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The Role of p-sets in Topology

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Abstract

The interior and closure operators in topological spaces play a vital role in the generalization of closed sets and open sets in topological spaces. These operators have application to Rough set theory, Data mining and Digital image processing. Thangavelu and Chandrasekhara Rao introduced the concept of p-sets in topology and studied the basic properties of p-sets. The purpose of this paper is to characterize the further properties of p-sets in topology.

1. Introduction

The interior and closure operators in topological spaces play a dominant role in the generalization of closed sets and open sets in topological spaces. These operators have application to Rough set theory, Data mining and Digital image processing. Thangavelu and Chandrasekhara Rao [6] introduced the concept of p-sets in topology and studied the basic properties of p-sets. The purpose of this paper is to characterize the further properties of p-sets in topology. The basic properties of the interior and the closure operators are discussed in section 2 and the section 3 is dealt with the topologies generated by the p-sets. In section 4, the p-interior and p-closure operators are introduced. Finally the p-sets in the real line are investigated in section 5.

2. Preliminaries

Throughout this section X is a topological space and A, B are the subsets of X. The notations *cl*A and *int*A denote the closure of A and interior of A in X respectively.
The following definitions and lemmas will be useful in sequel.

Definition 2. 1

i) Regular open [5] if A = *int cl* A and regular closed if A = *cl int* A.

ii) α-open[4] if $A \subseteq \mathit{int}\ \mathit{cl}\ \mathit{int}\ A$ and α-closed if $\mathit{cl}\ \mathit{int}\ \mathit{cl}\ A \subseteq A$.
iii) β-open [1] if $A \subseteq \mathit{cl}\ \mathit{int}\ \mathit{cl}\ A$ and β-closed if $\mathit{int}\ \mathit{cl}\ \mathit{int}\ A \subseteq A$.
iv) Pre-open [3] if $A \subseteq \mathit{int}\ \mathit{cl}\ A$ and pre-closed if $\mathit{cl}\ \mathit{int}\ A \subseteq A$.
v) A p-set [6] if $\mathit{cl}\ \mathit{int}\ A \subseteq \mathit{int}\ \mathit{cl}\ A$.

A subset of a topological space X is clopen if it is both open and closed.. Similarily regular clopen, α-clopen, β-clopen, pre-clopen sets are defined.

Lemma 2. 2 [2]
If A is a subset of X then
(i) $\alpha\ \mathit{int}\ \alpha\ \mathit{cl}\ A = \mathit{int}\ \mathit{cl}\ A$
(ii) $\alpha\ \mathit{cl}\ \alpha\ \mathit{int}\ A = \mathit{cl}\ \mathit{int}\ A$
where α interior of A is denoted by $\alpha\ \mathit{int}\ A$ and α closure of A is denoted by $\alpha\ \mathit{cl}\ A$.

Lemma 2. 3 [6]
If A is clopen and B is a p-set then $A\cap B$ is a p-set.

Lemma 2. 4 [6]
A is a p-set if and only if $X- A$ is a p-set.

3. Topology generated by p-sets
Lemma 3. 1
If A is a subset of a topological X, then the following are equivalent
(i) A is clopen
(ii) A is regular clopen
(iii) A is α-clopen.

Thus we have the following diagram

Diagram 3. 2
Regular clopen $\Leftrightarrow$ clopen $\Leftrightarrow$ α-clopen
$\Downarrow$ $\Downarrow$
p-set $\Leftarrow$ pre – clopen

Proposition 3. 3
Let (X, τ) be a topological space. Suppose A and B are p-sets. If A is clopen then $N = \{\phi, A\cap B, A, A-(A\cap B), X\}$ is a topology and every member of N is a p-set.

Proof
$N = \{\phi, A\cap B, A, A-\{A\cap B\}, X\}$ is clearly a topology on X. Clearly ϕ, A, X are all p-sets. Since A is clopen and B is a p-set, by using Lemma 2. 3, $A\cap B$ is a p-set. Since complement of a p-set is again a p-set, $X-(A\cap B)$ is a p-set that implies $A \cap (X-(A\cap B))$ is also a p-set. This proves that $A-(A\cap B) = A \cap (X-(A\cap B))$ is a p-set.

Hence every member of N is a p-set.
In the rest of the section, $p(\tau)$ denotes the collection of p-sets in (X, τ).

