International Journal of Applied Engineering Research ISSN 0973-4562 Volume 10, Number 14 (2015) pp 34699-34703
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Duality Theorem for Discrete Hartley Transform

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Abstract- This paper presents a new property called the Duality Theorem for Continuous Hartley Transform (CHT). Hartley Transform is an integral transform related to the renowned Fourier Transform but has remained in the oblivion from mathematicians and engineers alike. Many of the properties of the Hartley Transform are akin to those of the Fourier Transform subject to minor differences. A formal derivation of the Duality Theorem corresponding to Continuous Hartley Transform is given which was hitherto not mentioned or derived in the literature. The usage of Duality Theorem helps in finding the time-domain function from the Hartley frequency domain and vice versa thereby reducing considerable labour involved in integration and computation time.

Keywords- duality, kernel, integral, orthogonal

1. Introduction

The Discrete Hartley Transform (DHT) is a variant of the Discrete Fourier Transform (DFT) which is renowned as the largely used transforms in the arenas of Communication Engineering and Signal Processing [1]. Discrete Hartley Transform (DHT) was developed by Ronald N. Brace well, a famous Australian physicist, engineer, and mathematician, in the 1980s. DHT is a discretized version of the Continuous Hartley Transform (CHT) invented by the U.S. electronics researcher Ralph Vinton Lyon Hartley (1888 A.D. – 1970 A.D.)in 1942. Though it existed in the technical literature, it remained in the oblivion for a number of years until Brace well invented and published a discretized version of the same in 1983 [2]. Hartley also invented the Hartley Oscillator and also contributed towards the development of Information Theory during its stage of infancy. In this paper, a new property called the Duality Theorem for the Discrete Hartley Transform is formally proposed and derived, followed by its illustration with a suitable example.

2. The Continuous Hartley Transform

The Continuous Hartley Transform (CHT) or the Infinite Hartley Transform is an orthogonal integral transform. Originally, Hartley defined the CHT of a continuous – time function$f(t)$ as

$$H(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t).cas(\omega t)\ dt \qquad \dots (1)$$

Here, ω is the angular frequency in rad/sec [3]. Note that $cas(\alpha) = \cos\alpha + sin\alpha = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\alpha - \frac{\pi}{4}\right)$ is the cosine-and-sine or Hartley kernel. In Signal Processing terms, this transform takes a signal (function) from the time-domain to the Hartley spectral domain (frequency domain). Eq. (1) is called the Analysis Equation or the Forward Equation/Transform of the CHT. The Inverse Continuous Hartley Transform (ICHT) was again defined by Hartley originally as

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(\omega).cas(\omega t)\ d\omega \qquad \dots (2)$$

Eq. (2) is called the Synthesis Equation or the Backward Equation/Transform of the CHT. The Hartley transform has the convenient property of being its own inverse. Hence, it is an Involution Integral or Transform [4].

The properties of the $cas[.]$ function follow directly from Trigonometry and its definition as phase-shifted trigonometric functions as given below.

$$cas\theta_2 = \sqrt{2}\sin\left(\theta_2 + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\theta_2 - \frac{\pi}{4}\right) \qquad \dots (3)$$

It has angle-addition identity as given below.

$$cas(\theta_1 + \theta_2) = \{\cos\theta_1 cas\theta_2 + \sin\theta_1 cas(-\theta_2)\} \qquad \dots (4)$$

Also, we can write the above formula as

$$cas(\theta_1+\theta_2) = \{\cos\theta_2 cas\theta_1 + \sin\theta_2 cas\,(-\theta_1)\} \quad ...(5)$$

The first-order derivative of $cas\theta_1$ w.r.t. θ_1 is given by the following expression,

$$cas'(\theta_1) = \cos\theta_1 - \sin\theta_1 = cas(-\theta_1) \quad ...(6)$$

Thus, the definition of the Continuous Hartley Transform defined Eqs. (1) can be alternatively defined as

$$H(\omega) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(t).sin(\omega t + \frac{\pi}{4})\,dt \quad \ldots(7)$$

$$H(\omega) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(t).cos(\omega t - \frac{\pi}{4})\,dt \quad \ldots(8)$$

