^2)\left(\sin at + \frac{2}{a^2}\right) = 0.$$

Hence, we have a constant scalar curvature $S = -3a^2$. In fact,

$$f = u^{1/2} = \left(\sin at + \frac{2}{a^2}\right)^{1/2}$$

and

$$f' = \frac{a\cos at}{2}u^{-1/2}, f'' = \frac{-a^2\sin at}{2}u^{-1/2} - \frac{a^2\cos^2 at}{4}u^{-3/2}.$$

And so, from (4.7) we have

$$S = 6\left\{\left(\frac{f'}{f}\right)^2 - \frac{1}{f^2} + \frac{f''}{f}\right\}$$

$$= 6\left\{\frac{a^2\cos^2 at}{4}\frac{1}{u^2} - \frac{1}{u} - \frac{a^2\sin at}{2}\frac{1}{u} - \frac{a^2\cos^2 at}{4}\frac{1}{u^2}\right\}$$

$$= -3a^2$$

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