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An unreliablefeedbackretrial queue with multi optionalstages of serviceunder at most J vacations and non-persistent customers

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Abstract: This paper deals with the steady state analysis ofbatch arrival feedback retrial queue with *k*optional stages of service under at most *J* vacations. The non-persistent customers are allowed to balk and renege at particular time. After completion of the i^{th} stage of service,the customer may have the option to choose $(i+1)^{th}$ stage of service with probability θ_i, with probability p_i may join into orbit as feedback customer or may leave the system with probability $q_i = \begin{cases} 1-p_i-\theta_i, i=1,2,...k-1 \\ 1-p_i, i=k \end{cases}$.If the orbit is empty at the service completion of each stage service, the server takes at most *J* vacations until at leastone customer appears in the orbit on the server returns from a vacation. Busy server may get to breakdown and the service channel will fail for a short interval of time. By using the supplementary variable method, steady state probability generating function for system size, some system performance measures and numerical illustrations are discussed.

Keywords: Bernoulli feedback, retrial,non-persistent customers,*k* -optional service, Supplementary Variable Technique.

1. Introduction

Retrial queues or queues with repeated attempts have widely used to provide stochastic modeling of many problems arising in telecommunication and computer network. In retrial queues, if the server is busy at the arrival epoch and there is no waiting space is available then the whole batch joins the retrial group known as orbit, whereas if the server is free then one of the arriving units starts its service and the rest join the orbit. There is an extensive literature on the retrial queues. We refer the works by Artelijo,J.R. and Gomez-Corral, A [1],Artalejo[2],Yang., Templeton [25] ,and Falin., Templeton [10] as a few. In balking scenario, an arriving customer who finds the server busy has to leave the system or may join into the orbit. Later, after entering into orbit the reneging customers may decide to go to service area or leave the system. Such a scenario is prevalent at supermarket, reservation counters, call centers etc. Ke and Chang[14] investigated M/G/1 retrial queue with modified vacation policy by incorporating balking and reneging concepts. Some of the others like Wang and Li[24], Baruah et al. [19] and Rajadurai et al [20] developed a queueing model with non-persistent customers.In a vacation queueing system, the server may not be available for a period of time due to many reasons like, being checked for maintenance, scanning for new job, or taking break. This period of time, when the server is unavailable for primary customers is referred as a vacationKeilson and Servi [16] introduced the concept ofBernoulli vacation. If the system is empty, the assumption for their model is that the server must take another vacation. Doshi [9] presented a survey on queueing systems with vacations. Krishnakumar and Arivudainambi [17] have investigated a single server retrial queue with Bernoulli schedule and general retrial times. This system has potential applications in packet-switched networks. Later, Chang and Ke [14], Chen et al. [5] and Rajadurai [22] discussed a different *J* vacation queueing models.

In our model, a single server provides *K* optional stages of service. When the service of a customer is unsatisfied, it may be retried again and again until a successful service completion.Recently many authors developed a queueing models with two or more stages of service.Wang and Li[24]
1. have studied the single server retrial queueing system with second multi optional services. Krishnakumar et al. [18] examined the M/G/1 retrial queue with feedback including some applications, where the server is subjected to starting failure. Recently, Salehirad and Badmachizadeh [23] and Bagyam and Charika [3] have discussed about the concept of Bernoulli feedback.

In unreliable queueing system, while the server is working with any phase of service, it may breakdown at any instant and the service channel will fail for a short interval of time. The repair process does not start immediately after a breakdown and there is a delay time for repair to start.Ke and Choudhury [15] discussed about the *batch arrival retrial queueing system with two phases of service under the concept of breakdown and delaying repair. Choudhury and Deka [7] considered a single server queue with two phases of service and the server is subject to breakdown while providing service to the customers. Further, Choudhury and Deka [6] developed the previous model with the concept of Bernoulli vacation. Recently, Authors like Wang and Li [24], Radha J et al. [21], Mokaddis et al. [12] and

Choudhury et al. [8] discussed about the retrial queueing systems with the concept of breakdown and repair. In this paper, we investigate a steady state analysis of retrial queueing system with multi optional stage of service, Bernoulli feedback ,at most J vacations and non- persistent customers, where the server is subject to breakdown and repair.

The results of this paper finds applications in LAN, telephone systems, electronic mail services on internet, network and software designs of various computer communications systems, packet switched networks, production lines, In the operational model of WWW server for HTTP requests, call centers, inventory and production, maintenance and quality control in industrial organizations, etc.

2. Model Description

In this section, we consider a model for batch arrival feedback retrial queue with K optional stages of service and un persistent customers under at most J vacations where the server is subject to breakdowns and repair. The detailed description of the model is given as follows:

***Arrival process*:** Customers arrive in batches according to a compound Poisson process with rate λ. Let X_k denote the number of customers belonging to the k^{th} arrival batch, where X_k, k =1,2 ,3,…are with a common distribution $\Pr[X_k = n] = \chi_n$, n= 1,2,3… $X(z)$ denotes the probability generating function of X. The first and second moments are $E(X)$ and $E(X(X-1))$.

***Retrial process*:**We assume that there is no waiting space and therefore if an arriving batch of customers finds the server free, the arrival beings his service one from the batch and rest of them join into pool of blocked customers called an orbit with probability b or entire batch leaves the system with balking probability *1-b*. If an arriving customer finds the server being busy, vacation or breakdown, the arrivals either leave the service area with probability *b*and join the orbit or balk the system with probability *1-b*. If a primary customer arrives first, then the retrial customer cancels the attempt for service and either return to its position in the orbit with probability r or leaves the system with reneging probability r = *1-r.*Inter-retrial times have an arbitrary distribution $R(t)$ with corresponding Laplace-Stieltijes transform (LST) $R^*(\varphi)$

***Service process*:**The server provides k stages of service in succession. The First Stage Service (FSS) is followed by *i*stages of service. The service time for all the stages has a general distribution. It is denoted by the random variable $S_i$with distribution function $S_i(t)$ having LST $S_i^*(\varphi)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, (i= 1,2,…k).

Feedback rule:After completion of i^{th} stage of service the customer may go to $(i+1)^{th}$stage with probability θ_i or may join into the orbit as feedback customer with probability p_i or leaves the system with probability $q_i = 1-\theta_i - p_i$, for i = 1,2,…k-1. If the customer in the last k^{th} stage may join to the orbit with probability p_k or leaves the system with probability $q_k = 1 - p_k$. From this model, the service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST $S^*(\varphi) = \prod_{i=1}^{k} \Theta_{i-1} S_i^*(\varphi)$ and the expected value is $E(S) = \sum_{i=1}^{k} \Theta_{i-1} E(S_i)$, where $\Theta_i = \theta_1\theta_2...\theta_i$ and $\Theta_0 = 1$.

***Vacation process*:**Whenever the orbit is empty, the server leaves for a vacation of random length V. If no customer appears in the orbit when the server returns from a vacation, it leaves again for another vacation with the same length. Such pattern continues until it returns from a vacation to find atleast one customer found in the orbit or it has already taken J vacations. If the orbit is empty at the end of the J th vacation, the server remains idle for new arrivals in the system. At a vacation completion epoch the orbit is non empty, the server waits for the customers in the orbit. The vacation time V has distribution function $V(t)$ and LST$V^*(\varphi)$ with moments $E(V)$ and $E(V^2)$.

Breakdownprocess: While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time i.e. server is down for a short interval of time. The breakdowns i.e. server's life times are generated by exogenous Poisson processes with rates α_i for i^{th} stage respectivelyfor (i=1,2,…k).

Repair process:As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the primary customers till service channel is repaired. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G_i)distributions of the server for i stagesare assumed to be arbitrarily distributed with d.f.$G_i(x)$ and LST $G_i^*(\varphi)$for (i=1,2,…k).

Various stochastic processes involved in the system are assumed to be independent of each other.