Proposition 3. 4
Let (X, τ) be a space with point inclusion topology. Then $p(\tau)$ is a discrete topology on X.

Proof
Fix $a \in X$. Let $\tau = \{\phi\} \cup \{A \subseteq X : a \in A\}$. Let $B \subseteq X$.

$$\mathit{int}\, B = \begin{cases} \phi & \text{if } B = \phi \\ B & \text{if } a \in B \\ \phi & \text{if } a \notin B \end{cases} \quad \text{and } \mathit{cl}\, B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } a \in B \\ B & \text{if } a \notin B \end{cases}$$

$$\mathit{cl}\ \mathit{int}\, B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } a \in B \\ \phi & \text{if } a \notin B \end{cases} \quad \text{and } \mathit{int}\ \mathit{cl}\, B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } a \in B \\ \phi & \text{if } a \notin B \end{cases}$$

Then it follows $p(\tau) = \wp(X)$, the power set of X, that is the discrete topology on X.

Proposition 3. 5
Let (X, τ) be a space with set inclusion topology. Then $p(\tau)$ is a discrete topology on X.

Proof
Let $A \subseteq X$, $A \neq \phi$, $\tau_A = \{\phi\} \cup \{ B \subseteq X ; B \supseteq A\}$. Then $B \subseteq X$ is closed if and only if $X - B$ is open iff $X - B = \phi$ or $X - B \supseteq A$ iff $B = X$ or $B \subseteq X - A$ iff $B = X$ or $B \cap A = \phi$.

$$\text{int } B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ A & \text{if } B = A \\ B & \text{if } B \supset A \\ \phi & \text{if } B \cap A = \phi, B \neq \phi \\ \phi & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ \phi & \text{if } B = X \end{cases}$$

$$cl\,\mathrm{B} = \begin{cases} \phi & \text{if } \mathrm{B} = \phi \\ \mathrm{X} & \text{if } \mathrm{B} \subset \mathrm{A}, \mathrm{B} \neq \mathrm{A} \\ \mathrm{X} & \text{if } \mathrm{B} = \mathrm{A} \\ \mathrm{X} & \text{if } \mathrm{B} \supset \mathrm{A} \\ B & \text{if } \mathrm{B} \cap \mathrm{A} = \phi, \mathrm{B} \neq \phi \\ X & \text{if } \mathrm{B} \cap \mathrm{A} \neq \phi, \mathrm{B} \cap (\mathrm{X} - \mathrm{A}) \neq \phi \\ X & \text{if } \mathrm{B} = X \end{cases}$$

$$cl\,\mathrm{int}\,B = \begin{cases} \phi & \text{if } \mathrm{B} = \phi \\ \phi & \text{if } \mathrm{B} \subset \mathrm{A}, \mathrm{B} \neq \mathrm{A} \\ \mathrm{X} & \text{if } \mathrm{B} = \mathrm{A} \\ \mathrm{X} & \text{if } \mathrm{B} \supset \mathrm{A} \\ \phi & \text{if } \mathrm{B} \cap \mathrm{A} = \phi, \mathrm{B} \neq \phi \\ \phi & \text{if } \mathrm{B} \cap \mathrm{A} \neq \phi, \mathrm{B} \cap (\mathrm{X} - \mathrm{A}) \neq \phi \\ X & \text{if } \mathrm{B} = X \end{cases}$$

$$\mathrm{int}\,\mathrm{cl}\,B = \begin{cases} \phi & \text{if } \mathrm{B} = \phi \\ \phi & \text{if } \mathrm{B} \subset \mathrm{A}, \mathrm{B} \neq \mathrm{A} \\ \mathrm{X} & \text{if } \mathrm{B} = \mathrm{A} \\ \mathrm{X} & \text{if } \mathrm{B} \supset \mathrm{A} \\ \phi & \text{if } \mathrm{B} \cap \mathrm{A} = \phi, \mathrm{B} \neq \phi \\ \phi & \text{if } \mathrm{B} \cap \mathrm{A} \neq \phi, \mathrm{B} \cap (\mathrm{X} - \mathrm{A}) \neq \phi \\ X & \text{if } \mathrm{B} = X \end{cases}$$

Thus $p(\tau) = \wp(X)$

Proposition 3. 6

Let (X, τ) be a space with point exclusion topology. Then $p(\tau)$ is a topology on X.

