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Common Fixed Point In Intuitionistic $\mathcal{M}$-Fuzzy Metric Space for Compatible Mappings With Types (I) And (II)

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Abstract

In this paper, we give some definitions of compatible mappings of types (I) and (II) in intuitionistic $\mathcal{M}$-fuzzy metric spaces and prove some common fixed point theorems for compatible mapping with types (I) and (II) in complete intuitionistic $\mathcal{M}$-fuzzy metric spaces.

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1. Introduction

In 1992, Dhage [1] introduced the notion of generalized metric or D-metric spaces and proved several fixed point theorems in it. Since D-metric space not possess some topological properties (see [6, 7, 8]), recently Sedghi and Shobe [13] introduced D^*-metric space as a probable modification of D-metric space and studied some topological properties which are not valid in D-metric spaces. Based on D^*- metric concepts, they [12, 14] define $\mathcal{M}$-fuzzy metric space and proved a common fixed point theorem in it. Soleimani [16] introduced intuitionistic $\mathcal{M}$-fuzzy metric space proved some properties and theorems in it.

Mishra et al. [5] obtained some common fixed point theorems for compatible mappings in fuzzy metric spaces. Recently, Jungek et al. [4] introduced the concept of compatible mappings of type (A) in metric spaces, which is equivalent to the concept of compatible mappings under some conditions, and proved common fixed point theorems in metric spaces. Cho [10] introduced the concept of compatible mappings of type (α) in fuzzy metric spaces.

In 1999, Pathak et al. [9] introduced the concept of compatible mappings of types (I) and (II) in metric spaces. sedghi et al. [10] introduced the concept of compatible mappings of types (I) and (II) in fuzzy metric spaces.

In this paper we some new definitions of compatible mappings of types (I) and (II) in intuitionistic $\mathcal{M}$-fuzzy metric space and common fixed point theorem in intuitionistic $\mathcal{M}$-fuzzy metric space for compatible mapping with types (I) and (II) is proved.

2. Preliminaries

Now, we give basic definitions and their properties as follows:

Definition 2.1. [11] A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(1) $*$ is commutative and associative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Example 2.2. Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.3. [11] A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t-conorm if $\diamond$ is satisfying the following conditions:

(1) $\diamond$ is commutative and associative,

(2) $\diamond$ is continuous,

(3) $a \diamond 1 = a$ for all $a \in [0, 1]$,

(4) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Example 2.4. Two typical examples of continuous t-conorm are $a \diamond b = \max\{a, b\}$ and $a \diamond b = \min\{1, a + b\}$.

Definition 2.5. [16] A 5-tuple $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is called an intuitionistic $\mathcal{M}$-fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathcal{M}, \mathcal{N}$ are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

(1) $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \le 1$,

(2) $\mathcal{M}(x, y, z, t) > 0$,

(3) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(4) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function,

(5) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(6) $\mathcal{M}(x, y, z, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

(7) $\mathcal{N}(x, y, z, t) > 0$,

(8) $\mathcal{N}(x, y, z, t) = 0$ if and only if $x = y = z$,

(9) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function,

(10) $\mathcal{N}(x, y, a, t) \diamond \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s)$,

(11) $\mathcal{N}(x, y, z, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Then $(\mathcal{M}, \mathcal{N})$ is called an intuitionistic $\mathcal{M}$-fuzzy metric on X. The functions $\mathcal{M}(x, y, z, t)$ and $\mathcal{N}(x, y, z, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Lemma 2.6. [16] Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic $\mathcal{M}$-fuzzy metric space. For any $x, y \in X$ and $t > 0$, we have

(1) $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

(2) $\mathcal{N}(x, x, y, t) = \mathcal{N}(x, y, y, t)$.

Definition 2.7. [16] Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic $\mathcal{M}$-fuzzy metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\longrightarrow\infty} x_n = x$) if $\lim_{n\longrightarrow\infty} \mathcal{M}(x, x, x_n, t) = 1$ and $\lim_{n\longrightarrow\infty} \mathcal{N}(x, x, x_n, t) = 0$ for all $t > 0$.