Likewise, the definition of the Infinite Continuous Hartley Transform defined Eqs. (2) can be alternatively defined as

$$f(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H(\omega).sin(\omega t + \frac{\pi}{4})\,dt \quad \ldots(9)$$

$$f(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H(\omega).cos(\omega t - \frac{\pi}{4})\,dt \quad \ldots(10)$$

Thus, $f(t)$ and $H(\omega)$ are said to form an Infinite or Continuous Hartley Transform pair [5]. This is represented mathematically as

$$f(t) \overset{CHT}{\longleftrightarrow} H(\omega)$$

The above symbology is largely adopted and used in the Signal Processing literature.

3. Relationship between the Continuous Fourier Transform and Continuous Hartley Transform

The Continuous Hartley Transform (CHT) or the Infinite Hartley Transform is closely related to the Continuous Time Fourier Transform or Infinite Fourier Transform or the Complex Fourier Transform. CHT differs from the classic Fourier transform in the choice of the kernel. It is well-known that the Continuous Time Fourier Transform is a complex integral transform due to the use of the complex exponential, $e^{-j\omega t}$ or $e^{-j2\pi ft}$ as its kernel. But the Continuous Hartley Transform (CHT), unlike the Continuous Time Fourier Transform, is a real transform because its kernel $cas(\theta) = \cos\theta + \sin\theta$ is real. Even its backward or inverse transform is also real [6].

The Continuous – Time Fourier Transform (CTFT) of a continuous – time function $f(t)$ is defined as,

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad \ldots(11)$$

Eq. (11) corresponds to the Analysis Equation or the Forward Equation/Transform of CTFT.

The Inverse Fourier Transform of $F(\omega)$ is defined by the following equation,

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \quad \ldots(12)$$

Eq. (12) corresponds to the Synthesis Equation or the Backward Equation/Transform of CTFT. The CTFT can be directly obtained by CHT by the following expression,

$$F(\omega) = \frac{1}{2}[H(\omega)+H(-\omega)] - j\frac{1}{2}[H(\omega)-H(-\omega)] \quad \ldots(13)$$

That is, the real and imaginary parts of the Fourier transform are simply given by the even and odd parts of the Hartley transform, respectively. Conversely, for a real-valued function $f(t)$, the Hartley transform is given from the Continuous – Time Fourier transform's real and imaginary parts:

$$H(\omega) = F_R(\omega) - F_I(\omega) = Re[F\{f(t(1+j))\}] \quad \ldots(14)$$

Here, F is the Fourier Transform operator, and $Re[.]$ depicts the real part of the entity. It is easy to extend the definition of Continuous Hartley Transform (CHT) and its inverse to N −dimensions [7].

4. The Discrete Hartley Transform (DHT)

The Discrete Hartley Transform (DHT) of a discrete – time signal, $x(n)$, is defined as

$$X_H(k) = DHT[x(n)] = \sum_{n=0}^{N-1} x(n)cas\left(\frac{2\pi kn}{N}\right) \quad \ldots(15)$$

$\nabla\ 0 \le n,k \le N-1$.Also, note that $cas(\frac{2\pi kn}{N})$ is the cosine-and-sine kernel which is simply given by the formula,

$$cas\left(\frac{2\pi kn}{N}\right) = cos\left(\frac{2\pi kn}{N}\right) + sin\left(\frac{2\pi kn}{N}\right) \quad \ldots(16)$$

It is interesting to note that

$$cas\left(-\frac{2\pi kn}{N}\right) = cos\left(\frac{2\pi kn}{N}\right) - sin\left(\frac{2\pi kn}{N}\right) \quad \ldots(17)$$

Eq. (15) is called the analysis equation or the forward equation/transform of DHT [8].The Inverse Discrete Hartley Transform (IDHT) of $X_H(k)$ is defined by the relation,

$$x(n) = IDHT[X_H(k)] = \frac{1}{N}\sum_{n=0}^{N-1} X_H(k)cas\left(\frac{2\pi kn}{N}\right),\ \nabla\ 0 \le n,k \le N-1. \quad \ldots(18)$$