Proof

Fix $a \in X$. Let $\tau = \{X\} \cup \{A \subseteq X, a \notin A\}$. Let $B \subseteq X$.

$$\mathrm{int}\,B = \begin{cases} \phi & \text{if } \mathrm{B} = \phi \\ B - a & \text{if } \mathrm{a} \in \mathrm{B} \\ \mathrm{B} & \text{if } \mathrm{a} \notin \mathrm{B} \\ \phi & \text{if } \mathrm{B} = \{a\} \\ X - a & \text{if } \mathrm{B} = X \end{cases}$$

$$cl\,B=\begin{cases}\phi & \text{if } B=\phi\\ B & \text{if } a\in B\\ B\cup\{a\} & \text{if } a\notin B\\ \{a\} & \text{if } B=\{a\}\\ X & \text{if } B=X\end{cases}$$

$$cl\,\mathrm{int}\,B=\begin{cases}\phi & \text{if } B=\phi\\ B & \text{if } a\in B\\ B\cup\{a\} & \text{if } a\notin B\\ \phi & \text{if } B=\{a\}\\ X & \text{if } B=X\\ X & \text{if } B=X-a\end{cases}$$

$$\mathrm{int}\,cl\,B=\begin{cases}\phi & \text{if } B=\phi\\ B-\{a\} & \text{if } a\in B\\ B & \text{if } a\notin B\\ \phi & \text{if } B=\{a\}\\ X & \text{if } B=X\\ X & \text{if } B=X-a\end{cases}$$

$p(\tau) = \{\phi, X, \{a\}, X-a\}$ is a topology.

Proposition 3. 7

Let (X, τ) be a space with set exclusion topology. Then $p(\tau)$ is a topology on X.

Proof

Let $A \subseteq X$, $A \neq \phi$.

Let $\tau = \{X\} \cup \{B \subseteq X: B\cap A=\phi\} = \{X\} \cup \{B \subseteq X: B \subseteq X-A\}$. Let $B \subseteq X$ is closed iff $X-B$ is open iff $X-B = X$ or $X-B \subseteq X-A$ iff $B = \phi$ or $B \supseteq A$.

$$\mathrm{int}\,B=\begin{cases}\phi & \text{if } B=\phi\\ \phi & \text{if } B\subset A, B\neq A\\ \phi & \text{if } B=A\\ B-A & \text{if } B\supset A\\ B & \text{if } B\cap A=\phi, B\neq\phi\\ B-A & \text{if } B\cap A\neq\phi, B\cap(X-A)\neq\phi\\ X & \text{if } B=X\end{cases}$$

$$clB = \begin{cases} \phi & \text{if } B = \phi \\ A & \text{if } B \subset A, B \neq A \\ A & \text{if } B = A \\ B \cup A & \text{if } B \supset A \\ B \cup A & \text{if } B \cap A = \phi, B \neq \phi \\ B \cup A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$cl\operatorname{int} B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ \phi & \text{if } B = A \\ B \cup A & \text{if } B \supset A \\ B \cup A & \text{if } B \cap A = \phi, B \neq \phi \\ B \cup A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$\operatorname{int} clB = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ \phi & \text{if } B = A \\ B - A & \text{if } B \supset A \\ B - A & \text{if } B \cap A = \phi, B \neq \phi \\ B - A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$p(\tau) = \{\phi, X\} \cup \{B: B \subseteq A\}$ is a topology.

Proposition 3. 8
Let X be an infinite set. $\tau = \{\phi\} \cup \{A \subseteq X: X - A \text{ is finite}\}$. $p(\tau)$ is a topology on X.

Proof
Let $B \subseteq X$.

$$\operatorname{int} B = \begin{cases} B & \text{if } B = \phi \text{ or } X \\ B & \text{if } B \text{ is infinite, } X - B \text{ is finite} \\ \phi & \text{if } B \text{ is finite} \\ \phi & \text{if } B \text{ is infinite, } X - B \text{ is infinite} \end{cases}$$

$$\text{cl B} = \begin{cases} \text{B} & \text{if B} = \phi \text{ or X} \\ \text{X} & \text{if B is infinite, X} - \text{B is finite} \\ B & \text{if B is finite} \\ \text{X} & \text{if B is infinite, X} - \text{B is infinite} \end{cases}$$

$$\text{cl int B} = \begin{cases} \text{B} & \text{if B} = \phi \text{ or X} \\ \text{X} & \text{if B is infinite, X} - \text{B is finite} \\ \phi & \text{if B is finite} \\ \phi & \text{if B is infinite, X} - \text{B is infinite} \end{cases}$$

$$\text{int cl B} = \begin{cases} \text{B} & \text{if B} = \phi \text{ or X} \\ \text{X} & \text{if B is infinite, X} - \text{B is finite} \\ \phi & \text{if B is finite} \\ \text{X} & \text{if B is infinite, X} - \text{B is infinite} \end{cases}$$

Then $p(\tau)=\wp(X)$ is a topology.