In the steady state, we assume that $R(0)=0, R(\infty)=1, S_i(0)=0, S_i(\infty)=1, V(0)=0, V(\infty)=1, i=1,2,...k$ are continuous at $x = 0$ and $G_i(0)=0, G_i(\infty)=1$ are continuous at $y = 0$, $(1 \le i \le k)$. The state of the system at time $t$is $R^0(t), S_i^0(t), V_j^0(t)$and $G_i^0(t)$be the elapsed times respectively, the elapsed retrial time, the elapsed service timeson i^{th} stage,the elapsed vacation time on j^{th}vacation $(for j=1,2,...J)$,the elapsed repair times on i^{th} stage, $(1 \le i \le k)$.Further, introduce the random variable,

$$C(t)=\begin{cases} 0, \text{ if the server is idle at time } t, \\ 1, \text{ if the server is busy on } i^{th} \text{ stage at time } t, \\ 2, \text{ if the server is repair on } i^{th} \text{ stage at time } t. \\ 3, \text{ if the server is on vacation with the first vacation at time } t. \\ 4, \text{if the server is on vacation with the second vacation at time } t. \\ . \\ j+2, \text{if the server is on vacation with the j}^{th} \text{ vacation at time } t. \\ . \\ J+2, \text{if the server is on vacation with the J}^{th} \text{ vacation at time } t \end{cases}$$

The state of system at time t can be described by the bivariate Markov process $\{C(t), N(t); t \ge 0\}$ where $C(t)$ denotes the server state $(0,1,2,3,4,...j+2,...J+2)$ depending if the server is idle, busy on i^{th} stage, repair on i^{th} stage and 1 st vacation,…and J th vacation respectively. $N(t)$ corresponding to the number of customers in orbit at time t.So that the functions $a(x)$, $\mu_i(x)$, $\gamma(x)$ and $\xi_i(y)$ are the conditional completion rates for retrial, service, vacation and repair respectively ($1 \le i \le k$).

$$a(x)dx = \frac{dR(x)}{1-R(x)}, \mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)},$$

$$\gamma(x)dx = \frac{dV(x)}{1-V(x)} \text{ and } \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)}.$$

Then define $B_i^* = S_1^* S_2^* ... S_i^*$ and $B_0^* = 1$.The first moment M_{1i} and second moment M_{2i} of B_i^* are given by

$$M_{1i} = \lim_{z\to1} dB_i^*[A_i(z)]\Big/dz = \sum_{j=1}^{i} \lambda b E(X) E(S_j)\left(1+\alpha_j E(G_j)\right)$$

$$M_{2i} = \lim_{z\to1} d^2 B_i^*[A_i(z)]\Big/dz^2 = \sum_{j=1}^{i} \begin{bmatrix} \lambda b E(X(X-1)) E(S_j)\left(1+\alpha_j E(G_j)\right) \\ +\alpha_j \left(\lambda b E(X)\right)^2 E(S_j) E(G_j^2) \\ +\left(\lambda b E(X)\right)^2 E(S_j^2)\left(1+\alpha_j E(G_j)\right)^2 \end{bmatrix}$$

where

$A_i(z) = b(z) + \alpha_i \left(1 - G_i^*(b(z))\right)$ and $b(z) = \lambda b\left(1 - X(z)\right)$

Let $\{t_n;\ n = 1,2,…\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends or repair period ends. The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system.

Theorem 2.1: The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if, $\rho < 1$. Where $\rho = (E(X) - \bar{r})\left(1 - R^*(\lambda)\right) + \omega,$

where $\omega = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} - \sum_{i=1}^{k-1} \Theta_i M_{1i}$.

3.Steady state distribution

In this section, we first develop the steady state difference-differential equations for the retrial system by treating the elapsed retrial time, the elapsed service time, the elapsed repair time and the elapsed vacation time as supplementary variables. Then we derive the probability generating functions for the server state and the number of customers in the system/orbit .The following probabilities are used in sequent sections:

$P_0(t)$is the probability that the system is empty at time t.

$P_n(x,t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed retrial time of the test customers undergoing retrial is x

$\Pi_{i,n}(x,t), (1 \le i \le k)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed service time on i^{th} stage of the test customer undergoing service is x.

$Q_{j,n}(x,t), (j=1,2,...J)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed vacation time on j^{th} vacation is x.

$R_{i,n}(x,y,t), (1 \le i \le k)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed service time of the test customer undergoing service is x and the elapsed repair time on i^{th} stage of server is y.

For the process $\{N(t),\ t \ge 0\}$, we define the probabilities $P_0(t) = P\{C(t)=0,\ N(t)=0\}$ and the probability densities for $t \ge 0, (x,y) \ge 0, (1 \le i \le k), (1 \le j \le J)$ and $n \ge 0$

$$P_n(x,t)dx = P\left\{C(t)=0, N(t)=n,\ x \le R^0(t) < x+dx\right\}.$$

$$\Pi_{i,n}(x,t)dx = P\left\{C(t)=1,\ N(t)=n,\ x \le S_i^0(t) < x+dx\right\}.$$

$$Q_{j,n}(x,t)dx = P\left\{C(t)=j+2,\ N(t)=n,\ x \le V_j^0(t) < x+dx\right\}.$$

$$R_{i,n}(x,y,t)dy = P\left\{C(t)=2,\ N(t)=n,\ y \le G_i^0(t) < y+dy \big/ S_i^0(t)=x\right\}.$$

We assume that the stability condition is fulfilled in the sequel and so that we can set for $t\geq 0$, $x\geq 0$, $y\geq 0$, $n\geq 0$, for i = 1,2,…k. and $j=1,2,\ldots k$.

$$P_0=\lim_{t\to\infty}P_0(t),\ P_n(x)=\lim_{t\to\infty}P_n(x,t),\ \Pi_{i,n}(x)=\lim_{t\to\infty}\Pi_{i,n}(x,t),$$
$$Q_{j,n}(x)=\lim_{t\to\infty}Q_{j,n}(x,t),\ R_{i,n}(x,y)=\lim_{t\to\infty}R_{i,n}(x,y,t),\ \text{for } t\geq 0.$$

3.1 Steady state equations

By the method of supplementary variable technique (Kelison et al. [16]), we obtain the following system of equations that govern the dynamics of the system behavior for (i=1,2,..k)

$$\lambda bP_0=\int_0^\infty Q_{j,0}(x)\gamma(x)dx \qquad (3.1)$$

$$\frac{dP_n(x)}{dx}+[\lambda+a(x)]P_n(x)=0,\ n\geq 1 \qquad (3.2)$$

$$\frac{d\Pi_{i,0}(x)}{dx}+[b\lambda+\alpha_i+\mu_i(x)]\Pi_{i,0}(x)=\int_0^\infty \xi_i(y)R_{i,0}(x,y)dy, n=0 \qquad (3.3)$$

$$\frac{d\Pi_{i,n}(x)}{dx}+[b\lambda+\alpha_i+\mu_i(x)]\Pi_{i,n}(x)=b\lambda\sum_{k=1}^{n}\chi_k\Pi_{i,n-k}(x)+\int_0^\infty \xi_i(y)R_{i,n}(x,y)dy, n\geq 1 \qquad (3.4)$$

$$\frac{dQ_{j,0}(x)}{dx}+[b\lambda+\gamma(x)]Q_{j,0}(x)=0,\ n=0,\ j=1,2,\ldots J \qquad (3.5)$$

$$\frac{dQ_{j,n}(x)}{dx}+[b\lambda+\gamma(x)]Q_{j,n}(x)=b\lambda\sum_{k=1}^{n}\chi_k Q_{j,n-k}(x),\ n\geq 1,\ j=1,2,\ldots J \qquad (3.6)$$

$$\frac{dR_{i,0}(x,y)}{dy}+[b\lambda+\xi_i(y)]R_{i,0}(x,y)=0,\ n=0 \qquad (3.7)$$

$$\frac{dR_{i,n}(x,y)}{dy}+[b\lambda+\xi_i(y)]R_{i,n}(x,y)=b\lambda\sum_{k=1}^{n}\chi_k R_{i,n-k}(x,y),\ n\geq 1 \qquad (3.8)$$