(2) $\{x_n\}$ is called a Cauchy sequence if for all $t > 0$ and $p > 0$,

$$\lim_{n\longrightarrow\infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t) = 1 \;\; and \;\; \lim_{n\longrightarrow\infty} \mathcal{N}(x_{n+p}, x_{n+p}, x_n, t) = 0.$$

(3) An intuitionistic $\mathcal{M}$-fuzzy metric in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.8. [16] Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing and $\mathcal{N}(x, y, z, t)$ is non-increasing with respect to t, for all $x, y, z \in X$.

Lemma 2.9. **[17]** Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic $\mathcal{M}$-fuzzy metric space. If we define $E_{\lambda,\mathcal{M}} : X^3 \to \mathbf{R}^+ \cup \{0\}$ by

$$E_{\lambda,\mathcal{M}}(x, y, z) = \inf\{t > 0 : \mathcal{M}(x, y, z, t) > 1 - \lambda\}$$

and $E_{\lambda,\mathcal{N}} : X^3 \to \mathbf{R}^+ \cup \{0\}$ by

$$E_{\lambda,\mathcal{N}}(x, y, z) = \sup\{t > 0 : \mathcal{N}(x, y, z, t) < \lambda\}$$

for all $\lambda \in (0, 1)$ and $x, y, z \in X$, then

(1) For all $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$, such that

$$\begin{aligned} E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \le {} & E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots \\ & + E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

and

$$\begin{aligned} E_{\mu,\mathcal{N}}(x_1, x_1, x_n) \ge {} & E_{\lambda,\mathcal{N}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{N}}(x_2, x_2, x_3) + \cdots \\ & + E_{\lambda,\mathcal{N}}(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X$.

(2) The sequence $\{x_n\}_{n\in\mathbf{N}}$ is convergent in intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ if and only if $E_{\lambda,\mathcal{M}}(x_n, x_n, x) \to 0$ and $E_{\lambda,\mathcal{N}}(x_n, x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence if and only if it is a Cauchy sequence with $E_{\lambda,\mathcal{M}}$ and $E_{\lambda,\mathcal{N}}$.

Theorem 2.10. **[17]** Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic $\mathcal{M}$-fuzzy metric space. If

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \ge \mathcal{M}(x_0, x_0, x_1, k^n t) \text{ and } \mathcal{N}(x_n, x_n, x_{n+1}, t) \le \mathcal{N}(x_0, x_0, x_1, k^n t)$$

for some $k > 1$ and for every $n \in \mathbf{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

3. Main Results

In this section, we present the main result in this paper. And prove some common fixed point theorems for compatible mapping with types (I) and (II) in complete intuitionistic $\mathcal{M}$-fuzzy metric spaces.

Definition 3.1. Let A and S be mappings from a intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ into itself. Then pair (A, S) is said to be compatible of type (I) if, for all $t > 0$

$$\lim_{n\longrightarrow\infty} \mathcal{M}(ASx_n, x, x, t) \le \mathcal{M}(Sx, x, x, t),$$

and

$$\lim_{n\longrightarrow\infty} \mathcal{N}(ASx_n, x, x, t) \geq \mathcal{N}(Sx, x, x, t),$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} Ax_n = \lim_{n\longrightarrow\infty} Sx_n = x \in X$.

Definition 3.2. Let A and S be mapping from a intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ into itself. Then pair (A, S) is said to be compatible of type (II) if and if (S, A) is compatible of type (I).

Proposition 3.3. Let A and S be mappings from a intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ into itself. Suppose that the pair (A, S) is compatible of type (I) (respectively, (II)) and $Az = Sz$ for some $z \in X$. Then for all $t > 0$,
$\mathcal{M}(Az, Az, SSz, t) \geq \mathcal{M}(Az, Az, ASz, t)$ and $\mathcal{N}(Az, Az, SSz, t) \leq \mathcal{N}(Az, Az, ASz, t)$
(respectively,
$\mathcal{M}(Sz, Sz, AAz, t) \geq \mathcal{M}(Sz, Sz, SAz, t) and \mathcal{N}(Sz, Sz, AAz, t) \leq \mathcal{N}(Sz, Sz, SAz, t)$).