Eq. (18) is called the synthesis equation or the backward equation/transform of DHT. The term $cas\left(\frac{2\pi kn}{N}\right)$ used in the definition of the analysis and synthesis equations of the DHT, is a real term due to which the DHT is a real transform. Thus, unlike in the DFT, where there is a change of sign in the kernel in the synthesis equation, in the case of the DHT, there is no such thing, which means that a single algorithm can be used to compute both the forward and backward transforms of DHT, which is a major advantage over the conventional DFT [8]. This is the equivalent to the case encountered in Continuous Hartley Transform. Hence, the DHT also is its own inverse. Thus, the DHT involves only real operations and hence, the computational load and memory requirement are considerably reduced by 50%, which stands out as a good merit over the conventional DFT. Efficient algorithms called the Fast Hartley Transforms (FHT) have been developed for computing the DHT and have been in use in many DSP applications [9]. Thus, $x(n)$ and $X_H(k)$ form a Discrete Hartley Transform (DHT) pair. This is mathematically represented as,

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 10, Number 14 (2015) pp 34699-34703
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$$x(n) \overset{DHT}{\longleftrightarrow} X_H(k)$$

5. Relationships between DHT and DFT

A. Relationship between the Analysis Equations of DFT and DHT

The DFT of a discrete – time sequence $x(n)$ is defined as

$$X(k) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{2\pi kn}{N}} \qquad \ldots (19)$$

$\nabla\ 0 \le n, k \le N-1$.

Using Euler's Theorem, $e^{\pm j\gamma} = cis(\pm\gamma) = \cos(\gamma) \pm j\sin(\gamma)$, we get,

$$X(k) = \sum_{n=0}^{N-1} x(n)\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right] \qquad \ldots (20)$$

Due to the absence of the j term in $X_H(k)$, DHT is purely real. Next, consider the following expression,

$e^{j\alpha} = cis(\alpha) = \cos(\alpha) + j\sin(\alpha)$.

It is well known that $\cos\alpha = \frac{e^{j\alpha}+e^{-j\alpha}}{2}$ and $\sin\alpha = \frac{e^{j\alpha}-e^{-j\alpha}}{2j}$

$$\Rightarrow cas\alpha = \cos\alpha + \sin\alpha = \frac{(1-j)e^{j\alpha}}{2} + \frac{(1+j)e^{-j\alpha}}{2} \qquad \ldots (21)$$

Next, we make the following assumptions

$$\chi = e^{j\alpha}, k = cas\alpha, \xi = \frac{(1-j)}{2}, \delta = \frac{1+j}{2} \qquad \ldots (22)$$

Substitution and simplification leads to the following quadratic equation [10],

$$\xi\chi^2 - k\chi + \delta = 0 \qquad \ldots (23)$$

Solving for χ, we get, $\chi = \frac{\rho}{2\xi} \pm \frac{1}{2\xi} j\,(\cos\alpha - \sin\alpha) \qquad \ldots (24)$

Considering only the negative sign and simplifying, we get,

$$\chi = e^{j\alpha} = \frac{1+j}{2} cas\alpha + \frac{1-j}{2} cas(-\alpha) \qquad \ldots (25)$$

Also, we have,

$$\chi^{-1} = \frac{1}{\chi} = e^{-j\alpha} = \frac{1+j}{2} cas(-\alpha) + \frac{1-j}{2} cas(\alpha) \qquad \ldots (26)$$

Now, the DFT of $x(n)$ is given by the well known equation

$$X(k) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{2\pi kn}{N}} \qquad \ldots (27)$$

Making use of Eq. (26) in the above equation, we get,

$$X(k) = \frac{1+j}{2} X_H(-k) + \frac{1-j}{2} X_H(k)$$

$$X(k) = \frac{1}{2}[X_H(k) + X_H(-k)] + j\frac{1}{2}[X_H(-k) - X_H(k)] \ldots (28)$$