The steady state boundary conditions at x = 0 and y = 0 are

$$P_n(0)=\begin{pmatrix}\sum_{i=1}^{k-1}q_i\int_0^\infty \Pi_{i,n}(x)\mu_i(x)dx+(1-p_k)\int_0^\infty \Pi_{k,n}(x)\mu_k(x)dx\\ +\sum_{j=1}^{J}\int_0^\infty Q_{j,n}(x)\gamma(x)dx+\sum_{i=1}^{k}p_i\int_0^\infty \Pi_{i,n-1}(x)\mu_i(x)dx,\ n\geq 1\end{pmatrix} \qquad (3.9)$$

$$\Pi_{1,0}(0)=\int_0^\infty P_1(x)a(x)dx+\lambda\bar{r}\int_0^\infty P_1(x)dx+b\lambda\chi_1P_0,\ n=0 \qquad (3.10)$$

$$\Pi_{1,n}(0)=\begin{pmatrix}\int_0^\infty P_{n+1}(x)a(x)dx+\lambda r\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+1}(x)dx\\ +\lambda\bar{r}\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+2}(x)dx+b\lambda\chi_{n+1}P_0,\ n\geq 1\end{pmatrix} \qquad (3.11)$$

$$\Pi_{i,n}(0)=\theta_{i-1}\int_0^\infty \Pi_{i-1,n}(x)\mu_{i-1}(x)dx,\ n\geq 1,\ (2\leq i\leq k) \qquad (3.12)$$

$$Q_{1,0}(0)=\sum_{i=1}^{k-1}q_i\int_0^\infty \Pi_{i,0}(x)\mu_i(x)dx+(1-p_k)\int_0^\infty \Pi_{k,0}(x)\mu_k(x)dx,\ n=0 \qquad (3.13)$$

$$Q_{j,n}(0)=\int_0^\infty Q_{j-1,n}(x)\gamma(x)dx, n=0,\ j=2,3,\ldots J \qquad (3.14)$$

$$R_{i,n}(x,0)=\alpha_i\Pi_{i,n}(x),\ n\geq 0 \qquad (3.15)$$

The normalizing condition is

$$P_0+\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx+\sum_{n=0}^{\infty}\begin{pmatrix}\sum_{i=1}^{k}\left(\int_0^\infty \Pi_{i,n}(x)dx+\int_0^\infty\int_0^\infty R_{i,n}(x,y)dxdy\right)\\ +\sum_{j=1}^{J}\int_0^\infty Q_{j,n}(x)dx\end{pmatrix}=1 \qquad (3.16)$$

Steady state solutions

To solve the above equations, then we define the generating functions for $|z|\leq 1, i=1,2,\ldots k$

$$P(x,z)=\sum_{n=1}^{\infty}P_n(x)z^n;\ P(0,z)=\sum_{n=1}^{\infty}P_n(0)z^n;$$
$$\Pi_i(x,z)=\sum_{n=0}^{\infty}\Pi_{i,n}(x)z^n;\Pi_i(0,z)=\sum_{n=0}^{\infty}\Pi_{i,n}(0)z^n;$$
$$Q_j(x,z)=\sum_{n=0}^{\infty}Q_{j,n}(x)z^n;\ Q_j(0,z)=\sum_{n=0}^{\infty}Q_{j,n}(0)z^n;$$
$$R_i(x,y,z)=\sum_{n=0}^{\infty}R_{i,n}(x,y)z^n;\ R_i(x,0,z)=\sum_{n=0}^{\infty}R_{i,n}(x,0)z^n$$

Now multiplying the steady state equation and steady state boundary condition (3.2) - (3.15) by z^n and summing over n, (n = 0,1,2...,$1\leq i\leq k$ *and j=1,2,…J*)

$$\frac{\partial P(x,z)}{\partial x}+[\lambda+a(x)]P(x,z)=0 \qquad (3.17)$$

$$\frac{\partial \Pi_i(x,z)}{\partial x}+[b\lambda(1-X(z))+\alpha_i+\mu_i(x)]\Pi_i(x,z)=\int_0^\infty \xi_i(y)R_i(x,y,z)dy \qquad (3.18)$$

$$\frac{\partial Q_j(x,z)}{\partial x}+[b\lambda(1-X(z))+\gamma(x)]Q_j(x,z)=0 \quad (3.19)$$

$$\frac{dR_i(x,y,z)}{dy}+[b\lambda(1-X(z))+\xi_i(y)]R_i(x,y,z)=0 \quad (3.20)$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P(0,z)=\left(\begin{array}{l}\sum_{i=1}^{k}\left\{(p_i z+q_i)\int_0^\infty \Pi_i(x,z)\mu_i(x)dx\right\}\\ +\int_0^\infty Q_j(x,z)\gamma(x)dx-b\lambda P_0-Q_{j,0}(0)\end{array}\right) \quad (3.21)$$

$$\Pi_1(0,z)=\left(\begin{array}{l}\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\lambda r\frac{X(z)}{z}\int_0^\infty P(x,z)dx\\ +\lambda\bar{r}\frac{X(z)}{z^2}\int_0^\infty P(x,z)dx+\frac{b\lambda X(z)}{z}P_0\end{array}\right) \quad (3.22)$$

$$\Pi_i(0,z)=\theta_{i-1}\int_0^\infty \Pi_{i-1}(0,z)\mu_{i-1}(x)dx,(i=2,3,...k). \quad (3.23)$$

$$R_i(x,0,z)=\alpha_i\Pi_i(x,z) \quad (3.24)$$

Solving the partial differential equations (3.17)-(3.20), it follows that for ($1 \le i \le k$)

$$P(x,z)=P(0,z)[1-R(x)]e^{-\lambda x} \quad (3.25)$$

$$\Pi_i(x,z)=\Pi_i(0,z)[1-S_i(x)]e^{-A_i(z)x} \quad (3.26)$$

$$Q_j(x,z)=Q_j(0,z)[1-V(x)]e^{-b(z)x} \quad (3.27)$$

$$R_i(x,y,z)=R_i(x,0,z)[1-G_i(y)]e^{-b(z)y} \quad (3.28)$$

where $A_i(z)=b(z)+\alpha_i\left(1-G_i^*(b(z))\right)$ and $b(z)=b\lambda\left(1-X(z)\right)$

From (3.5) we obtain

$$Q_{j,0}(x)=Q_{j,0}(0)[1-V(x)]e^{-b\lambda x} \quad (3.29)$$

Multiplying with equation (3.29) by $\gamma(x)$ on both sides for $j = J$ and integrating with respect to x from 0 to ∞,

then from (3.1) we have, $Q_{j,0}(0)=\frac{b\lambda P_0}{V^*(b\lambda)}$ (3.30)

From Eq.(3.30) and solving (3.14) and(3.29) over the range $j = J-1, J-2, \ldots 1$,we get

$$Q_{j,0}(0)=\frac{b\lambda P_0}{\left[V^*(b\lambda)\right]^{J-j+1}},\ j=1,2,...J-1 \quad (3.31)$$

From (3.14),(3.30) and(3.31), we get

$$Q_j(0,z)=\frac{b\lambda P_0}{\left[V^*(b\lambda)\right]^{J-j+1}},\ j=1,2,...J \quad (3.32)$$

Integrating Eq. (3.29) from 0 to ∞ and using (3.30) and (3.31) again, we finally obtain

$$Q_{j,0}(0,z)=\frac{P_0(1-V^*(b\lambda))}{\left[V^*(b\lambda)\right]^{J-j+1}},\ j=1,2,...J \quad (3.33)$$

Note that $Q_{j,0}$represents the steady-state probability that no customer appear while the server is on the j th vacation. Let us define Q_0 as the probability that no customer appear in the system while the server is on vacation. Then

$$Q_0=\frac{P_0\left(1-\left[V^*(b\lambda)\right]^J\right)}{\left[V^*(b\lambda)\right]^J},\ j=1,2,...J \quad (3.34)$$

Inserting (3.25) in (3.22), we obtain

$$\Pi_1(0,z)=\frac{P(0,z)}{z}\left[R^*(\lambda)+\left(r+\frac{\bar{r}}{z}\right)X(z)\left(1-R^*(\lambda)\right)\right]+\frac{b\lambda X(z)}{z}P_0 \quad (3.35)$$

Inserting (3.34) in (3.23), we obtain

$$\Pi_i(0,z)=\Theta_{i-1}\Pi_1(0,z)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right),\ (i=2,3,...k) \quad (3.36)$$

Inserting (3.28) in (3.24), we obtain

$$R_i(x,0,z)=\alpha_i\Pi_i(0,z)[1-S_i(x)]e^{-A_i(z)x} \quad (3.37)$$

Using (3.26) and (3.32) in (3.21), finally we get,

$$P(0,z)=\left(\begin{array}{l}\sum_{i=1}^{k}\left\{(p_i z+q_i)\Pi_i(0,z)\left(S_i^*\left[A_i(z)\right]\right)\right\}\\ +Q_j(0,z)V^*\left[b(z)\right]-b\lambda P_0-\frac{b\lambda P_0}{\left[V^*(b\lambda)\right]^{J-j+1}}\end{array}\right) \quad (3.38)$$

Solving (3.32),(3.35),(3.36) and (3.38), we get

$$P(0,z)=b\lambda P_0\times\left\{\frac{X(z)\left\{\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right\}+z\left(N(z)-1\right)}{z-R_1\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\} \quad (3.39)$$

$$where, R_1=R^*(\lambda)+\left(r+\frac{\bar{r}}{z}\right)X(z)\left(1-R^*(\lambda)\right)$$

where $N(z)=\frac{\left(1-\left[V^*(b\lambda)\right]^J\right)}{\left[V^*(b\lambda)\right]^J\left(1-\left[V^*(b\lambda)\right]\right)}\left(V^*(b(z))-1\right)$

Using (3.39) in (3.35), we get,

$$\Pi_1(0,z) = b\lambda P_0 \times \left\{ \frac{(N(z)-1)R_1 + X(z)}{z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}} \right\} \quad (3.40)$$

$$\Pi_i(0,z) = b\lambda P_0 \Theta_{i-1}(B_{i-1}^*(A_{i-1}(z))) \left\{ \frac{(N(z)-1)R_1 + X(z)}{z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}} \right\} \quad (3.41)$$

Using (3.41) in (3.37), we get

$$R_i(x,0,z) = \alpha_i \Theta_{i-1} \Pi_i(0,z)[1 - S_i(x)] e^{-A_i(z)x} \quad (3.42)$$

Using (3.25)-(3.28) , (3.39)-(3.42) and (3.32), then we get the limiting probability generating functions $P(x,z), \Pi_i(x,z), Q_j(x,z)$ and $R_i(x,y,z)$. We summarize the above results in the following theorem 3.1.

