International Journal of Applied Engineering Research ISSN 0973-4562 Volume 10, Number 17 (2015) pp 38078-38080

$\mathfrak{B}$ -Core Theorems in Ultrametric Fields

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Abstract
In this paper, K denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in K. Here, we have defined $\mathfrak{B}-core$ or Banach core and proved a few theorems on the $\mathfrak{B}-core$ in such fields.

Keywords: Core of a sequence, $\mathfrak{B}-core$, Regular matrix, f-regular matrix, Normal matrix.

AMS subject classifications: 40A05, 40C05, 46S10.

Introduction

Let K be a complete, non-trivially valued, non-archimedean field. Let

$$Ax = (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \ , \qquad n = 0,1,2,...,$$

it being assumed that the series on the right converge and $(Ax)_n$ is called the A-transform of $x = \{x_k\}$. The infinite matrix $A = (a_{nk})$, $n, k = 0,1,2, ...$ is said to be regular if $(Ax)_n$ converges whenever $x = \{x_k\}$ converges and have the same limit.

Let $x = \{x_k\}, x_k \in K, k = 0,1,2 ...$, we denote by $C_n(x)$, $n = 0,1,2 ...$ the smallest closed convex set containing $x_n, x_{n+1}, ...$ and call

$$\mathcal{K}(x) = \bigcap_{n=0}^{\infty} C_n(x)$$

the core of x.

For the infinite matrix $B = (b_{nk})$, $n, k = 0,1,2, ...$ similarly we define,

$$Bx = (Bx)_n = \sum_{k=0}^{\infty} b_{nk}x_k \ , \qquad n = 0,1,2, ...$$

Matrix $B = (b_{nk})$, $n, k = 0,1,2, ...$ is called normal if it is a lower semi-triangular matrix with non-zero diagonal entries. Whenever B is normal, B has a reciprocal. Denote its reciprocal by

$B^{-1} = ({b_{nk}}^{-1})$.

Definition 1. 1.

Let K be a complete, non-trivially valued, non-archimedean field. For every bounded sequence $x = \{x_k\}$, we define the $\mathfrak{B}-core$ or Banach core as

$$\mathfrak{B}(x) = \bigcap_{u \in K} B_x(u),$$

where

$$B_x(u) = \left\{ w \in K : |w - u| \leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} |x_k - u| \right\}.$$

Theorem 1. 1.

An infinite matrix $A = (a_{nk})$ $n, k = 0,1,2, ...$ is such that $\mathcal{K}(Ax) \subset \mathfrak{B}(x)$ if and only if A is regular and satisfies $\lim_{n\to\infty} sup_{k\geq 0}|a_{nk}| = 1$.

Proof : Necessary part:

Assume A is regular and $\lim_{n\to\infty} sup_{k\geq 0}|a_{nk}| = 1$. (1)

To prove $\mathcal{K}(Ax) \subset \mathfrak{B}(x)$

Let $y \in \mathcal{K}(Ax)$. By definition, we have that

$$|y - u| \leq \lim_{n\to\infty} \sup_n \left| \sum_{k=0}^{\infty} a_{nk}x_k - u \right|$$

$$\leq \lim_{n\to\infty} \sup_n \left| \sum_{k=0}^{\infty} a_{nk}(x_k - u) \right| \quad (since\ A\ is\ regular)$$

$$\leq \lim_{n\to\infty} \sup_n \sup_{k\geq 0} |a_{nk}|\, |x_k - u|$$

$$\leq \sup_n |x_k - u| \qquad using\ (1)$$

$$\leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} |x_k - u|$$

$$|y - u| \leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} |x_k - u|$$

$\Rightarrow y \in \mathfrak{B}(x)$.

Therefore, $\mathcal{K}(Ax) \subset \mathfrak{B}(x)$.

Sufficient Part:

Assume $\mathcal{K}(Ax) \subset \mathfrak{B}(x)$.

To prove that (1) holds.

But by definition, $\mathfrak{B}(x) \subset \mathcal{K}(x)$.

Therefore, $\mathcal{K}(Ax) \subset \mathcal{K}(x)$ (2)

In view of (2), we have A is regular and that $\lim_{n\to\infty} sup_{k\geq 0}|a_{nk}| = 1$.

This completes the proof of the theorem.