Since $X(k)$ is complex, we can write the above equation as,

$$X(k) = X_R(k) + jX_I(k) \qquad \ldots (29)$$

Thus, the DFT coefficients in terms of DHT coefficients are given by the following two equations [11],

$$X_R(k) = \frac{1}{2}[X_H(k) + X_H(-k)] \qquad \ldots (30)$$

$$X_I(k) = \frac{1}{2}[X_H(-k) - X_H(k)] \qquad \ldots (31)$$

Subtracting Eq. (31) from Eq. (30), we get the DHT coefficients which are expressed in terms of DFT.

$$X_H(k) = X_R(k) - X_I(k) \qquad \ldots (32)$$

B. Relationship between the Synthesis Equations (Inverses) Of DFT and DHT

If $x_H(n) = IDHT[X_H(k)]$, then it is possible to express the $IDHT$ of $X_H(k)$ in terms of the IDFT of $X(k)$.

$$x_H(n) = x_R(n) - x_I(n) \qquad \ldots (33)$$

Here, $x_R(n)$ and $x_I(n)$ are the real and imaginary parts of $x(n)$ respectively. Note that $x_R(n) = IDFT[X_R(k)]$ and $x_I(n) = IDFT[X_I(k)]$ respectively.

DHT finds applications in a variety of domains such as Speech Processing, Image Processing, Biomedical Signal Processing, Data Compression, amongst others [12].

6. Duality Theorem for DHT

Statement:

If $x(n)$ and $X_H(k) = X(k)$ form a Discrete Hartley Transform pair, i.e., if $X(k) = DHT[x(n)]$, then, $DHT[X(n)] = Nx(k)$, i.e., $X(n)$ and $Nx(k)$ form a Discrete Hartley Transform pair.

Proof:

We present two methods for proving the Duality Theorem for DHT. The first method is by using the definitions of the DHT and its inverse. The second method is by using the Duality Theorem of DFT for proving the Duality Theorem of DHT.

A. First Method

By definition, it follows that the Discrete Hartley Transform of a function $x(n)$ and its inverse are given by the equations,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, cas\left(\frac{2\pi kn}{N}\right) \qquad \ldots (34)$$

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)\, cas\left(\frac{2\pi kn}{N}\right) \qquad \ldots (35)$$

Cross multiplying in Eq. (35), we get,

$$Nx(n) = \sum_{n=0}^{N-1} X(k)\, cas\left(\frac{2\pi kn}{N}\right) \qquad \ldots (36)$$

Replacing n by k and vice versa in Eq. (36), we get,

$$Nx(k) = \sum_{k=0}^{N-1} X(n)\, cas\left(\frac{2\pi nk}{N}\right)$$

$$Nx(k) = \sum_{k=0}^{N-1} X(n)\, cas\left(\frac{2\pi kn}{N}\right) = DHT[X(n)] \qquad \ldots (37)$$

Comparing Eq. (37) with Eqs. (34) and (35), we see that the RHS of it is nothing but the Discrete Hartley Transform of $X(n)$, with the time – and frequency – variables being interchanged respectively. Thus, we can formally write the Duality Theorem of Discrete Hartley Transform in a compact form.

$$DHT[X(n)] = Nx(k) \qquad \ldots (38)$$

Eq. (38) is the final step in the derivation of the Duality Theorem for Discrete Hartley Transform.

B. Second Method

The Duality Theorem for DFT is given by,

$$Nx((-k))_N = Nx(N-k) = DFT[X(n)] \qquad \ldots (39)$$

$$Nx(N-k) = \sum_{n=0}^{N-1} X(n) e^{-j2\pi kn/N} \qquad \ldots (40)$$

Since $x((-k))_N = x(N-k)$ is complex, we can write the above expression as,

$$Nx_R(N-k) + jNx_I(N-k)$$
$$= \sum_{n=0}^{N-1} X(n)\cos\left(\frac{2\pi kn}{N}\right) - j\sum_{n=0}^{N-1} X(n)\sin\left(\frac{2\pi kn}{N}\right) \quad \dots(41)$$

where,

$$Nx_R(N-k) = \sum_{n=0}^{N-1} X(n)\cos\left(\frac{2\pi kn}{N}\right) \quad \dots(42)$$

and

$$Nx_I(N-k) = -\sum_{n=0}^{N-1} X(n)\sin\left(\frac{2\pi kn}{N}\right) \quad \dots(43)$$