Theorem 3.1. *Under the stability condition ρ < 1, the stationary distributions of the number of customers in the orbit and the server's state has the following PGF's (for 1≤ i ≤ k)*

$$P(x,z) = \left(\begin{array}{l} b\lambda P_0 (1-R(x)) e^{-\lambda x} \\ \times \left\{ \dfrac{X(z)\left\{ \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\} \right\} + z(N(z)-1)}{z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}} \right\} \end{array} \right) \quad (3.43)$$

$$\Pi_i(x,z) = \left(\begin{array}{l} b\lambda P_0 \left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)(1 - S_i(x)) e^{-A_i(z)x} \\ \times \left\{ \dfrac{\Theta_{i-1}\left((N(z)-1)R_1 + X(z)\right)}{z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}} \right\} \end{array} \right) \quad (3.44)$$

$$Q_j(x,z) = \frac{b\lambda P_0}{\left[V^*(b\lambda)\right]^{J-j+1}} (1 - V(x)) e^{-b(z)x}, \; j = 1,2,...J \quad (3.45)$$

$$R_i(x,y,z) = \alpha_i \Pi_i(0,z)[1 - S_i(x)] e^{-A_i(z)x} \times [1 - G_i(y)] e^{-b(z)y} \quad (3.46)$$

where $A_i(z) = b(z) + \alpha_i\left(1 - G_i^*(b(z))\right)$ and $b(z) = b\lambda(1 - X(z))$

.

Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 3.2. *Under the stability condition ρ < 1, the stationary distributions of the number of customers in the system when server being idle, busy on i^{th} stage, vacation on j^{th} stage of vacation, under repair on i^{th} stage (for 1≤ i ≤ k,j=1,2,..J) are given by*

$$P(z) = \lambda P_0 \left(1 - R^*(\lambda)\right) \left\{ \frac{X(z)\left\{ \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\} \right\} + z(N(z)-1)}{z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}} \right\} \quad (3.47)$$

$$\Pi_i(z) = \left(\begin{array}{l} b\lambda P_0 \left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right) \\ \times \left\{ \dfrac{\Theta_{i-1}\left((N(z)-1)R_1 + X(z)\right)}{z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}} \right\} \end{array} \right) \quad (3.48)$$

$$Q_j(z) = \frac{P_0\left(1 - V^*(b(z))\right)}{\left[V^*(b\lambda)\right]^{J-j+1}(1 - X(z))} \quad (3.49)$$

$$R_i(z) = \left(\begin{array}{l} \dfrac{\alpha_i b\lambda P_0 \left(1 - S_i^*(A_i(z))\right)\left(1 - G_i^*(b(z))\right)}{A_i(z) b(z)} \\ \times \left\{ \dfrac{\Theta_{i-1}\left((N(z)-1)R_1 + X(z)\right)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)}{\left(z - R_1 \sum_{i=1}^{k} \left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right) \right\}\right)} \right\} \end{array} \right) \quad (3.50)$$

where

$$P_0 = \frac{1}{\beta}\left\{ 1 - (E(X) - \bar{r})\left(1 - R^*(\lambda)\right) - \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k} p_i \Theta_{i-1} + \sum_{i=1}^{k-1} \Theta_i M_{1i} \right\} \quad (3.51)$$

$$\beta = \left(\begin{array}{l} \left(1 + \dfrac{N^1(1)}{E(X)}\right)\left(1 - \omega - (\bar{r} - E(X)(1-b))\left(1 - R^*(\lambda)\right)\right) \\ -b(1-\omega)\left(1 - R^*(\lambda)\right) + \lambda b \left\{ \sum_{i=1}^{k} \Theta_{i-1} E(S_i)\kappa\left(1 - \alpha_i E(G_i)\right) \right\} \end{array} \right)$$

$$\kappa = E(X) + N'(1) - (E(X) - \bar{r})\left(1 - R^*(\lambda)\right)$$

Proof. Integrating the above (3.43) - (3.45) equations with respect to x and define the partial probability generating functions as, for ($1 \le i \le k$ and $j=1,2,...J$)

$$P(z) = \int_0^\infty P(x,z)dx, \; \Pi_i(z) = \int_0^\infty \Pi_i(x,z)dx, \; Q_j(z) = \int_0^\infty Q_j(x,z)dx.$$

Integrating the equation (3.46) with respect to x and y define the partial probability generating functions as, for($1 \le i \le k$), $R_i(x,z) = \int_0^\infty R_i(x,y,z)dy, \; R_i(z) = \int_0^\infty R_i(x,z)dx.$ Since, the only unknown is P_0 the probability that the server is idle when no customer in the orbit and it can be determined using the normalizing condition($1 \le i \le k$). Thus, by setting $z = 1$ in

(3.47) – (3.50) and applying L – Hospitals rule whenever necessaryand we get $P_0 + P(1) + \sum_{j=1}^{J} Q_j(1) + \sum_{i=1}^{k} \left(\Pi_i(1) + R_i(1)\right) = 1.$

Theorem3.3. *Under the stability condition ρ < 1, probability generating function of number of customers in the system and orbit size distribution at stationary point of time is*

$$K(z) = \frac{Nr(z)}{Dr(z)} \qquad (3.52)$$

$$Nr(z) = P_0 \left(\begin{array}{l} \left(\begin{array}{l} z\left\{ \sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1 - S_i^{*}(A_i(z))\right)\left((N(z)-1)R_1 + X(z)\right) \right\} \\ -N(z)\left(z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right) \end{array} \right) \\ +\left[1 - X(z)\right] \left(\begin{array}{l} \left[z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right] + \\ b\left(1 - R^{*}(\lambda)\right) \left[\begin{array}{l} \left\{ \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1} X(z) \left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right\} \\ +z\left(N(z)-1\right) \end{array} \right] \end{array} \right) \end{array} \right)$$

$$Dr(z) = \left[1 - X(z)\right]\left(z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right),$$

$$where R_1 = R^{*}(\lambda) + \left(r + \frac{\bar{r}}{z} \right) X(z)\left(1 - R^{*}(\lambda)\right)$$

$$H(z) = \frac{Nr(z)}{Dr(z)} \qquad (3.53)$$

$$Nr(z) = P_0 \left\{ \begin{array}{l} \left(\begin{array}{l} \left\{ \sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1 - S_i^{*}(A_i(z))\right)\left((N(z)-1)R_1 + X(z)\right) \right\} \\ -N(z)\left(z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right) \end{array} \right) \\ +\left[1 - X(z)\right] \left(\begin{array}{l} \left[z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right] + \\ b\left(1 - R^{*}(\lambda)\right) \left[\begin{array}{l} \left\{ \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1} X(z) \left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right\} \\ +z\left(N(z)-1\right) \end{array} \right] \end{array} \right) \end{array} \right\}$$

$$where R_1 = R^{*}(\lambda) + \left(r + \frac{\bar{r}}{z} \right) X(z)\left(1 - R^{*}(\lambda)\right)$$

where P_0 is given in Eq. (3.51).

Proof. Theprobability generating function of the number of customer in the system ($K(z)$) is obtained by using $K(z) = P_0 + P(z) + \sum_{j=1}^{J} Q_j(z) + z\sum_{i=1}^{k} \left(\Pi_i(z) + R_i(z)\right)$. The PGF of number of customer in the orbit ($H(z)$) is obtained by using $H(z) = P_0 + P(z) + \sum_{j=1}^{J} Q_j(z) + \sum_{i=1}^{k} \left(\Pi_i(z) + R_i(z)\right)$ Substituting (3.47) – (3.50) in the above results, then the equations (3.52) and (3.53) can be obtained by direct calculation.

