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Trigonometric Algebras Euclidean, Spherical, Minkowski, Hyperbolic Spaces

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Abstract

The discovery of non-Euclidean geometry developed geometry dramatically. These new mathematical ideas were the basis for such concepts as the general relativity of a century ago and the physical theory of today. The idea of curvature is a mathematical idea. Plane hyperbolic geometry is the simplest example of a negatively curved space. Spherical geometry has even more practical applications.

Riemann was the first geometer who really sorted out a concept in geometry. He made a general study of curvature of spaces in all dimensions. In two dimensions:

Euclidean geometry is flat. It is curvature zero and any triangle angle sum is 180 degrees.

The non-Euclidean geometry of Lobachevsky is negatively curved, and any triangle angle sum is smaller than 180 degrees. The geometry of the sphere is positively curved, and any triangle angle sum is bigger than 180 degrees [1],[2],[3],[4],[5],[6],[7],[8].

I will show to combine Euclidean, Spherical, Minkowski, Hyperbolic geometries on four dimensional Trigonometric Algebra. Trigonometric Algebra is a four dimensional hypercomplex number theory. It is a noncommutative and associative algebra. It is an isomorphism with Quaternion algebra. My intuitive conception and observation of position and motion suggest that the position of geometry in space can only be specified relative to some other geometry, chosen as a reference. Likewise, the motion of geometry can only be specified relative to some reference geometry.

Introduction

The foundations of Euclidean geometry are five postulates concerning points and lines. A point is an abstraction of the notion of a position in space. A line is an abstraction of the path of a light beam connecting two nearby points. Thus, any two points determine a unique line passing through them. This is Euclid's first postulate. The second postulate states that a line segment can be extended without limit in either direction. This is rather less intuitive and requires an imaginative conception of space as being infinite in extent. The third postulate states that, given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center, thereby recognizing the special importance of the circle and the use of straight-edge and compass to construct planar figures. The fourth postulate states that all right angles are equal, thereby acknowledging our perception of perpendicularity and its uniformity. The fifth and final postulate states that if two lines are drawn in the plane to intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate, stating that, given a line and a point not on the line, there exists one and only one straight line in the same plane that passes through the point and never intersects the first line, no matter how far the lines are extended. The parallel postulate is somewhat contrary to our physical perception of distance perspective, where in fact two lines constructed to run parallel seem to converge in the far distance. While any geometric construction that does not exclusively rely on the five postulates of Euclid can be called non-Euclidean, the two basic non-Euclidean geometries, hyperbolic and elliptic, accept the first four postulates of Euclid, but use their own versions of the fifth. Incidentally, Euclidean geometry is sometimes called parabolic. The parallel postulate of Euclid has many implications, for example, that the sum of the angles of a triangle is 180^0. Not surprisingly, this and other implications do not hold in non-Euclidean geometries. Classical, Newtonian mechanics assumes that the geometry of space is Euclidean, The development of Euclidean geometry essentially relies on our intuition that every line segment joining two points has a length associated with it. Length is measured as a multiple of some chosen unit [1],[2],[3],[4],[5],[6],[7].

Geometric Spaces

EUCLIDEAN 3-SPACE, E^3

Definition 1-1:

Euclidean 3-space, $E^3 = \{x = (x^1, x^2, x^3) : x^1, x^2, x^3 \in R\}$

Definition 1-2:

Dot Product;

For $x, y \in E^3$,

$$x \cdot y = x^1 y^1 + x^2 y^2 + x^3 y^3$$

$$x \cdot x = |x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

$$d_E(x, y) = |x - y|$$

Proposition 1-3:

d_E is metric.

Corollary 1-4:

Schwarz Inequality is $|x \cdot y| \le |x| \cdot |y|$

Equality can be by including a multiple, $\cos(\theta(x,y))$, in the inequality.

$$x \cdot y = |x| \cdot |y| \cdot \cos(\theta(x,y))$$

Where $\theta(x,y)$ is the angle between x and y.