Proof. See Proposition 7 of [18]. ■

Let ϕ be the set of all continuous and increasing functions
$\phi : [0, 1]^{12} \longrightarrow [0, 1]$ in any coordinate and $\phi(s, s, \ldots, s) > s$ for every $s \in [0, 1)$.
And
Let ψ be the set of all continuous and decreasing functions $\psi : [0, 1]^{12} \longrightarrow [0, 1]$ in any coordinate and $\psi(s, s, \ldots, s) < s$ for every $s \in [0, 1)$.

Example 3.4. Let $\phi : [0, 1]^{12} \longrightarrow [0, 1]$ is define by

$$\phi(x_1, x_2, \cdots, x_{12}) = (\min\{x_i\})^h,$$

for some $h \in [0, 1)$. And Let $\psi : [0, 1]^{12} \longrightarrow [0, 1]$ is define by

$$\psi(x_1, x_2, \cdots, x_{12}) = (\max\{x_i\})^h,$$

for some $h \in [0, 1)$.

Theorem 3.5. Let A, B, S and T be self-mappings of a intuitionistic $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ satisfying:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a complete intuitionistic $\mathcal{M}$-fuzzy metric space of X,

(ii) $\mathcal{M}(Ax, By, Bz, t)$ (ii.1)

$$\geq \phi \begin{pmatrix} \mathcal{M}(Sx, Ty, Tz, kt), & \mathcal{M}(Sx, By, Tz, kt), & \mathcal{M}(Sx, Ty, Bz, kt), \\ \mathcal{M}(Sx, By, By, kt), & \mathcal{M}(Ty, By, Bz, kt), & \mathcal{M}(Ty, Ty, Bz, kt), \\ \mathcal{M}(Ty, By, By, kt), & \mathcal{M}(Ty, Bz, Bz, kt), & \mathcal{M}(By, Ty, Tz, kt), \\ \mathcal{M}(By, By, Tz, kt), & \mathcal{M}(By, Tz, Tz, kt), & \mathcal{M}(Tz, Bz, Bz, kt), \end{pmatrix}$$

and$\mathcal{N}(Ax, By, Bz, t)$ (ii.2)

$$\leq \psi \begin{pmatrix} \mathcal{N}(Sx, Ty, Tz, kt), & \mathcal{N}(Sx, By, Tz, kt), & \mathcal{N}(Sx, Ty, Bz, kt), \\ \mathcal{N}(Sx, By, By, kt), & \mathcal{N}(Ty, By, Bz, kt), & \mathcal{N}(Ty, Ty, Bz, kt), \\ \mathcal{N}(Ty, By, By, kt), & \mathcal{N}(Ty, Bz, Bz, kt), & \mathcal{N}(By, Ty, Tz, kt), \\ \mathcal{N}(By, By, Tz, kt), & \mathcal{N}(By, Tz, Tz, kt), & \mathcal{N}(Tz, Bz, Bz, kt), \end{pmatrix}$$

if the mappings A, B, S and T satisfy any one of the following conditions:

(iii) The pairs (A, S) and (B, T) are compatible of type (II) and A or B is continuous,

(iv) The pairs (A, S) and (B, T) are compatible of type (I) and S, T is continuous.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},$$

for $a = 0, 1, 2, \cdots$.

If we set $d_m(t) = \mathcal{M}(x_m, x_m, x_{m+1}, t)$ for all $t > 0$, then we prove that $\{d_m(t)\}$ is increasing with respect to m.