4. Performance Measures

In this section, we obtain some probabilities when the system is in different status. Various performance measures like the mean number of customers in the orbit (L_q) and the reaming measures like the mean number of customers in the system (L_s), the average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are required during the analysis of unreliable queueing model. Note that (3.51) gives the steady state probability that the server is idle but available in the system. From (3.511) we have ρ< 1 which is stability condition. It follows from (3.47)-(3.50) that the probabilities of the server sate are as follows in theorem 4.1:

Theorem 4.1.

If the system satisfies the stability condition ρ< 1, then we get the following probabilities,

1)Let P be the steady state probability that the server is idle during the retrial time, $P = \frac{b\left(1 - R^{*}(\lambda)\right)}{\beta}\left(E(x) + N'(1) + \omega - 1\right)$

2) Let Π_i be the steady-state probability that the server is busy on i^{th} stage,

$$\Pi_i = \sum_{i=1}^{k} \Pi_i = \frac{1}{\beta}\sum_{i=1}^{k}\left\{ \Theta_{i-1}\lambda b E(S_i)\kappa \right\}$$

3) Let Q_j be the steady state probability that the server is on vacation with j th stage

$$Q_j = \sum_{j=1}^{J} Q_j = \frac{1}{\beta}\left\{ 1 - \omega - (E(X) - \bar{r})\left(1 - R^{*}(\lambda)\right) \right\} \frac{N'(1)}{E(X)}$$

4) Let R_i be the steady state probability that the server is under repair on i^{th} stage,

$$R_i = \sum_{i=1}^{k} R_i = \frac{1}{\beta}\sum_{i=1}^{k}\left\{ \alpha_i \Theta_{i-1} b\lambda\kappa E(S_i) E(G_i) \right\}$$

Proof. Noting that

$$P = \lim_{z\to 1} P(z), \quad \sum_{i=1}^{k}\Pi_i = \lim_{z\to 1}\sum_{i=1}^{k}\Pi_i(z),$$

$$\sum_{j=1}^{J} Q_j = \lim_{z\to 1}\sum_{j=1}^{J} Q_j(z) \text{ and } \sum_{i=1}^{k} R_i = \lim_{z\to 1}\sum_{i=1}^{k} R_i(z).$$

The stated formula follows by direct calculation.

Theorem 4.2. Let L_s, L_q, W_s and W_q be the mean number of customers in the system, the mean number of customers in the orbit, average time a customer spends in the system and average time a customer spends in the orbit using Little's formula respectively, then under the stability condition, we have

$$L_q = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_q''(1) = -2\begin{pmatrix} \kappa\left\{\sum_{i=1}^{k}\Theta_{i-1}b\lambda E(X)E(S_i)\left(1+\alpha_i E(G_i)\right)\right\} \\ +\left\{1-\omega-(E(X)-\bar{r})\left(1-R^*(\lambda)\right)\right\}\left(N'(1)+E(X)\right) \\ +b\left(1-R^*(\lambda)\right)E(X)\delta_1 \end{pmatrix}$$

$$Nr_q'''(1) = 3\begin{pmatrix} -\sum_{i=1}^{k}\Theta_{i-1}\begin{pmatrix} \kappa\begin{pmatrix} \lambda bE(X(X-1))E(S_i)\left(1+\alpha_i E(G_i)\right) \\ +\alpha_i\left(\lambda bE(X)\right)^2 E(S_i)E(G_i^2) \\ +\left(\lambda bE(X)\right)^2 E(S_i^2)\left(1+\alpha_i E(G_i)\right)^2 \\ +2\lambda bE(X)E(S_i)\left(1+\alpha_i E(G_i)\right)M_{1i-1} \end{pmatrix} \\ +\lambda bE(X)E(S_i)\left(1+\alpha_i E(G_i)\right) \\ \left(\left(1-R^*(\lambda)\right)\begin{pmatrix} E(X(X-1))-2\bar{r}E(X)+2\bar{r}- \\ 2\left(1-R^*(\lambda)\right)(E(X)-\bar{r})N'(1) \\ -N''(1)-E(X(X-1)) \end{pmatrix}\right) \end{pmatrix} \\ -N''(1)\left(1-\rho\right)+\begin{pmatrix} N'(1)\left[\tau+\left(1-R^*(\lambda)\right)(\delta_2-b\delta_3)\right] \\ -E(X(X-1))\left[1-\rho+b\left(1-R^*(\lambda)\right)\delta_1\right] \end{pmatrix} \end{pmatrix}$$

$$Dr_q''(1) = -2E(X)\left(1-\rho\right)$$

$$Dr_q''' = 3\begin{Bmatrix} E(X)\begin{pmatrix} \tau+2(E(X)-\bar{r})(1-R^*(\lambda))\omega \\ +\left(1-R^*(\lambda)\right)\left(E(X(X-1))\right)-\bar{r}(2-E(X)) \end{pmatrix} \\ -E(X(X-1))(1-\rho) \end{Bmatrix}$$

where
$$\tau = \sum_{i=1}^{k}\Theta_{i-1}M_{2i} + 2\sum_{i=1}^{k}p_i\Theta_{i-1}M_{1i} - \sum_{i=1}^{k-1}\Theta_i M_{2i},$$
$$\delta_1 = E(X)+N'(1)+\omega-1,$$
$$\delta_2 = \left(E(X(X-1))-2\bar{r}E(\mathrm{X})+2\bar{r}+2\omega E(X)-2\omega\bar{r}\right),$$
$$\delta_3 = \left(E(X(X-1))+2\omega E(X)+\tau+\mathrm{N}''(1)+2\,\mathrm{N}'(1)\right)$$

$$L_s = P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_s'''(1) = Nr_q'''(1) - 6\sum_{i=1}^{k}\Theta_{i-1}\lambda b\kappa E(X)E(S_i)\left(1+\alpha_i E(G_i)\right)$$

$$W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}$$

Proof. The mean number of customers in the orbit (L_q) under steady state condition is obtained by differentiating (3.48) with respect to z and evaluating at $z = 1$

$$L_q = \lim_{z\to 1}\frac{d}{dz}H(z) = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

The mean number of customers in the system (L_s) under steady state condition is obtained by differentiating (3.47) with respect to z and evaluating at $z = 1$

$$L_s = \lim_{z\to 1}\frac{d}{dz}K(z) = P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

The average time a customer spends in the system (W_s) and orbit (W_q) under steady- state condition due to Little's formula is, $L_s = \lambda W_s$ and $L_q = \lambda W_q$.

5. Stochastic decomposition and special cases

Stochastic decomposition has been widely observed among M/G/1 type queueing models with server vacations by Fuhrman and Cooper [11]. A key result in these analyses is that the number of customers in the system in steady-state at a random point in time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard queueing system (in steady-state) at a random point in time, the other random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled. Stochastic decomposition has also been observed to hold for some M/G/1 retrial queueing models by Krishnakumar and Arivudainambi [17].

Let $K(z)$ be the stationary system size distribution of $M^{[X]}/G/1$ feedback retrial queue with multi stage service, balking & reneging, at most J vacation policyand random breakdown is the convolution of two independent random variables $\chi(z)$ and $\phi(z)$.

The mathematical version of the stochastic decomposition law is $K(z) = \chi(z).\phi(z)$.

(i) The system size distribution of $M^{[x]}/G/1$ feedback queueing system with multi stage service, balking , reneging and service interruption. (represented in first term of $K(z)$),

(ii) The conditional distribution of the number of customers in the vacation system at random point in time given the server is idle (represented in second term of $K(z)$).