Definition 1-5:
Cross Product;
For $x, y \in E^3$,

$$x \times y = \begin{vmatrix} i & j & k \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}$$

Theorem 1-6:

1 $x \times y = -y \times x$

2 $(x \times y) \cdot z = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}$

$(x \times y) \cdot z = (z \times x) \cdot y = (y \times z) \cdot x$

3 $x \times (y \times z) = (x \cdot z)y - (x \cdot y)z$

4 $(x \times y) \cdot (z \times w) = \begin{vmatrix} x \cdot z & x \cdot w \\ y \cdot z & y \cdot w \end{vmatrix}$

Property 4 combined with

$x \cdot y = |x| \cdot |y| \cdot \cos(\theta(x,y))$ yields

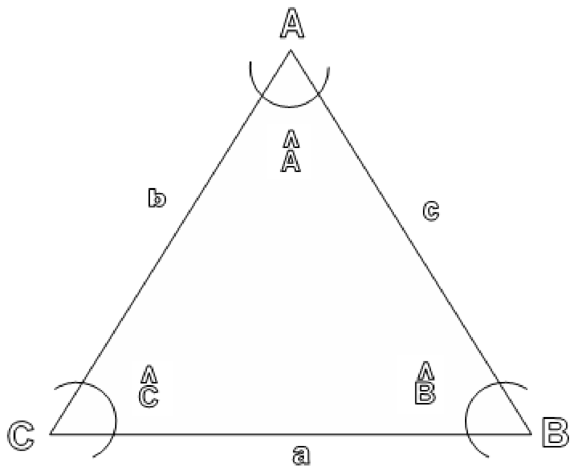
$|x \times y| = |x| \cdot |y| \cdot \sin(\theta(x,y))$ [1],[2],[3],[4],[5],[6],[7].

Euclidean Triangles In E^3

Triangles in E^3 consist of 3 points, $x, y, z \in E^3$ and the geodesics connecting points.

Geodesics are "straight lines" between points. In E^3, geodesics are straight lines.

SIDES	$[x,y]$	$[y,z]$	$[z,x]$
LENGTHS	$a = d_E(x,y)$	$b = d_E(y,z)$	$c = d_E(z,x)$
ANGLES	A	B	C
GEODESICS	$f:[0,a]\to E^3$	$g:[0,b]\to E^3$	$h:[0,c]\to E^3$



Graphic 1: Euclidean triangle.

Euclidean Law of Sines

Euclidean Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Euclidean Law of Cosines

Euclidean Law of Cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Spherical 2-Space, S^2

Defition 1-1:

Unit Sphere, S^2

$$S^2 = \{x \in E^3 : |x| = 1\}$$

$$x \cdot y = |x||y|\cos(\theta(x,y)) = \cos(\theta(x,y)) \text{ and}$$

$$|x \times y| = |x||y|\sin(\theta(x,y)) = \sin(\theta(x,y))$$

On S^2, the geodesic between two points is the shortest are of the great circle passing through the points. This gives distance between two points on the sphere to be

$$d_S(x,y) = \theta(x,y) = \cos^{-1}(x \cdot y)$$

Then

$$0 \le d_S(x,y) \le \pi$$

$$d_S(x,y) = \pi \Leftrightarrow y = -x \text{ is antipodal}$$

Proposition 2-2:

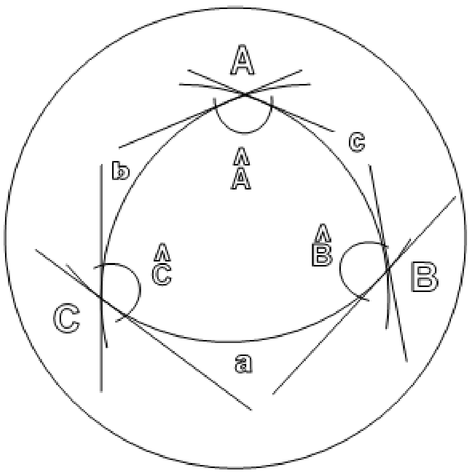
d_S is metric [1],[2],[3],[4],[5],[6],[7].

Spherical Triangles in S^2

Triangles in S^2 consist 3 points, $x, y, z \in S^2$ and the geodesics connecting the points.