Setting $m = 2n$, then by putting $x = x_{2n}$, $z = y = x_{2n+1}$ in (ii), we have

$$\begin{aligned} d_{2n}(t) &= \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t) = \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \\ &\geq \phi \begin{pmatrix} \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt), \mathcal{M}(Sx_{2n}, Bx_{2n+1}, Tx_{2n+1}, kt), \\ \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Sx_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt), \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Tx_{2n+1}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt), \\ \mathcal{M}(Bx_{2n+1}, Bx_{2n+1}, Tx_{2n+1}, kt), \\ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt), \\ \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt) \end{pmatrix} \end{aligned}$$

$$= \phi \begin{pmatrix} \mathcal{M}(y_{2n-1}, y_{2n}, y_{2n}, kt), \mathcal{M}(y_{2n-1}, y_{2n+1}, y_{2n}, kt), \mathcal{M}(y_{2n-1}, y_{2n}, y_{2n+1}, kt), \\ \mathcal{M}(y_{2n-1}, y_{2n+1}, y_{2n+1}, kt), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, kt), \mathcal{M}(y_{2n}, y_{2n}, y_{2n+1}, kt), \\ \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, kt), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, kt), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, kt), \\ \mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n}, kt), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, kt), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, kt) \end{pmatrix}$$

$$\geq \phi \begin{pmatrix} d_{2n-1}(kt), & \mathcal{M}(y_{2n-1}, y_{2n+1}, y_{2n}, kt), & \mathcal{M}(y_{2n-1}, y_{2n}, y_{2n+1}, kt), \\ d_{2n-1}(kt), & d_{2n}(kt), & d_{2n}(kt) \\ d_{2n}(kt), & d_{2n}(kt), & d_{2n}(kt) \\ d_{2n}(kt), & d_{2n}(kt), & d_{2n}(kt), \end{pmatrix}$$

that is,

$$d_{2n}(t) \geq \phi \begin{pmatrix} d_{2n-1}(kt), & d_{2n-1}(kt) * d_{2n}(kt), & d_{2n-1}(kt) * d_{2n}(kt), \\ d_{2n-1}(kt), & d_{2n}(kt), & d_{2n}(kt) \\ d_{2n}(kt), & d_{2n}(kt), & d_{2n}(kt) \\ d_{2n}(kt), & d_{2n}(kt), & d_{2n}(kt) \end{pmatrix}.$$

We claim that, for all $n = 1, 2, \cdots$, $d_{2n}(t) \geq d_{2n-1}(kt)$. In fact, if $d_{2n}(t) < d_{2n-1}(kt)$, then, we have

$$\begin{aligned} d_{2n}(t) \geq\ & \phi\ (d_{2n}(kt), d_{2n}(kt) * d_{2n}(kt), d_{2n}(kt) \\ & * d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt), d_{2n}(kt) \\ & , d_{2n}(kt), d_{2n}(kt)) \\ >\ & d_{2n}(kt), \end{aligned}$$

which is a contradiction. Hence $d_{2n}(t) \geq d_{2n-1}(kt)$ for all $n = 1, 2, \cdots$ and $t > 0$.

Similarly, for $m = 2n + 1$, we have $d_{2n+1}(t) \geq d_{2n}(kt)$ and so $\{d_n(t)\}$ is increasing sequence, that is

$$\mathcal{M}(y_n, y_n, y_{n+1}, kt) \geq \mathcal{M}(y_{n-1}, y_{n-1}, y_n, kt) \geq \cdots \geq \mathcal{M}(y_0, y_0, y_1, k^n t).$$

Similarly, if we set $d_{m'}(t) = \mathcal{N}(x_{m'}, x_{m'}, x_{m'+1}, t)$ for all $t > 0$, then we prove that $\{d_{m'}(t)\}$ is decreasing with respect to m'.

For $m' = 2n$, we have $d_{2n}(t) \leq d_{2n-1}(kt)$ and for $m' = 2n + 1$, we have $d_{2n+1}(t) \leq d_{2n}(kt)$, so $\{d_n(t)\}$ is decreasing sequence, that is

$$\mathcal{N}(y_n, y_n, y_{n+1}, kt) \leq \mathcal{N}(y_{n-1}, y_{n-1}, y_n, kt) \leq \cdots \leq \mathcal{N}(y_0, y_0, y_1, k^n t).$$

Hence by Theorem (2.10), $\{y_n\}$ is Cauchy sequence and the completeness of X, $\{y_n\}$ converges to a point u in X.

So, we have

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = u,$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = u.$$

Now, suppose that T is continuous and pairs (A, S), (B, T) are compatible of type (I).