From Eq. (32), the relationship between DFT and DHT is given by the relation,

$$X_H(k) = X_R(k) - X_I(k) \quad \dots(44)$$

where, it is understood that $X_R(k) = Re\{X(k)\}$ and $X_I(k) = Im\{X(k)\}, 0 \le n, k \le N-1$.

Applying the above criteria to the duality, we get,

$$Nx_H(k) = Nx_R(N-k) - Nx_I(N-k) \quad \dots(45)$$

Substitution and simplification yields,

$$Nx_H(k) = \sum_{n=0}^{N-1} X(n)\left[\cos\left(\frac{2\pi kn}{N}\right) + \sin\left(\frac{2\pi kn}{N}\right)\right]$$
$$Nx_H(k) = \sum_{n=0}^{N-1} X(n)\,\mathrm{cas}\left(\frac{2\pi kn}{N}\right)$$
$$\Rightarrow Nx_H(k) = DHT[X(n)]$$

Writing $x_H(k) = x(k)$ in the above equation, we get,

$$Nx(k) = DHT[X(n)] \quad \dots(46)$$

Eq. (46) is the final step in the derivation of the Duality Theorem for Discrete Hartley Transform. Comparing Eqs. (38) and (46), we see that the results obtained in both the methods are one and the same. This completes the proof of the Duality Theorem for DHT.

We shall verify the above theorem with a simple example in the next section.

7. Illustration of the Duality Theorem for DHT

Consider the discrete – time exponential signal, given by the function,

$$x(n) = a^n u(n) \quad \dots(47)$$

Taking DHT on both sides of Eq. (47), we get,

$$X_H(k) = X(k) = \sum_{n=0}^{N-1} a^n u(n)\, cas\left(\frac{2\pi kn}{N}\right)$$
$$X(k) = \sum_{n=0}^{N-1} a^n \cos\left(\frac{2\pi kn}{N}\right) + \sum_{n=0}^{N-1} a^n \sin\left(\frac{2\pi kn}{N}\right)$$

Simplifying using Euler's theorem, we get,

$$= \sum_{n=0}^{N-1} a^n \left\{\frac{e^{+\frac{2\pi kn}{N}} + e^{-\frac{2\pi kn}{N}}}{2}\right\} + \sum_{n=0}^{N-1} a^n \left\{\frac{e^{+\frac{2\pi kn}{N}} - e^{-\frac{2\pi kn}{N}}}{2j}\right\}$$

This results in four finite geometric summations. Using the well – known formula,

$$\sum_{n=0}^{M} \alpha^n = \begin{cases} \frac{1-\alpha^{M+1}}{1-\alpha}, |\alpha| \ne 1 \\ M+1, |\alpha| = 1 \end{cases},$$

noting that $e^{\pm j2\pi k} = 1$, and simplifying, we get,

$$X(k) = \frac{1-a^N}{2}\left[\frac{1}{1-ae^{\frac{j2\pi k}{N}}} + \frac{1}{1-ae^{-\frac{j2\pi k}{N}}}\right]$$
$$+ \frac{1-a^N}{2j}\left[\frac{1}{1-ae^{\frac{j2\pi k}{N}}} - \frac{1}{1-ae^{-\frac{j2\pi k}{N}}}\right]$$

The above equation can be written as

$$\Rightarrow X(k) = \frac{(1-a^N)\left[1-a\left\{\cos\left(\frac{2\pi k}{N}\right) - \sin\left(\frac{2\pi k}{N}\right)\right\}\right]}{1-2a\cos\left(\frac{2\pi k}{N}\right)+a^2} \quad \dots(48)$$