SIDES	$[x,y]$	$[y,z]$	$[z,x]$
LENGTHS	$a = \theta(x,y)$	$b = \theta(y,z)$	$c = \theta(z,x)$
ANGLES	A	B	C
GEODESICS	$f:[0,a] \to S^2$	$g:[0,b] \to S^2$	$h:[0,c] \to S^2$

$\theta(y \times z, z \times x) = \pi - A$	$\theta(y \times z, x \times z) = A$
$\theta(z \times x, x \times y) = \pi - B$	$\theta(z \times x, y \times x) = B$
$\theta(x \times y, y \times z) = \pi - C$	$\theta(x \times y, z \times y) = C$



Graphic 2: Spherical triangle.

Spherical Law of Sines

$$(y\times z)\times(z\times x)=((y\times z)\cdot x)z-((y\times z)\cdot z)x=((y\times z)\cdot x)z$$

$$|(y\times z)\times(z\times x)|=|((y\times z)\cdot x)z|$$

$$|y\times z||z\times x|\sin\theta(y\times z,z\times x)=|((y\times z)\cdot x)||z|$$

$$\sin b\sin c\sin(\pi-A)=|((y\times z)\cdot x)|$$

$$\sin b\sin c\sin A=|((y\times z)\cdot x)|$$

Similarly

$$(z\times x)\times(x\times y)=((z\times x)\cdot y)x$$

$$(x\times y)\times(y\times z)=((x\times y)\cdot z)y$$

Taking the norm of the reamining two equalities, noticing the right hand sides of each are equal, yields

$$\sin b\sin c\sin A=\sin c\sin a\sin B=\sin a\sin b\sin C$$

Spherical Law of Sines

Spherical Law of Sines

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}$$

Spherical Law of Cosines

Spherical Law of Cosines

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

Minkowski 3- Space, M^3

Definition 3-1:

Minkowski 3-space;

$$M^3=\{x:x=(x^1,x^2,x^3)\}$$

Definition 3-2:

Boxdot Product;

For $x,y\in M^3$,

$$x\nabla y=x^1y^1+x^2y^2-x^3y^3$$

$$x\nabla x=\|x\|^2$$

$$d_M(x,y) = \|x - y\|$$

$x \in M^3$ is called time-like if $x\nabla x < 0$

$x \in M^3$ is called space-like if $x\nabla x > 0$

$x \in M^3$ is called light-like if $x\nabla x = 0$

We will be mainly concerned with time-like vectors for the remainder of the time.

Proposition 3-3:
d_M is metric.

Corolarly 3-4:
For x and y timelike vectors,

$$x\nabla y \geq \|x\|\|y\|$$

Equality can be attained by including a multiple, $\cosh(\theta(x,y))$, in the equality.

$$x\nabla y = \|x\|\|y\|\cosh(\theta(x,y))$$

where $\theta(x,y)$ is the hyperbolic angle between x and y.

Definition 3-5:
Boxcroos Product;

For $x, y \in M^3$,

$$x\Delta y = \begin{vmatrix} i & j & k \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}$$

Theorem 3-6:
Properties of Vectors in Minkowski 3-Space

1. If x, y are positive time-like vectors, then $x\Delta y$ is space-like.
2. If u, v are space-like vectors, then the following are equivalent:
    a. The vectros u and v satisfy the inequality $|u\nabla v| < \|u\|\|v\|$
    b. $u\Delta v$ is the time-like.
    c. The vector subspace V spanned by u and v is space-like.Every nonzero vector is space-like.

3. If u, v are space-like vectors spaning a space-like vector space,then

$$u\nabla v = \|u\|.\|v\|\cos(\theta(u,v))$$

$$\|\|u\Delta v\|\| = \|u\|.\|v\|\sin(\theta(u,v))$$

where $\left|\|u\|^2\right| = -(u\nabla u)$

Theorem 3-7:
Properties of ∇ and Δ

1. $x\Delta y = -y\Delta x$
2. $(x\Delta y)\nabla z = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}$
3. $x\Delta(y\Delta z) = -((x\nabla z)y - (x\nabla y)z)$
4. $(x\Delta y)\nabla(z\Delta w) = -\begin{vmatrix} x\nabla z & x\nabla w \\ y\nabla z & y\nabla w \end{vmatrix}$

For x and y time-like vectors, property 4 combined with

$x\nabla y = \|x\|.\|y\|\cosh(\theta(x,y))$ yields

$\|x \times y\| = -\|x\|.\|y\|\sinh(\theta(x,y))$ [1],[2],[3],[4],[5],[6],[7].