Hence, we have

$$\lim_{n\to\infty} TTx_{2n+1} = Tu,$$

$$\mathcal{M}(Tu, u, u, t) \geq \lim_{n\to\infty} \mathcal{M}(BTx_{2n+1}, u, u, t)$$

and

$$\mathcal{N}(Tu, u, u, t) \leq \lim_{n\to\infty} \mathcal{N}(BTx_{2n+1}, u, u, t).$$

Now, we putting $x = y = x_{2n}$ and $z = Tx_{2n+1}$ in the inequality $(ii, 1)$ and using the compatibility of type (I), we have

$$\begin{aligned}
&\mathcal{M}(Ax_{2n}, Bx_{2n}, BTx_{2n+1}, t)\\
&\geq \phi\left(\begin{array}{l}
\mathcal{M}(Sx_{2n}, Tx_{2n}, TTx_{2n+1}, kt), \mathcal{M}(Sx_{2n}, Bx_{2n}, TTx_{2n+1}, kt),\\
\mathcal{M}(Sx_{2n}, Tx_{2n}, BTx_{2n+1}, kt),\\
\mathcal{M}(Sx_{2n}, Bx_{2n}, Bx_{2n}, kt), \mathcal{M}(Tx_{2n}, Bx_{2n}, BTx_{2n+1}, kt),\\
\mathcal{M}(Tx_{2n}, Tx_{2n}, BTx_{2n+1}, kt),\\
\mathcal{M}(Tx_{2n}, Bx_{2n}, Bx_{2n}, kt), \mathcal{M}(Tx_{2n}, BTx_{2n+1}, BTx_{2n+1}, kt),\\
\mathcal{M}(Bx_{2n}, Tx_{2n}, TTx_{2n+1}, kt),\\
\mathcal{M}(Bx_{2n}, Bx_{2n}, TTx_{2n+1}, kt), \mathcal{M}(Bx_{2n}, TTx_{2n+1}, TTx_{2n+1}, kt),\\
\mathcal{M}(TTx_{2n+1}, BTx_{2n+1}, BTx_{2n+1}, kt),
\end{array}\right)
\end{aligned}$$

so

$$\begin{aligned}
&\mathcal{M}(u, u, BTx_{2n+1}, t)\\
&\geq \phi\left(\begin{array}{l}
\mathcal{M}(u, u, Tu, kt), \mathcal{M}(u, u, Tu, kt), \mathcal{M}(u, u, BTx_{2n+1}, kt),\\
\mathcal{M}(u, u, u, kt), \mathcal{M}(u, u, BTx_{2n+1}, kt), \mathcal{M}(u, u, BTx_{2n+1}, kt),\\
\mathcal{M}(u, u, u, kt), \mathcal{M}(u, BTx_{2n+1}, BTx_{2n+1}, kt), \mathcal{M}(u, u, Tu, kt),\\
\mathcal{M}(u, u, Tu, kt), \mathcal{M}(u, Tu, Tu, kt), \mathcal{M}(Tu, BTx_{2n+1}, BTx_{2n+1}, kt)
\end{array}\right)
\end{aligned}$$

$$\geq \phi\left(\begin{array}{l}
\mathcal{M}(u, u, Tu, kt/2), \mathcal{M}(u, u, Tu, kt/2), \mathcal{M}(u, u, BTx_{2n+1}, kt/2),\\
\mathcal{M}(u, u, u, kt/2), \mathcal{M}(u, u, BTx_{2n+1}, kt/2), \mathcal{M}(u, u, BTx_{2n+1}, kt/2),\\
\mathcal{M}(u, u, u, kt/2), \mathcal{M}(u, BTx_{2n+1}, BTx_{2n+1}, kt/2), \mathcal{M}(u, u, BTx_{2n+1}, kt/2),\\
\mathcal{M}(u, u, Tu, kt/2), \mathcal{M}(u, Tu, Tu, kt/2), \mathcal{M}(Tu, BTx_{2n+1}, BTx_{2n+1}, kt/2),
\end{array}\right)$$