Now, from Eq. (6), it follows that
$\cos\left(\frac{2\pi k}{N}\right) - \sin\left(\frac{2\pi k}{N}\right) = \mathrm{cas}\left(-\frac{2\pi k}{N}\right) = cas'\left(\frac{2\pi k}{N}\right)$.
Thus, we have,

$$X(k) = \frac{(1-a^N)\left[1-a\,\mathrm{cas}\left(-\frac{2\pi k}{N}\right)\right]}{1-2a\cos\left(\frac{2\pi k}{N}\right)+a^2} = \frac{(1-a^N)\left[1-a\,cas'\left(\frac{2\pi k}{N}\right)\right]}{1-2a\cos\left(\frac{2\pi k}{N}\right)+a^2} \quad \dots(49)$$

Next, consider the computation of the IDHT of

$$X(k) = a^k u(k) \quad \dots(50)$$

Taking IDHT on both sides of Eq. (50), we get,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} a^k u(k)\, cas\left(\frac{2\pi kn}{N}\right)$$
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} a^k \cos\left(\frac{2\pi kn}{N}\right) + \frac{1}{N}\sum_{k=0}^{N-1} a^k \sin\left(\frac{2\pi kn}{N}\right)$$

Simplifying using Euler's theorem, we get,

$$x(n) = \frac{1}{2N}\sum_{k=0}^{N-1}\left(ae^{\frac{j2\pi n}{N}}\right)^k + \frac{1}{2N}\sum_{k=0}^{N-1}\left(ae^{-\frac{j2\pi n}{N}}\right)^k$$
$$+ \frac{1}{2jN}\sum_{k=0}^{N-1}\left(ae^{\frac{j2\pi n}{N}}\right)^k - \frac{1}{2jN}\sum_{k=0}^{N-1}\left(ae^{-\frac{j2\pi n}{N}}\right)^k$$

Using the finite geometric series formula,

$$\sum_{n=0}^{M} \alpha^n = \begin{cases} \frac{1-\alpha^{M+1}}{1-\alpha}, |\alpha| \ne 1 \\ M+1, |\alpha| = 1 \end{cases},$$

and also noting that $e^{\pm j2\pi k} = 1$, we get,

$$x(n) = \frac{1-a^N}{2N}\left[\frac{1}{1-ae^{\frac{j2\pi n}{N}}} + \frac{1}{1-ae^{-\frac{j2\pi n}{N}}}\right]$$
$$+ \frac{1-a^N}{2jN}\left[\frac{1}{1-ae^{\frac{j2\pi n}{N}}} - \frac{1}{1-ae^{-\frac{j2\pi n}{N}}}\right]$$

Simplifying, we get,

$$x(n) = \frac{(1-a^N)\left(1-a\cos\left(\frac{2\pi n}{N}\right)\right)}{N\left[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2\right]} + \frac{(1-a^N)\,a\sin\left(\frac{2\pi n}{N}\right)}{N\left[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2\right]}$$
$$\Rightarrow x(n) = \frac{(1-a^N)\left[1-a\left\{\cos\left(\frac{2\pi n}{N}\right) - \sin\left(\frac{2\pi n}{N}\right)\right\}\right]}{N\left[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2\right]}$$

Now, in view of Eq. (6), it follows that
$\cos\left(\frac{2\pi n}{N}\right) - \sin\left(\frac{2\pi n}{N}\right) = \mathrm{cas}\left(-\frac{2\pi n}{N}\right) = cas'\left(\frac{2\pi n}{N}\right)$.
Thus, we have,

$$x(n) = \frac{(1-a^N)\left[1-a\,\mathrm{cas}\left(-\frac{2\pi n}{N}\right)\right]}{N\left[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2\right]} = \frac{(1-a^N)\left[1-a\,cas'\left(\frac{2\pi n}{N}\right)\right]}{N\left[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2\right]} \quad \dots(51)$$