4. p-interior and p-closure

Definition 4. 1

Let A be a subset of a topological space X. The p-interior of A denoted by p-*int*(A) is the union of all p-sets contained in A and the p-closure of A denoted by p-cl(A) is the intersection of all p-sets containing A.

Since the collection of all p-sets is not closed under union and intersection it follows that p-*int*(A) need not be a p-set and p-*cl*(A) need not be a p-set. But p-*int*(A) $\subseteq$ A $\subseteq$ p-*cl*(A) is always true for any subset A of a topological space.

Proposition 4. 2

i) p-*int*(ϕ) = ϕ, p-*cl*(ϕ) = ϕ
ii) p-*int*(X) = X, p-*cl*(X) = X
iii) p-*int* A $\subseteq$ A $\subseteq$ p-*cl* A
iv) A $\subseteq$ B $\Rightarrow$ p-*int* A $\subseteq$ p-*int* B and p-*cl* A $\subseteq$ p-*cl* B.
v) p-*int* (A $\cap$ B) $\subset$ p-*int* A $\cap$ p-*int* B.
vi) p-*cl* (A $\cap$ B) $\subseteq$ p-*cl* A $\cap$ p-*cl* B
vii) p-*cl* (A $\cup$ B) $\supseteq$ p-*cl* A $\cup$ p-*cl* B
viii) p-*int* (A $\cup$ B) $\supseteq$ p-*int*A $\cup$ p-*int*B
ix) p-*int*(p-*int* A) $\subseteq$ p-*int* A
x) p-*cl*(p-*cl* A) $\supseteq$ p-*cl* A
xi) p-*int*(p-*cl* A) $\supseteq$ p-*int* A
xii) p-*cl*(p-*int* A) $\subseteq$ p-*cl* A

Proof

(i), (ii), (iii), (iv) are obvious.

Since A $\cap$ B $\subseteq$ A, A $\cap$ B $\subseteq$ B, by using (iv), we have p-*int* (A $\cap$ B) $\subset$ p-*int* A $\cap$ p-

int B and p-*cl* (A $\cap$ B) $\subseteq$ p-*cl* A $\cap$ p-*cl* B. Also since A $\subseteq$ A$\cup$B and B $\subseteq$A $\cup$B it follows from (iv) that p-*cl* A $\cup$ p-*cl* B $\subseteq$ p-*cl* (A $\cup$ B) and p-*int*A $\cup$ p-*int*B $\subseteq$ p-*int*(A$\cup$B). This proves (v), (vi), (vii) and (viii), Since p-*int* A $\subseteq$ A, A$\subseteq$ p-*cl* A and p-*int* A $\subseteq$ p-*cl* A it follows from (iv) that p-*int*(p-*int* A) $\subseteq$ p-*int* A, p-*cl*(p-*cl* A) $\supseteq$ p-*cl* A. p-*int*(p-*cl* A) $\supseteq$ p-*int* A and p-*cl*(p-*int* A)$\subseteq$ p-*cl* A. This proves (ix), (x), (xi) and (xii). This completes the proof of the proposition.

However, the reverse inclusions in (v), (vi), (vii), (viii), (ix), (x), (xi) and (xii) of Proposition 4. 2 are not generally true. The following results can be easily established.

Proposition 4. 3

If A is a p-set then p-*int* A = A = p-*cl* A.

Proposition 4. 4

If A is a p-set then p-*cl* p-*int* A = A = p-*int* p-*cl* A.

Proposition 4. 5

If A is a p-set and B is clopen(resp. α-clopen, regular clopen) then p-*int* (A $\cap$ B) = p-*int* A $\cap$ p-*int* B and p-*cl* (A $\cap$ B) = p-*cl* A $\cap$ p-*cl* B.