thus it follows that

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{M}(BTx_{2n+1}, BTx_{2n+1}, Tu, kt) \geq &\lim_{n\to\infty} \mathcal{M}(BTx_{2n+1}, BTx_{2n+1}, u, kt/2)\\
&* \lim_{n\to\infty} \mathcal{M}(u, u, Tu, kt/2),
\end{aligned}$$

and so

$$\lim_{n\to\infty} \mathcal{M}(BTx_{2n+1}, BTx_{2n+1}, Tu, kt) \geq \lim_{n\to\infty} \mathcal{M}(BTx_{2n+1}, BTx_{2n+1}, u, kt/2).$$

Hence, since $\phi(t, t, t, t, t) > t$ by the above inequalities, we have

$$\mathcal{M}(u, u, \lim_{n\to\infty} BTx_{2n+1}, t) > \mathcal{M}(u, u, \lim_{n\to\infty} BTx_{2n+1}, kt/2),$$

which is a contradictions, it follows that $\lim BTx_{2n+1} = u$. Now, using the compatibility of type (I), we have

$$\mathcal{M}(Tu, u, u, kt) \geq \lim_{n\to\infty} \mathcal{M}(u, u, BTx_{2n+1}, kt) = 1,$$

Similarly, we putting $x = y = x_{2n}$ and $z = Tx_{2n+1}$ in the inequality $(ii, 2)$ and using the compatibility of type (I), we have

$$\begin{aligned}
&\mathcal{N}(u, u, BTx_{2n+1}, t) \\
\leq\ & \psi\begin{pmatrix} \mathcal{N}(u, u, Tu, kt), \mathcal{N}(u, u, Tu, kt), \mathcal{N}(u, u, BTx_{2n+1}, kt), \\ \mathcal{N}(u, u, u, kt), \mathcal{N}(u, u, BTx_{2n+1}, kt), \mathcal{N}(u, u, BTx_{2n+1}, kt), \\ \mathcal{N}(u, u, u, kt), \mathcal{N}(u, BTx_{2n+1}, BTx_{2n+1}, kt), \mathcal{N}(u, u, Tu, kt), \\ \mathcal{N}(u, u, Tu, kt), \mathcal{N}(u, Tu, Tu, kt), \mathcal{N}(Tu, BTx_{2n+1}, BTx_{2n+1}, kt) \end{pmatrix}
\end{aligned}$$

$$\leq \psi\begin{pmatrix} \mathcal{N}(u, u, Tu, kt/2), \mathcal{N}(u, u, Tu, kt/2), \mathcal{N}(u, u, BTx_{2n+1}, kt/2), \\ \mathcal{N}(u, u, u, kt/2), \mathcal{N}(u, u, BTx_{2n+1}, kt/2), \mathcal{N}(u, u, BTx_{2n+1}, kt/2), \\ \mathcal{N}(u, u, u, kt/2), \mathcal{N}(u, BTx_{2n+1}, BTx_{2n+1}, kt/2), \mathcal{N}(u, u, BTx_{2n+1}, kt/2), \\ \mathcal{N}(u, u, Tu, kt/2), \mathcal{N}(u, Tu, Tu, kt/2), \mathcal{N}(Tu, BTx_{2n+1}, BTx_{2n+1}, kt/2), \end{pmatrix}$$

thus it follows that

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{N}(BTx_{2n+1}, BTx_{2n+1}, Tu, kt) \leq\ & \lim_{n\to\infty} \mathcal{N}(BTx_{2n+1}, BTx_{2n+1}, u, kt/2) \\
& * \lim_{n\to\infty} \mathcal{N}(u, u, Tu, kt/2),
\end{aligned}$$

and so

$$\lim_{n\to\infty} \mathcal{N}(BTx_{2n+1}, BTx_{2n+1}, Tu, kt) \leq \lim_{n\to\infty} \mathcal{N}(BTx_{2n+1}, BTx_{2n+1}, u, kt/2).$$

Hence, since $\psi(t, t, t, t, t) < t$ by the above inequalities, we have

$$\mathcal{N}(u, u, \lim_{n\to\infty} BTx_{2n+1}, t) < \mathcal{N}(u, u, \lim_{n\to\infty} BTx_{2n+1}, kt/2),$$

which is a contradictions, it follows that $\lim_{n\to\infty} BTx_{2n+1} = u$. Now, using the compatibility of type (I), we have

$$\mathcal{N}(Tu, u, u, kt) \leq \lim_{n\to\infty} \mathcal{N}(u, u, BTx_{2n+1}, kt) = 0,$$

and so it follows that $Tu = u$.

