International Journal of Applied Engineering Research ISSN 0973-4562 Volume 10, Number 17 (2015) pp 37338-37339

The Pseudospectral Method for the Optimal Control Problem

T. Raghunathan
School of Electrical Engineering, VIT University, Vellore-632014, India
Email: raghunathan.t@vit.ac.in

Abstract
The dynamic optimization or optimal control problem is more difficult to solve than the static optimization problem. This is because the conventional, classical method of solving the optimal control problem is as a two point boundary value problem, which needs split boundary conditions to be satisfied. This paper is a review of a recent, promising method known as the pseudospectral method for solving the optimal control problem. The states and control are parameterized as Chebyshev polynomials, and the solution found using the well developed static optimization theory. To demonstrate the effectiveness of the new method, an analytically solvable optimal control problem is solved by the Chebyshev polynomial method, which compares favourably with the analytical solution.

Keywords: optimal control, dynamic optimization, Chebyshev polynomials, static optimization

I. INTRODUCTION

Optimal control problems appear in diverse fields such as engineering, agriculture, fishing, economics and medicine. However, while formulating an optimal control problem is an easy task, solving the optimal control problem is a formidable task, with the exception of the simplest cases. This paper presents a recent, promising method of solving optimal control problems, named as pseudospectral method.

Optimization problems have been noted and sometimes solved from ancient times, like in the founding of the Carthage city by Princess Dido [1]. A subclass of optimization is dynamic optimization, or optimal control. The year of birth of optimal control is generally accepted as 1696 CE, when Johann Bernoulli posed his brachistochrone (in Latin, *brachistos* = shortest, *chronos* = time) problem to the 'most learned' men of his times, and challenged them to solve it, in the journal *Acta Eruditorum* [2]. Among the intellectual giants who solved the problem correctly were Isaac Newton, Leibnitz, Bernoulli's own brother Jakob Bernoulli, Marquis de l'Hospital, and Tschirnhaus [3]. From the efforts of Leibnitz arose the newer branch of calculus of variations or variational calculus, distinct from ordinary calculus, of which too, Leibnitz was the foremost pioneer.

This paper is organized as follows: Section II introduces the optimal control problem, and its solution as a two point boundary value problem, Section III explains the pseudospectral method, Section IV contains a simple optimal control problem that is solved by the pseudospectral method, and Section V concludes the paper with suggestions for future work.

II. THE OPTIMAL CONTROL PROBLEM

The static optimization problem can be defined by

$$\min_{X} y = f(X) \quad \text{s.t. } h(X) = 0 \tag{1}$$

The dynamic optimization or optimal control problem can be defined by

$$\min_{U} J = \int V(X, U, t)\,dt \quad \text{s.t. } \dot{X} = f(X, U, t) \tag{2}$$

It is to be noted that there is some similarity between the two optimization problems (1) and (2), as well as a lot of differences. The solution of static optimization problem is very mature now, and will not be discussed any further. From here on, only the optimal control problem would be treated. The solution of (2) requires calculus of variations or variational calculus, as opposed to ordinary calculus to solve (1). The solution steps, well covered in references such as [1], are given below:

(i) Form the Hamiltonian,

$$H(X(t), \lambda(t), U(t), t) = V + \lambda^T f \tag{3}$$

(ii) Stationarity/optimality condition is given by

$$H_U = \left(\frac{\partial H}{\partial U}\right)_* = 0 \tag{4}$$

Solving (4), obtain

$$U^*(t) = h(X^*(t), \lambda^*(t), t) \tag{5}$$

(iii) Substitute this in H (step (i)), to obtain

$$H^*(X^*(t), \lambda^*(t), t) \tag{6}$$

(iv) Solve the $2n$ Euler Lagrange equations given by

State equation:

$$\dot{X} = H_\lambda^* = f(X^*(t), \lambda^*(t), t) \tag{7}$$

Co-state equation: $\dot{\lambda} = -H_X$ (8)

with the given boundary conditions to get $X^*(t), \lambda^*(t), U^*(t)$.

It is to be noted that, solving these $2n$ differential equations is very difficult since the boundary conditions are split: half of these are at initial time t_0 and the other half at final time t_f. In other words, this is two point boundary value problem (TPBVP, or simply, BVP).