Corollary 4. 6

If A is pre-clopen and B is clopen then p-*int* (A $\cap$ B) = p-*int* A $\cap$ p-*int* B and p-*cl* (A $\cap$ B) = p-*cl* A $\cap$ p-*cl* B.

Proposition 4. 7

If A is a p-set and B is clopen(resp. α-clopen, regular clopen) then p-*int* (B–A) = B–A = p-*cl* (B–A).

Corollary 4. 8

If A is pre-clopen and B is clopen then p-*int* (B–A) = B–A = p-*cl* (B–A).

Proposition 4. 9

If A is a p-set and B is clopen(resp. α-clopen, regular clopen) then p-*int* (A $\cup$ B) = p-*int* A $\cup$ p-*int* B and p-*cl* (A $\cup$ B) = p-*cl* A $\cup$ p-*cl* B.

Corollary 4. 10

If A is pre-clopen and B is clopen then p-*int* (A $\cup$ B) = p-*int* A $\cup$ p-*int* B and p-*cl* (A $\cup$ B) = p-*cl* A $\cup$ p-*cl* B.

Proposition 4. 11

If A is a p-set and B is clopen(resp. α-clopen, regular clopen) then

i) p-*int* (p-*cl*(A$\cap$B)) = p-*int* (p-*cl*(A)) $\cap$ p-*int* (p-*cl*(B)).

ii) p-*cl*(p-*int* (A$\cap$B)) = p-*cl* (p-*int*(A)) $\cap$ p-*cl*(p-*int*(A $\cap$ B)).

iii) p-*int*(p-*cl*(A$\cup$B)) = p-*int*(p-*cl*(A)) $\cup$ p-*int*(p-*cl*(B)).

iv) p-*cl* (p-*int*(A $\cup$B)) = p-*cl*(p-*int*(A))$\cup$ p-*cl*(p-*int*(B))

Corollary 4. 12

If A is pre-clopen and B is clopen then

i) p-int (p-cl($A \cap B$)) = p-int (p-cl(A)) $\cap$ p-int (p-cl(B)).

ii) p-cl(p-int ($A \cap B$)) = p-cl (p-int(A)) $\cap$ p-cl(p-int($A \cap B$)).

iii) p-int(p-cl($A \cup B$)) = p-int(p-cl(A)) $\cup$ p-int(p-cl(B)).

iv) p-cl (p-int($A \cup B$)) = p-cl(p-int(A))$\cup$ p-cl(p-int(B))

Proposition 4. 13

For any subset $Y \subseteq X$, p-int(Y) = X– p-cl(X– Y).

Proposition 4. 14

(i) p-int(p-int A) $\subseteq$ p-int(p-cl(p-intA)) $\subseteq$ p-cl(p-int A) $\subseteq$ p-cl(p-int(p-clA)) $\subseteq$p-cl(p-cl A)

(ii) p-int(p-int A) $\subseteq$ p-int(p-cl(p-intA)) $\subseteq$ p-int(p-cl A) $\subseteq$ p-cl(p-int(p-clA)) $\subseteq$p-cl(p-clA)

Proposition 4. 15

Let (X, τ) be topological space. Then X is a union of two non empty disjoint p-sets if and only if it has a non trival proper p-set.

Proposition 4. 16

Suppose $A \subseteq B$ with cl A = cl B. If B is a p-set then A is a p-set.

Proposition 4. 17

Suppose $A \subseteq B$ with int A = int B. If A is a p-set then B is a p-set.

5. p-sets in the real line

Any generalization of sets in topology must have applications to the real line R^1. In this section the nature of p-sets in the real line is briefly discussed.

Proposition 5. 1

(i) Every singleton set in R^1 is a p-set.

(ii) Every subset of the set of all rational numbers is a p-set in R^1.

(iii) Every subset of the set of all irrational numbers is a p-set in R^1.

(iv) For every real x, R–{x} is a p-set in R^1.

(v) The intervals of positive length are not p-sets in R^1.

Proposition 5. 2

If p-int(A) = A then A need not be a p-set.

Proof

In R^1, take A = [a, b]. Then from Proposition 5. 1 (i), it follows that p-int(A) = A and from Proposition 5. 1 (v), A is not a p-set.

Proposition 5. 3
If p-*cl*(A) = A then A need not be a p-set.

Proof
In R^1, take A = [a, b]. Then from Proposition 5. 1 (iv), it follows that p-*cl*(A) = A and from Proposition 5. 1 (v), A is not a p-set.

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