If $Bu \neq u$ and replacing x, y by x_{2n} and z by u in inequality $(ii, 1)$ and letting $n \to \infty$, we have

$$\mathcal{M}(Ax_{2n}, Bx_{2n}, Bu, t) \geq \phi\begin{pmatrix} \mathcal{M}(Sx_{2n}, Tx_{2n}, Tu, kt), \mathcal{M}(Sx_{2n}, Bx_{2n}, Tu, kt), \\ \mathcal{M}(Sx_{2n}, Tx_{2n}, Bu, kt), \\ \mathcal{M}(Sx_{2n}, Bx_{2n}, Bx_{2n}, kt), \mathcal{M}(Tx_{2n}, Bx_{2n}, Bu, kt), \\ \mathcal{M}(Tx_{2n}, Tx_{2n}, Bu, kt), \\ \mathcal{M}(Tx_{2n}, Bx_{2n}, Bx_{2n}, kt), \mathcal{M}(Tx_{2n}, Bu, Bu, kt), \\ \mathcal{M}(Bx_{2n}, Tx_{2n}, Tu, kt), \\ \mathcal{M}(Bx_{2n}, Bx_{2n}, Tu, kt), \mathcal{M}(Bx_{2n}, Tu, Tu, kt), \\ \mathcal{M}(Tu, Bu, Bu, kt), \end{pmatrix}$$

so

$$\mathcal{M}(u,u,Bu,t) \geq \phi\begin{pmatrix} \mathcal{M}(u,u,u,kt), \mathcal{M}(u,u,u,kt), \mathcal{M}(u,u,Bu,kt), \\ \mathcal{M}(u,u,u,kt), \mathcal{M}(u,u,Bu,kt), \mathcal{M}(u,u,Bu,kt), \\ \mathcal{M}(u,u,u,kt), \mathcal{M}(u,Bu,Bu,kt), \mathcal{M}(u,u,u,kt), \\ \mathcal{M}(u,u,u,kt), \mathcal{M}(u,u,u,kt), \mathcal{M}(u,Bu,Bu,kt), \end{pmatrix}$$

and so, letting $n \to \infty$, we have

$$\mathcal{M}(Bu,u,u,t) > \mathcal{M}(u,u,Bu,kt),$$

Similarly, if $Bu \neq u$ and replacing x, y by x_{2n} and z by u in inequality $(ii, 2)$ and letting $n \to \infty$, we have

$$\mathcal{N}(Bu,u,u,t) < \mathcal{N}(u,u,Bu,kt),$$

which is a contradiction. Thus $Bu = u$.

Since $B(X) \subseteq S(X)$. There exist $w \in X$ such that $Sw = u = Bu$.

If $Aw \neq u$ then, by putting $x = w$ and $y = z = u$ in the inequalities $(ii, 1)$ and $(ii, 2)$, we have