Definition 1. 2.
The infinite matrix $A = (a_{nk})\,, n, k = 0,1,2, \dots$ is said to be *f*-regular if A is conservative and $\lim Ax = f(\lim x)$.

$$(ie)\ \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}x_k = f(l) \quad \text{where} \quad \lim_{k\to\infty} x_k = l.$$

Theorem 1. 2.
An infinite matrix $A = (a_{nk})\ n, k = 0,1,2, \dots$ is such that $\mathfrak{B}(Ax) \subset \mathfrak{B}(x)$ if and only if A is *f*-regular and satisfies $\lim_{n\to\infty} \sup_{k\geq 0} |a_{nk}| = 1.$

Proof: Necessary part:
Let $x = \{x_k\}$ be a bounded sequence. Let u be a limit point of *x*. ie., $\lim_{k\to\infty} x_k = u.$

If y is any point in $\mathfrak{B}(Ax)$, then by the definition,

$$|y - f(u)| \leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} \left|\sum_{k=0}^{\infty} a_{nk}x_k - f(u)\right|$$

Let us assume A is f-regular and satisfies

$$\lim_{n\to\infty} \sup_{k\geq 0} |a_{nk}| = 1. \qquad (3)$$

To prove $\mathfrak{B}(Ax) \subset \mathfrak{B}(x)$.

Let $y \in \mathfrak{B}(Ax)$, Also A is *f*-regular implies lim Ax = *f* (lim x) = *f*(u)

$$\text{ie.}, \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}x_k = f(u)$$

$$|y - f(u)| \leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} \left|\sum_{k=0}^{\infty} a_{nk}x_k - f(u)\right|$$

$$\leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} \left|\sum_{k=0}^{\infty} a_{nk}x_k - \sum_{k=0}^{\infty} a_{nk}u\right|$$

$$\left(\text{since} \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}u = f(u)\right)$$

$$\leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} \left|\sum_{k=0}^{\infty} a_{nk}(x_k - u)\right|$$

$$\leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} \sup_{k\geq 0} |a_{nk}||x_k - u|$$

$$\leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} |x_k - u| \quad using\ (3)$$

$$|y - f(u)| \leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} |x_k - u| \qquad (4)$$

Now consider that,

$$|y - u| = |y - f(u) + f(u) - u|$$
$$\leq \max\{|y - f(u)|, |f(z) - u|\}$$
$$= |y - f(u)| \quad (since\ f(u) \to u\ as\ n \to \infty)$$

$$|y - u| \leq \lim_{p\to\infty} \sup_{n\leq k\leq n+p} |x_k - u| \quad from\ (4)$$

$$\Rightarrow y \in \mathfrak{B}(x)$$
$$\Rightarrow \mathfrak{B}(Ax) \subset \mathfrak{B}(x).$$

Sufficient Part:
Assume $\mathfrak{B}(Ax) \subset \mathfrak{B}(x)$.
To prove that (3) holds.
Let $x = \{x_k\}$ be a bounded sequence that converges to a limit u.

$$\lim_{k\to\infty} x_k = u \ \ or \ \ \lim_{k\to\infty}(x_k - u) = 0.$$

Since $\mathfrak{B}(Ax) \subset \mathfrak{B}(x)$, A-transform of $\{x_k\}$ also converges to a limit.

$$\Rightarrow \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}(x_k - u) = \sum_{k=0}^{\infty} \lim_{k\to\infty} a_{nk}(x_k - u) = 0$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}x_k = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}u$$

$$\Rightarrow \lim \text{Ax} = f(\lim \text{x}) = f(\text{u})$$

$$\Rightarrow A\ is\ f - regular.$$

$$To\ prove \lim_{n\to\infty} \sup_{k\geq 0} |a_{nk}| = 1.$$

Since A is *f*-regular, $\lim \text{Ax} = f(\lim \text{x})$.

$$ie., \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}x_k = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}u$$

$$\lim_{n\to\infty} \left|\sum_{k=0}^{\infty} a_{nk}(x_k - u)\right| = 0$$

But,

$$\left|\sum_{k=0}^{\infty} a_{nk}(x_k - u)\right| \leq max\{|a_{n0}(x_0 - u)|, |a_{n1}(x_1 - u)|, \dots |a_{nk}(x_k - u)|, \dots\}$$

$$\lim_{n\to\infty} \sup_{k\geq 0} |a_{nk}| = 1 \ \ (as\ x_k \to l\ as\ k \to \infty).$$

This completes the proof of the theorem.
Before giving the main results we state the following Lemma [9].