Hyperbolic 2- Space, H^2
Definition 4-1
Unit hyperboloid, H^2;

$$H^2 = \{x \in M^3 x : x\nabla x = -1\}$$

$x\nabla y = \|x\|.\|y\|\cosh(\theta(x,y)) = -\cosh(\theta(x,y))$ and

$$\|x\Delta y\| = \|x\|.\|y\|\sinh(\theta(x,y)) = \sinh(\theta(x,y))$$

On H^2*;* the geodesic is the branch of a hyperbola passing through the points. This gives the distance between two points on the hyperboloid to be

$$d_H(x,y) = \theta(x,y) = \cosh^{-1}(-x\nabla y)$$

$$0 \le d_H(x,y)$$

Proposition 4-2

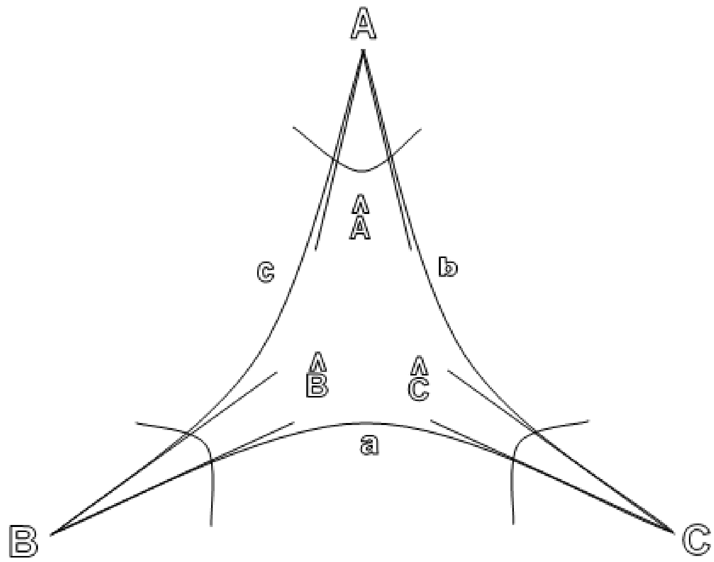
d_H is a metric [1],[2],[3],[4],[5],[6],[7].

Hyperbolic Triangles In H^2

Triangles in H^2 consist 3 points, $x, y, z \in H^2$ and the geodesics connecting the points.

SIDES	$[x, y]$	$[y, z]$	$[z, x]$
LENGTHS	$a = \theta(x, y)$	$b = \theta(y, z)$	$c = \theta(z, x)$
ANGLES	A	B	C
GEODESICS	$f : [0, a] \to H^2$	$g : [0, b] \to H^2$	$h : [0, c] \to H^2$

$\theta(y\Delta z, z\Delta x) = \pi - A$	$\theta(y\Delta z, x\Delta z) = A$
$\theta(z\Delta x, x\Delta y) = \pi - B$	$\theta(z\Delta x, y\Delta x) = B$
$\theta(x\Delta y, y\Delta z) = \pi - C$	$\theta(x\Delta y, z\Delta y) = C$



Graphic 3: Hyperbolic triangle.

Hyperbolic Law Of Sines

$$(y\Delta z)\Delta(z\Delta x) = -(((y\Delta z)\nabla x)z - ((y\Delta z)\nabla z)x) = -((y\Delta z)\nabla x)z$$

$$\|(y\Delta z)\Delta(z\Delta x)\| = \|-((y\Delta z)\nabla x)z\|$$

$$\|y\Delta z\|.\|z\Delta x\|\sin\theta(y\Delta z, z\Delta x) = |-((y\Delta z)\nabla x)|\|z\|$$

$$\sinh b \sinh c \sin(\pi - A) = |((y\Delta z)\nabla x)|(-z\nabla z)$$

$$\sinh b \sinh c \sin A = |((y\Delta z)\nabla x)|$$

Similarly

$$(z\Delta x)\Delta(x\Delta y) = -((z\Delta x)\nabla y)x$$
$$(x\Delta y)\Delta(y\Delta z) = -((x\Delta y)\nabla z)y$$