$$\begin{aligned}\mathcal{M}(Aw,Bu,Bu,t) &= \mathcal{M}(Aw,u,u,t) \\ &\geq \phi\begin{pmatrix} \mathcal{M}(Sw,Tu,Tu,kt), \mathcal{M}(Sw,Bu,Tu,kt), \mathcal{M}(Sw,Tu,Bu,kt), \\ \mathcal{M}(Sw,Bu,Bu,kt), \mathcal{M}(Tu,Bu,Bu,kt), \mathcal{M}(Tu,Tu,Bu,kt), \\ \mathcal{M}(Tu,Bu,Bu,kt), \mathcal{M}(Tu,Bu,Bu,kt), \mathcal{M}(Bu,Tu,Tu,kt), \\ \mathcal{M}(Bu,Bu,Tu,kt), \mathcal{M}(Bu,Tu,Tu,kt), \mathcal{M}(Tu,Bu,Bu,kt), \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(Aw,Bu,Bu,t) &= \mathcal{N}(Aw,u,u,t) \\ &\leq \psi\begin{pmatrix} \mathcal{N}(Sw,Tu,Tu,kt), \mathcal{N}(Sw,Bu,Tu,kt), \mathcal{N}(Sw,Tu,Bu,kt), \\ \mathcal{N}(Sw,Bu,Bu,kt), \mathcal{N}(Tu,Bu,Bu,kt), \mathcal{N}(Tu,Tu,Bu,kt), \\ \mathcal{N}(Tu,Bu,Bu,kt), \mathcal{N}(Tu,Bu,Bu,kt), \mathcal{N}(Bu,Tu,Tu,kt), \\ \mathcal{N}(Bu,Bu,Tu,kt), \mathcal{N}(Bu,Tu,Tu,kt), \mathcal{N}(Tu,Bu,Bu,kt) \end{pmatrix}.\end{aligned}$$

On making $n \to \infty$ we get $\mathcal{M}(Aw,u,u,t) = 1$ and $\mathcal{N}(Aw,u,u,t) = 0$ hence $Aw = u$ since the pair (A, S) is compatible of type (I) and $Aw = Sw = u$ by proposition (3.3), we have

$$\mathcal{M}(u,u,Su,t) \geq \mathcal{M}(u,u,Au,t),$$

and

$$\mathcal{N}(u,u,Su,t) \leq \mathcal{N}(u,u,Au,t).$$

Again, by putting $x = y = z = u$, we have

$$\begin{aligned}&\mathcal{M}(Au, Bu, Bu, t)\\ \geq\ & \phi\begin{pmatrix} \mathcal{M}(Su, Tu, Tu, kt), \mathcal{M}(Su, Bu, Tu, kt), \mathcal{M}(Su, Tu, Bu, kt), \\ \mathcal{M}(Su, Bu, Bu, kt), \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Tu, Tu, Bu, kt), \\ \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Bu, Tu, Tu, kt), \\ \mathcal{M}(Bu, Bu, Tu, kt), \mathcal{M}(Bu, Tu, Tu, kt), \mathcal{M}(Tu, Bu, Bu, kt), \end{pmatrix}.\end{aligned}$$

and

$$\begin{aligned}&\mathcal{N}(Au, Bu, Bu, t)\\ \leq\ & \psi\begin{pmatrix} \mathcal{N}(Su, Tu, Tu, kt), \mathcal{N}(Su, Bu, Tu, kt), \mathcal{N}(Su, Tu, Bu, kt), \\ \mathcal{N}(Su, Bu, Bu, kt), \mathcal{N}(Tu, Bu, Bu, kt), \mathcal{N}(Tu, Tu, Bu, kt), \\ \mathcal{N}(Tu, Bu, Bu, kt), \mathcal{N}(Tu, Bu, Bu, kt), \mathcal{N}(Bu, Tu, Tu, kt), \\ \mathcal{N}(Bu, Bu, Tu, kt), \mathcal{N}(Bu, Tu, Tu, kt), \mathcal{N}(Tu, Bu, Bu, kt) \end{pmatrix}.\end{aligned}$$

Hence, we have $\mathcal{M}(Au, u, u, t) = 1$ and $\mathcal{N}(Au, u, u, t) = 0$ so $Au = u$.

Therefore $Au = Bu = Su = Tu = u$. So A, B, S and have a fixed common point u.

Now to prove uniqueness, if possible $u' \neq u$ be another common fixed point of A, B, S and T. Then we putting $x = y = u'$ and $z = u$ we have.

$$\mathcal{M}(u', u', u, t) > \mathcal{M}(u', u', u, kt),$$

and

$$\mathcal{N}(u', u', u, t) < \mathcal{N}(u', u', u, kt),$$

which is a contradiction. Therefore, we have $u' = u$. This completes the proof of the theorem. ■

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