Next, we apply the Duality Theorem for DHT for the computation of the IDHT of

$$X(k) = a^k u(k) \quad \dots(52)$$

It is proved in Eq. (49) that

$$a^n u(n) \overset{DHT}{\longleftrightarrow} \frac{(1-a^N)\left[1-a\,\mathrm{cas}\left(-\frac{2\pi k}{N}\right)\right]}{1-2a\cos\left(\frac{2\pi k}{N}\right)+a^2} \quad \dots(53)$$

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We can also write

$$a^n u(n) \overset{DHT}{\longleftrightarrow} = \frac{(1-a^N)[1-a\, cas'(\frac{2\pi k}{N})]}{1-2a\cos\left(\frac{2\pi k}{N}\right)+a^2} \qquad \dots (54)$$

Applying Duality Theorem derived in the previous section to the above equation, we get,

$$Na^k u(k) \overset{DHT}{\longleftrightarrow} \frac{(1-a^N)\left[1-a\ \mathrm{cas}\left(-\frac{2\pi n}{N}\right)\right]}{1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2} \qquad \dots (55)$$

$$Na^k u(k) \overset{DHT}{\longleftrightarrow} = \frac{(1-a^N)[1-a\, cas'(\frac{2\pi n}{N})]}{1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2} \qquad \dots (56)$$

Dividing throughout by N in Eq. (55) , we get,

$$a^k u(k) \overset{DHT}{\longleftrightarrow} \frac{(1-a^N)[1-a\ \mathrm{cas}\left(-\frac{2\pi n}{N}\right)]}{N[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2]} \qquad \dots (57)$$

Or, we can also write

$$a^k u(k) \overset{DHT}{\longleftrightarrow} \frac{(1-a^N)[1-a\, cas'(\frac{2\pi n}{N})]}{N[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2]} \qquad \dots (58)$$

That is, $x(n) = IDHT[X(k)] = IDHT[a^k u(k)]$

$$\Rightarrow x(n) = \frac{(1-a^N)\left[1-a\ \mathrm{cas}\left(-\frac{2\pi n}{N}\right)\right]}{N[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2]} \qquad \dots (59)$$

Or, we can also write

$$\Rightarrow x(n) = \frac{(1-a^N)\left[1-a\, cas'\left(\frac{2\pi n}{N}\right)\right]}{N[1-2a\cos\left(\frac{2\pi n}{N}\right)+a^2]} \qquad \dots (60)$$

Eq. (60) is exactly same as Eq. (51) Thus, this example proves the veracity of the Duality Theorem corresponding to the Discrete Hartley Transform (DHT). If a DHT pair is known, then, it is possible to compute the IDHT of the unknown function which has a similar form to the function whose DHT is known by mere interchanging of the time – and frequency – domain variables. Hence, in such instances, the use of the Duality Theorem for DHT totally saves computation time by eradicating the need of finding the summation, thereby proving to be a versatile and indispensable tool in Signal Analysis, Image Processing, and Electromagnetic Field Theory. The Duality Theorem for DHT It thus saves computation time and computation cost to a maximum extent.

8. Conclusions

This paper gives a new property of Duality for the DHT, which has been not been mentioned and derived until now in the literature. Given a function whose DHT is known, by using this simple but beautiful property, we can find the Inverse DHT (IDHT) of a function whose shape resembles the time – domain function (whose DHT exists). Thus, by doing so, the actual labour involved in computing the inverse is eradicated thereby saving computation cost and time, which proves the usage of this duality property. This property can be used in real – time applications in Mathematics, Communications, and Signal Processing where it is common to encounter functions of the same shape in both time and frequency domains and the duality property fits in the bill amply in such instances which signifies its veracity.

9. Acknowledgement

We are grateful to Dr. Mohan Manghnani, Chairman, New Horizon Education Institution, for providing the necessary infrastructure and encouraging us to carry out this research work. We are also thankful to Dr. Manjunatha, Principal, New Horizon College of Engineering, for his constant encouragement. We are highly indebted to our colleagues for providing the necessary support to do this work. We are also grateful to our family members for enforcing enthusiasm and passion for research in us.

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