Lemma 1. 1.
Let $A = (a_{n,j})$ and $B = (b_{jk})$ be infinite matrices, where $a_{n,j}, b_{jk} \in K, \quad n, j, k = 0,1,2, \dots$.
For any bounded sequence $x = \{x_k\}$ there exists Ax whenever Bx is bounded if and only if the following conditions are satisfied for a fixed n.

$$(1)\ c_{nk} = \sum_{j=k}^{\infty} a_{nj}\, b_{jk}^{\ -1}, \qquad k = 0,1,2, \dots,$$

$$(2)\ \sup_{n,k} |c_{nk}| < \infty,$$

$$(3)\ \lim_{J\to\infty}\ \sup_{0\le k\le J}\left|\sum_{j=J+1}^{\infty} a_{nj}\, {b_{jk}}^{-1}\right| = 0.$$

Theorem 1. 3.
Let B be a normal matrix and A be any matrix. For any bounded sequence $x = \{x_k\}$ there exists A*x* whenever B*x* is bounded and that $\mathcal{K}(Ax) \subset \mathfrak{B}(Bx)$ it is necessary and sufficient that the following conditions are satisfied.

$(i)\ C = AB^{-1}\ exists,$

$(ii)\ C\ is\ regular,$

$$(iii)\ for\ a\ fixed\ n, \lim_{J\to\infty}\ \sup_{0\le k\le J}\left|\sum_{j=J+1}^{\infty} a_{nj}\, {b_{jk}}^{-1}\right| = 0,$$

$$(iv)\ \lim_{n\to\infty}\ \sup_{k\ge 0}|c_{nk}| = 1.$$

Proof: Necessary Part:
Assume $\mathcal{K}(Ax) \subset \mathfrak{B}(Bx)$.
For any bounded sequence $x = \{x_k\}$, B*x* is bounded and we write *y* = B*x*.
By lemma 1. 1, conditions (i) and (iii) hold.
Since $\mathcal{K}(Ax) \subset \mathfrak{B}(Bx)$, we have

$$\mathcal{K}(AB^{-1}y) \subset \mathfrak{B}(Bx)$$
$$\Rightarrow \mathcal{K}(Cy) \subset \mathfrak{B}(y), \qquad using\ (i)$$

Hence the conditions (ii) and (iv) hold, in view of Theorem 1. 1.
Sufficient Part:
Assume that the conditions (i) to (iv) hold.
Since C is regular and $\lim_{n\to\infty}\ \sup_{k\ge 0}|c_{nk}| = 1$,

We have,

$$\mathcal{K}(Cy) \subset \mathfrak{B}(y),\ \ by\ theorem\ 1$$
$$\Rightarrow \mathcal{K}(AB^{-1}y) \subset \mathfrak{B}(Bx), \quad from\ (i)$$
$$\Rightarrow \mathcal{K}(Ax) \subset \mathfrak{B}(Bx).$$

This completes the proof of the theorem.

Theorem 1. 4.
Let B be a normal matrix and A be any matrix. For any bounded sequence $x = \{x_k\}$ there exists A*x* whenever B*x* is bounded and that $\mathfrak{B}(Ax) \subset \mathfrak{B}(Bx)$ it is necessary and sufficient that the following conditions are satisfied.

$(i)\ C = AB^{-1}\ exists,$

$(ii)\ C\ is\ f - regular,$

$$(iii)\ \ for\ a\ fixed\ n, \lim_{J\to\infty}\ \sup_{0\le k\le J}\left|\sum_{j=J+1}^{\infty} a_{nj}\, {b_{jk}}^{-1}\right| = 0,$$

$$(iv)\ \lim_{n\to\infty}\ \sup_{k\ge 0}|c_{nk}| = 1.$$

Proof: **Necessary Part:**
Assume $\mathfrak{B}(Ax) \subset \mathfrak{B}(Bx)$.
For any bounded sequence $x = \{x_k\}$, B*x* is bounded and we write *y* = B*x*.
By lemma 1. 1, conditions (i) and (iii) hold.
Since $\mathfrak{B}(Ax) \subset \mathfrak{B}(Bx)$, we have

$$\mathfrak{B}(AB^{-1}y) \subset \mathfrak{B}(Bx)$$
$$\Rightarrow \mathfrak{B}(Cy) \subset \mathfrak{B}(y), \quad using\ (i)$$

Hence the conditions (ii) and (iv) hold, in view of Theorem 1. 2.

Sufficient Part:
Assume that the conditions (i) to (iv) hold.
Since C is *f*-regular and $\lim_{n\to\infty}\ \sup_{k\ge 0}|c_{nk}| = 1$,
We have,

$$\mathfrak{B}(Cy) \subset \mathfrak{B}(y),\ \ by\ Theorem\ 1.2$$
$$\Rightarrow \mathfrak{B}(AB^{-1}y) \subset \mathfrak{B}(Bx), \quad from\ (i)$$
$$\Rightarrow \mathfrak{B}(Ax) \subset \mathfrak{B}(Bx).$$

This completes the proof of the theorem.

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