The number of arrivals in the variant vacation system at a random point in time given that the server is on vacation or idle. In fact the second term can be also obtained through the vacation definition of our system,

i.e., $\phi(z) = \frac{N2(z)}{D2(z)} = \left(P_0 + P(z) + \sum_{j=1}^{J}Q_j(z)\right)\Big/\left(P_0 + P(1) + \sum_{j=1}^{J}Q_j(1)\right)$

$$N2(z) = P_0 \left\{ \begin{array}{l} \left(\begin{array}{l} z\left\{ \sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)\left(\left(N(z)-1\right)R_1 + X(z)\right) \right\} \\ -N(z)\left(z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right) \end{array} \right) \\ +\left[1-X(z)\right] \left\{ \begin{array}{l} \left[z - R_1 \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right] + \\ b\left(1-R^{*}(\lambda)\right)\left[\begin{array}{l} \left\{ \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1} X(z)\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right\} \\ +z\left(N(z)-1\right) \end{array} \right] \end{array} \right\} \end{array} \right\}$$

$$\times\left(z - \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right)$$

$$where R_1 = R^{*}(\lambda) + \left(r + \frac{\bar{r}}{z} \right) X(z)\left(1-R^{*}(\lambda)\right)$$

$$D2(z) = (1-\omega)(1-z)\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)\times Dr(z)$$

The first term can be obtained through the without vacation definition of our system.

$$\chi(z) = \left\{ \frac{(1-\omega)(1-z)\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)}{\left(z - \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \right)} \right\}$$

From above stochastic decomposition law, we observe that $K(z) = \chi(z).\phi(z)$ which conform that the decomposition result of Fuhrman and Cooper [11], also valid for this special vacation system.

5.1. Special cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

Case (i): *Single phase, No retrial, No balking and reneging, No Vacation and No breakdown*

Let $P\,[X = 1] = 1$, $R^{*}(\lambda)\to 1$, $P[V = 0] = 1$, $V^{*}(\lambda)\to 1$, $b = 1, r = 1$ and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to multi stage M/G/1 queueing system with Bernoulli feedback. The following results agree with Salehirad and Badamchizadeh [23].

$$K(z) = P_0 \left\{ \frac{\left(1-S_1^{*}(A_1(z))\right) + \sum_{i=2}^{k}\Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)}{z - \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\}} \right\}$$

Case (ii): *Single phase, No feedback, No balking & reneging, No vacation and No breakdown*

Let $P[X = 1] = 1$, $k=1$, $P\,[S_k = 0] = 1$, $\theta_1 = 0$, $P[V = 0] = 1$, $b = 1, r = 1$ and $\alpha_1 = \alpha_2 = 0$, our model can be reduced to M/G/1 retrial queue.

$$K(z) = \left\{ \frac{\left[R^{*}(\lambda) - \lambda E(S_0)\right] S_0^{*}[\lambda - \lambda z][z-1]}{z - \left[R^{*}(\lambda) + z(1-R^{*}(\lambda))\right]\left\{S_0^{*}\left[\lambda - \lambda z\right]\right\}} \right\};$$

$$L_q = \frac{\left\{ \lambda^2 E(S_0{}^2) + 2\lambda E(S_0)\left(1-R^{*}(\lambda)\right) \right\}}{2\left\{ R^{*}(\lambda) - \lambda E(S_0) \right\}}$$

The following form and results agree with Gomez-Corral [13].

Case (iii): *Single phase, No balking & reneging, No Vacation and No breakdown*

Let $P[X = 1] = 1$, $P[V = 0] = 1$, $V^{*}(\lambda)\to 1$, $b = 1, r = 1$ and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to Multi stage retrial queueing system with Bernoulli feedback. The following results agree with Bagyam and Chandrika [3].

$$K(z) = P_0 R^{*}(\lambda) \left(\begin{array}{l} \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\} \\ +z\left\{ \sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^{*}\left[A_{i-1}(z)\right]\right)\left(1-S_i^{*}(A_i(z))\right)-1 \right\} \end{array} \right)\times$$

$$\left\{ \frac{1}{z - \left[R^{*}(\lambda) + X(z)\left(1-R^{*}(\lambda)\right)\right]\sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^{*}\left[A_i(z)\right]\right) \right\}} \right\}$$

where

$$P_0 = \left\{ \frac{R^{*}(\lambda) - \sum_{i=1}^{k}\Theta_{i-1}M_{1i} - \sum_{i=1}^{k} p_i\Theta_{i-1} + \sum_{i=1}^{k-1}\Theta_{i-1}M_{1i}}{R^{*}(\lambda)\left\{ 1 - \sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_i)\left(1-\alpha_i E(G_i)\right) - \sum_{i=1}^{k}\Theta_{i-1}(p_i + M_{1i}) + \sum_{i=1}^{k-1}\Theta_{i-1}M_{1i} \right\}} \right\}$$

Case (iv): *Single phase, No feedback, No balking & reneging*

Let $P\,[X = 1] = 1$, $k=1$, $P\,[S_k = 0] = 1$, $\theta_1 = 0$ and $b = 1, r = 1$; our model can be reduced to M/G/1 retrial queue with at most J vacations and server breakdown.

$$K(z) = \left(\frac{R^{*}(\lambda) - \lambda E(S_0)}{N'(1) + R^{*}(\lambda)} \right)\times$$

$$\left\{ \frac{N(z)\left[R^{*}(\lambda) + z(1-R^{*}(\lambda))\right] + R^{*}(\lambda)(z-1)}{z - \left[R^{*}(\lambda) + z(1-R^{*}(\lambda))\right]\left\{S_0^{*}\left[A_0(z)\right]\right\}} \right\} S_0^{*}\left[A_0(z)\right]$$

The following form and results agree with Chen. P et al. [5].

6. Numerical illustration

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In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures of our system where all retrial times, service times, vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. We assume arbitrary values to the parameters such that the steady state condition is satisfied. The following tables give the computed values of various characteristics of our model like, probability that the server is idle P_0, the mean orbit size L_q, probability that server is idle during retrial rime, busy on k stages, vacation on j th stage of vacation and under repair on k stages respectively, P, Π_i, $Q_j$and R_i for (i=1,2,…k, j=1,2,...J) where exponential distribution is $f(x)=ve^{-vx}, x>0$, Erlang-2stage distribution is $f(x)=v^2xe^{-vx}, x>0$ and hyper-exponential distribution is $f(x)=cve^{-vx}+(1-c)v^2e^{-v^2x}, x>0$.

Table 1 shows that when retrial rate (a) increases, then the probability that server is idle P_0 increases, the mean orbit size L_q decreasing and probability that server is idle during retrial time P also decreasingfor the values of λ= 0.2; p_1= 0.2; μ_1 = 5;α_1 = 0.2; ξ_1 = 3; γ =5; c =0.8; k =2.;p_2= 0.4; μ_2 = 7;α_2 = 0.4; ξ_2 = 5;r = 0.5; b=0.5; θ_1=0.2; θ_2 = 0.2;J =2.Table 2 shows that when balking probability (1-b) increases, then the probability that server is idle P_0 increases, the mean orbit size L_q decreasing and probability that server is idle during retrial time P also decreasing for the values of λ= 0.5; p_1= 0.2; μ_1 = 5;α_1 = 0.2; ξ_1 = 3; γ =5; c =0.8; k =2.;p_2= 0.4; μ_2 = 7;α_2 = 0.4; ξ_2 = 5;r = 0.5; a=4; θ_1=0.2; θ_2 = 0.2;J =2. Table 3 shows that when the feedback (p_1) increases, then the probability that server is idle P_0 decreasing, the mean orbit size L_q increasing and probability that server is idle during retrial time P increasingfor the values of λ=0.5; a=4; μ_1 = 5; α_1 = 0.2; ξ_1 = 3; k=1; γ =5; J =1; c =0.8.

Table 4 shows that when number of vacations (J) increases, then the probability that server is idle P_0 decreases, the mean orbit size L_q and probability that server is vacation on second type service Q also increasing for the values of λ= 0.2; p_1= 0.2.;p_2= 0.4; μ_1 = 5; μ_2 = 7;α_1 = 0.2; α_2 = 0.4; ξ_1 = 3; ξ_2 = 5;γ =5; a = 4; c =0.8; k =2;r = 0.5; b=0.5; θ_1=0.2; θ_2 = 0.2. Table 5 shows that when vacation rate (γ) increases, then the probability that server is idle P_0 increases, the mean orbit size L_q and probability that server is Vacation on j th stage of vacation$Q$ also decreasing for the values of λ= 0.2; p_1= 0.2.;p_2= 0.4; μ_1 = 5; μ_2 = 7;α_1 = 0.2; ;α_2 = 0.4; ξ_1 = 3; ξ_2 = 5; a = 4; c =0.8; k =2;J=2; r = 0.5; b=0.5; θ_1=0.2; θ_2 = 0.2.