III. THE PSEUDOSPECTRAL METHOD

Spectral methods or pseudospectral methods (PSM) came into prominence in the 1970's, for solving PDE's and ODE's [4]. Their use in solving optimal control problems is more recent. The PSM for solution of the optimal control problem can be summarized as below [5]-[7]:

(i) Select the no. of grid points at which the control and the state are discretized.
(ii) Fix the location of the points [5].
(iii) Make polynomial approximation of the states and control at these grid points.

Compute the Chebyshev polynomial matrix T_n of the states and control.

(iv) Express the state equation as differentiation of the Chebyshev polynomial matrix T_n.
(v) Approximate the cost function J as per the Clenshaw-Curtis quadrature scheme [5] as $\hat{J}$.
(vi) Solve the resulting static optimization of minimizing $\hat{J}$ subject to the differential constraint described in (iv) above.

IV. RESULTS AND DISCUSSION

The pseudospectral method is demonstrated on the following problem:

$$\min_u J = \int_{-1}^{1} \frac{1}{2} u^2 dt \qquad (9)$$

$$\text{s.t. } \dot{x} = u, x(-1) = 0, x(1) = 4$$

The solution by Chebyshev polynomial method described in Section III is as below.

Step (i): Choose N =2, so that the no. of grid points, N+1 = 3

Step (ii): The location of these points are given by

$$t_i = -\cos\frac{\pi i}{N}, i = 0, 1, 2 \Rightarrow t_0 = -1,\ t_1 = 0,\ t_2 = 1 \qquad (10)$$

Step (iii): $$\hat{x} = \sum_{i=0}^{2} a_n T_n = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \qquad (11)$$

$$\hat{u} = \sum_{i=0}^{2} b_n T_n = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \qquad (12)$$

Step (iv): The differentiation matrix D is given by

$$D = \begin{bmatrix} -1.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & -2 & 1.5 \end{bmatrix} \qquad (13)$$

$$\dot{\hat{x}} = D\,\hat{x}$$

Step (v): The weights, w_0, w_1 and w_2 are given by

$$w_0 = w_2 = 1/3$$

$$w_1 = 4/3$$

The approximate cost function is given by

$$\hat{J} = \frac{1}{2} \sum_{i=0}^{2} w_i u_i$$

Step (vi): The Langrangian,

$$\bar{J} = \hat{J} + \lambda^T(\dot{\hat{x}} - \hat{f}) \qquad (14)$$

Solving this using the boundary conditions given in the problem,

$$a_0 = a_1 = 2, a_2 = 0$$

$$b_0 = 2,\ b_1 = b_2 = 0$$

The solution using the PSM or Chebyshev polynomial method,

$$\hat{u} = \begin{bmatrix} 2 - b_1 \\ 2 \\ 2 + b_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \text{ (since } b_1 = 0) \qquad (15)$$

The analytical solution is, $u^* = 2$, as can be found using the steps in Section II.

The results by the PSM and the analytical method are similar.

V. CONCLUSION

The new, promising pseudospectral method to solve the optimal control problem was reviewed. A typical optimal control problem was solved by the pseudospectral method and the solution compared with the analytical solution, to show its applicability.

There is enormous scope for future work using the pseudospectral method. More complicated optimal control problems occurring in practical problems may be solved by the pseudospectral method.

REFERENCES

[1] D. S. Naidu, "Optimal Control Systems," CRC Press, 2003, p. 11.

[2] W. Dunham, "The Challenge of Brachistochrone," http://www.jsoftware.com/papers/brachistochrone.htm

[3] H.J. Sussmann and J.C. Willems, "The brachistochrone problem and modern control theory," in proc. Contemporary trends in nonlinear geometric control theory and its applications, World Scientific Publishers, 2001

[4] L.N. Trefethen, "Spectral methods in Matlab," SIAM Books, 2000

[5] F. Fahroo and I. M. Ross, "Direct trajectory optimization by a Chebyshev pseudospectral method,", *Journal of Guidance, Control and Dynamics*, Vol. 25, No. 1, Jan-Feb 2002, pp. 160-166.

[6] Q. Gong, W. Kang, I. M. Ross, "A pseudospectral method for the optimal control of constrained feedback linearizable systems," *IEEE Trans. Automatic Control*, Vol 51, No. 7, July 2006, pp. 1115-1129.

[7] F. Fahroo and I. M. Ross, "Advances in pseudospectral methods for optimal control,", AIAA Guidance, Navigation and Control Conference, Honolulu, HI, Aug. 2008, pp. 1-23.