Table 1 The effect of Retrial rate (a) on P_0, Lq and P

Retrial rate	Exponential			Erlang – 2 stage			Hyper – Exponential		
A	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
1.00	0.9072	0.1572	0.0216	0.8105	0.3721	0.0492	0.9191	0.1364	0.0211
2.00	0.9173	0.1313	0.0111	0.8325	0.3041	0.0253	0.9301	0.1090	0.0098
3.00	0.9208	0.1225	0.0075	0.8401	0.2811	0.0170	0.9334	0.1008	0.0063
4.00	0.9226	0.1181	0.0056	0.8440	0.2696	0.1028	0.9350	0.0969	0.0047
5.00	0.9237	0.1154	0.0045	0.8463	0.2627	0.0103	0.9360	0.0947	0.0037

Table 2: The effect of Balking (1-b) on P_0, Lq and P

Balking Probability	Exponential			Erlang – 2 stage			Hyper – Exponential		
$1-b$	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.30	0.9066	0.1798	0.0073	0.8092	0.4289	0.0225	0.9220	0.1467	0.0011
0.40	0.9136	0.1476	0.0034	0.8250	0.3441	0.0119	0.9277	0.1208	0.0008
0.50	0.9226	0.1181	0.0018	0.8440	0.2696	0.0065	0.9350	0.0969	0.0006
0.60	0.9335	0.0909	0.0009	0.8664	0.2037	0.0034	0.9442	0.0748	0.0004
0.70	0.9466	0.0658	0.0004	0.8927	0.1450	0.0016	0.9551	0.0542	0.0002

Table 3: The effect of different feedback probabilities (p_1) on P_0, Lq and P

Feedback probability	Exponential			Erlang – 2 stage			Hyper – Exponential		
p_1	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.10	0.8876	0.0733	0.0066	0.7697	0.1749	0.0195	0.9059	0.0602	0.0051
0.20	0.8763	0.0870	0.0110	0.7472	0.2095	0.0287	0.8964	0.0715	0.0089
0.30	0.8619	0.1052	0.0167	0.7183	0.2573	0.1405	0.8842	0.0863	0.0136
0.40	0.8426	0.1305	0.0242	0.6798	0.3272	0.0563	0.8680	0.1067	0.0200
0.50	0.8157	0.1678	0.0348	0.6259	0.4396	0.0785	0.8452	0.1366	0.0289

Table 4: The effect of Number of vacations (J) on P_0, L_q and Q

Number of vacations (J)	Exponential			Erlang – 2 stage			Hyper – Exponential		
J	P_0	L_q	Q	P_0	L_q	Q	P_0	L_q	Q
1.00	0.9416	0.1164	0.0192	0.8799	0.2635	0.0366	0.9394	0.0974	0.0279
2.00	0.9226	0.1181	0.0380	0.8440	0.2696	0.0717	0.9120	0.1003	0.0550
3.00	0.9039	0.1198	0.0564	0.8096	0.2754	0.1052	0.8855	0.1032	0.0812
4.00	0.8857	0.1214	0.0745	0.7767	0.2810	0.1373	0.8597	0.1060	0.1067
5.00	0.8678	0.1230	0.0921	0.7451	0.2864	0.1681	0.8348	0.1087	0.1343

Table 5: The effect of vacation Rate (γ) on P_0, L_q and Q

Vacation rate	Exponential			Erlang – 2 stage			Hyper – Exponential		
γ	P_0	L_q	Q	P_0	L_q	Q	P_0	L_q	Q
1.00	0.7789	0.1635	0.1799	0.5926	0.4048	0.3169	0.7845	0.1417	0.1812
2.00	0.8666	0.1294	0.0933	0.7413	0.3058	0.1718	0.8819	0.1065	0.0848
3.00	0.8974	0.1222	0.0629	0.7972	0.2832	0.1173	0.9120	0.1003	0.0550
4.00	0.9131	0.1195	0.0380	0.8262	0.2742	0.0890	0.9265	0.0981	0.0406
5.00	0.9226	0.1181	0.0380	0.8440	0.2696	0.0717	0.9350	0.0969	0.0322

Table 6: The effect of Repair rate (ξ_1) on P_0, L_q and R

Repair rate	Exponential			Erlang – 2 stage			Hyper – Exponential		
ξ_1	P_0	L_q	R	P_0	L_q	R	P_0	L_q	R
1.00	0.8037	0.3698	0.0019	0.7513	0.5631	0.0157	0.8338	0.3010	0.0007
2.00	0.8100	0.3408	0.0010	0.7687	0.5037	0.0078	0.8398	0.2757	0.0003
3.00	0.8121	0.3333	0.0007	0.7745	0.4878	0.0052	0.8415	0.2701	0.0002
4.00	0.8132	0.3300	0.0005	0.7774	0.4807	0.0040	0.8423	0.2677	0.0002
5.00	0.8138	0.3281	0.0004	0.7791	0.4766	0.0032	0.8428	0.2663	0.0001

Table 6 shows that when repair rate on the first stage (ξ_1) increases, then the probability that server is idle P_0 increases, the mean orbit size L_q and probability that server is on repair R also decreasing for the values of λ=

0.2; p_1= 0.2.;p_2= 0.4; μ_1 = 5; ; μ_2 = 7;α_1 = 0.2; ;α_2 = 0.4; γ = 5;ξ_2 = 5; a = 4; c =0.8; k =2; r = 0.5; b=0.5; θ_1=0.2; θ_2 = 0.2;J =2. Table 7 shows that when the reneging probability (r) increases, then the probability that server is idle P_0 decreasing, the mean orbit size L_q decreasing and probability that server is idle during retrial time P decreasing for the values of λ=0.2; a=4; μ_1 = 5; α_1 = 0.2; ξ_1 = 3; p_1=0.2; k=1; γ=5; J =1; c =0.8.

Table 7: The effect of Reneging Probability (r) on P_0, L_q and P

Reneging Probanility	Exponential			Erlang – 2 stage			Hyper – Exponential		
R	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.10	0.9742	0.0323	0.0037	0.9434	0.0729	0.0087	0.9788	0.0267	0.0031
0.20	0.9681	0.0321	0.0037	0.0317	0.0723	0.0086	0.9736	0.0266	0.0030
0.30	0.9622	0.0320	0.0037	0.9202	0.0717	0.0086	0.9685	0.0265	0.0030
0.40	0.9562	0.0319	0.0037	0.9088	0.0711	0.0085	0.9634	0.0264	0.0030
0.50	0.9503	0.0317	0.0037	0.8975	0.0705	0.0085	0.9584	0.0263	0.0030
0.60	0.9444	0.0316	0.0036	0.8863	0.0699	0.0084	0.9534	0.0262	0.0030

For the effect of the parameters a, p_1, γ, J,r, and ξ_1 on the system performance measures, two dimensional graphs are drawn in Figure 1-7. Figure 1 shows that the idle probability P_0 increases for the increasing the value of the retrial rate (a). Figure 2 shows that the mean orbit size L_q increasing for the increasing the value of the feedback probability (p_1). Figure 3 shows that the mean orbit size L_q increases for the increasing the value of J. Figure 4 shows that the mean orbit size L_q decreases for the increasing the value of the vacation rate (γ). Figure 5 shows that the mean orbit size L_q increases for the increasing the value of reneging probability (r). Figure 6 shows that the mean orbit size L_q decreases for the increasing the value of repair rate on FSS (ξ_1). Figure 7 shows that the mean orbit size L_q increases for the increasing the value of balking probability (b).

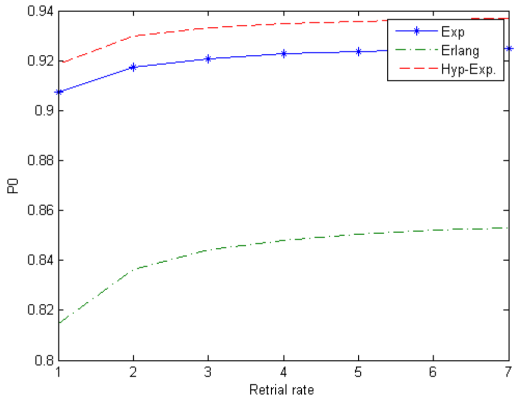


***Figure 1** P_0 verses a*

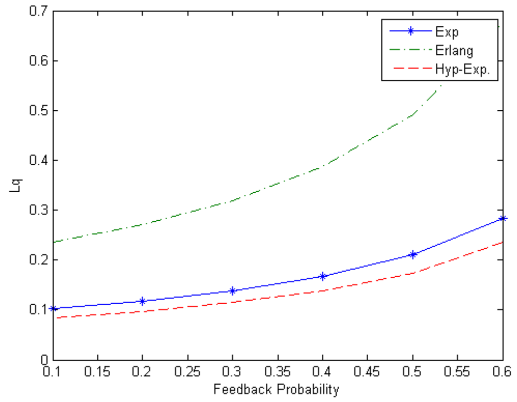


***Figure 2:** L_q verses p_1*

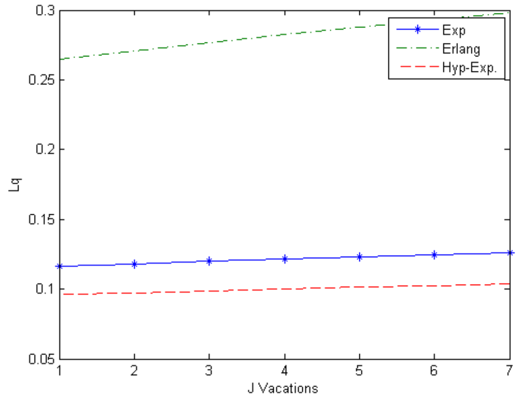


Figure 3 *L_q verses J*

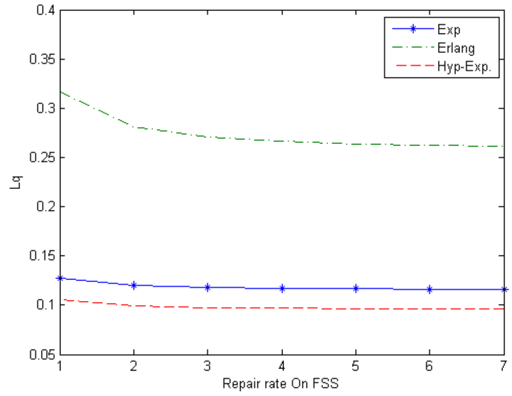


Figure 6 *L_q verses ξ_1*

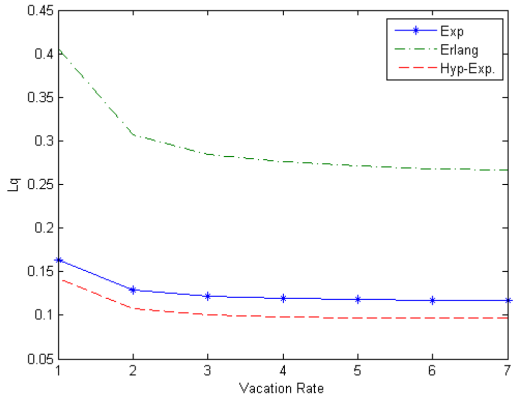


Figure 4*L_q verses γ*

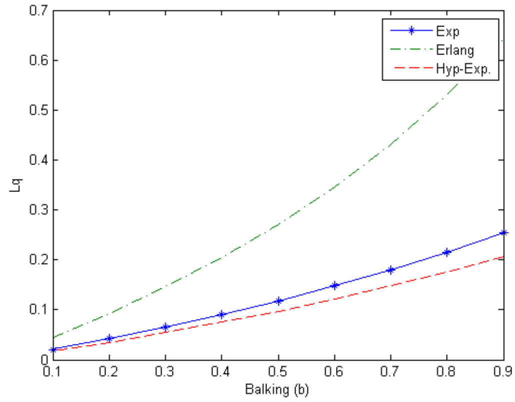


Figure 7: *L_q verses b*

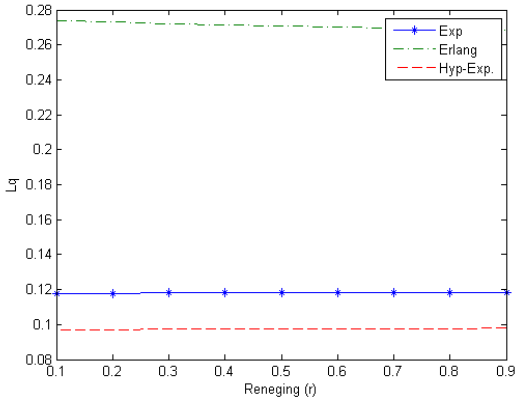
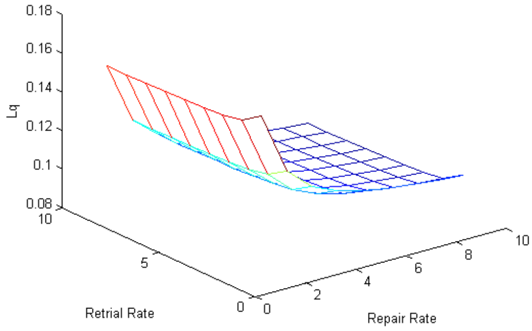


Figure 5 *L_q verses*



*r**Figure 8:*** *L_q verses a and ξ_1*

Three dimensional graphs are illustrated in Figure 8-12. In Figure 8, the surface displays upward trend as expected for increasing the value of the retrial rate (a) and repair rate on FSS (ξ_1) against the mean orbit size L_q. In Figure 9, the surface which represents the mean orbit size L_q displays an upward trend as expected for increasing the value of the non-balking probability $b$and repair rate on FSS (ξ_1). In Figure 10, the surface which represents the mean orbit size L_q displays an upward trend as expected for increasing the value of the non-reneging probability r

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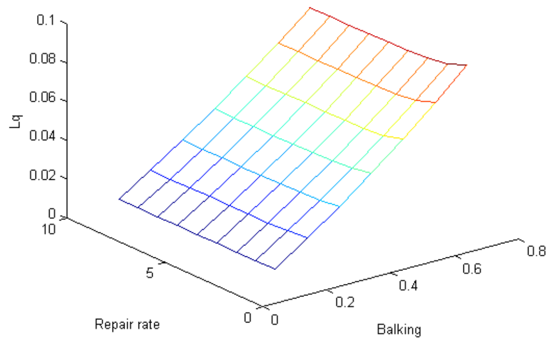


Figure 9: *L_q verses ξ_1 and b*

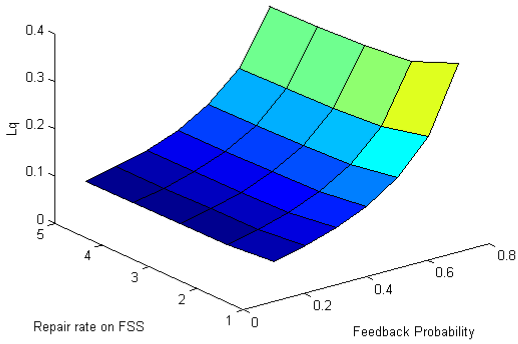


Figure 11: *L_q verses ξ_1 and p_1*

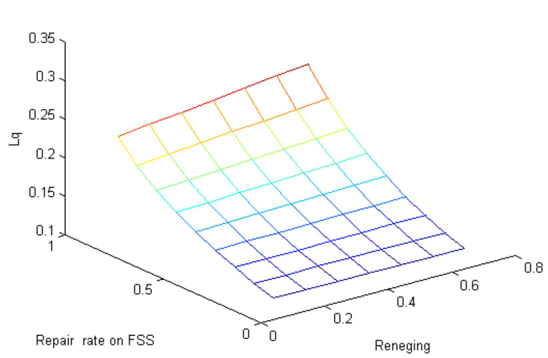
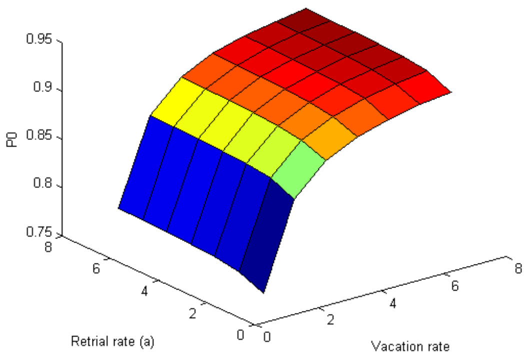


Figure 10: *L_q verses $\xi_1$and*



*r**Figure 12:** P_0 verses a and γ*

and repair rate on FSS (ξ_1). In Figure 11, the mean orbit size $L_q$increases for increasing the value of the repair rate on FSS (ξ_1) and the feedback probability (p_1).The idle probability (P_0) decreases for increasing the value of retrial rate (a) and the increasing value of (γ) is shown in Figure 12.

7. Conclusion

In this paper, we have studied a batch arrival feedback retrial queueing system with balking, reneging, at most J vacations where the server is subject to server breakdowns and repair. A single server provides multi stages of service. The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The

performance measures like, the mean number of customers in the system/orbit, the average waiting time of customer in the system/orbit and some system probabilities were obtained. Finally, the general decomposition law is shown to hold good for this model. The analytical results are validated with the help of numerical illustrations.

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