Taking the norm of the reamining two equalities, noticing the right hand sides of each are equal, yields

$$\sinh b \sinh c \sin A = \sinh c \sinh a \sin B = \sinh a \sinh b \sin C$$

Hyperbolic Law of Sines
Hyperbolic Law of Sines

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$$

Hyperbolic Law of Cosines
Hyperbolic Law of Cosines

$$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

Trigonometric Algebras

Definition 2-1
Let

Euclidean Law of Sines is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Spherical Law of Sines is

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Hyperbolic Law of Sines is

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$$

G is a trigonometric space.
A is an angle and its length is a

The form of a trigonometry number is $\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$
$\xi = \{\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3 \in G \,|\, (\sin A), (a), (\sin a), (\sinh a) \in R\}$

$$\xi = \left\{ \begin{array}{l} \Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3 \in G \left| e_1^{\ 2} = e_2^{\ 2} = e_3^{\ 2} = -1, e_1 e_2 = e_3, \right. \\ e_3 e_1 = e_2, e_2 e_3 = e_1, e_1 e_2 e_3 = -1 \end{array} \right\}$$

B is an angle and its length is b

The form of a trigonometry number is $\Psi = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3$

$$\xi = \{\Psi = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3 \in G | (\sin B), (b), (\sin b), (\sinh b) \in R\}$$

$$\xi = \left\{ \begin{array}{l} \Psi = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3 \in G \left| e_1^{\ 2} = e_2^{\ 2} = e_3^{\ 2} = -1, e_1 e_2 = e_3, \right. \\ e_3 e_1 = e_2, e_2 e_3 = e_1, e_1 e_2 e_3 = -1 \end{array} \right\}$$

C is an angle and its length is c

The form of a trigonometry number is $\Psi = (\sin C) + (c)e_1 + (\sin c)e_2 + (\sinh c)e_3$

$$\xi = \{\Psi = (\sin C) + (c)e_1 + (\sin c)e_2 + (\sinh c)e_3 \in G | (\sin C), (c), (\sin c), (\sinh c) \in R\}$$

$$\xi = \left\{ \begin{array}{l} \Psi = (\sin C) + (c)e_1 + (\sin c)e_2 + (\sinh c)e_3 \in G \left| e_1^{\ 2} = e_2^{\ 2} = e_3^{\ 2} = -1, e_1 e_2 = e_3, \right. \\ e_3 e_1 = e_2, e_2 e_3 = e_1, e_1 e_2 e_3 = -1 \end{array} \right\}$$

Trigonometric Algebras Operators

Product

Let

$$\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$$

$$\xi = \{\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3 \in G | (\sin A), (a), (\sin a), (\sinh a) \in R\}$$

$$\xi = \left\{ \begin{array}{l} \Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3 \in G \left| e_1^{\ 2} = e_2^{\ 2} = e_3^{\ 2} = -1, e_1 e_2 = e_3, \right. \\ e_3 e_1 = e_2, e_2 e_3 = e_1, e_1 e_2 e_3 = -1 \end{array} \right\}$$

$$\Psi_1 = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$$

$$\Psi_2 = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3$$

Multiplication is generally noncommutative $\Psi_1 \times \Psi_2 \neq \Psi_2 \times \Psi_1$

$\Psi_1 \times \Psi_2$	e_1	e_2	e_3
e_1	-1	e_3	$-e_2$
$1e_2$	$-e_3$	-1	e_1
e_3	e_2	$-e_1$	-1

Conjugate

The conjugate of $\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$ is

$$\overline{\Psi} = (\sin A) - (a)e_1 - (\sin a)e_2 - (\sinh a)e_3$$

$$\xi = \{\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3 \in G \,|\, (\sin A), (a), (\sin a), (\sinh a) \in R\}$$

$$\overline{\xi} = \{\overline{\Psi} = (\sin A) - (a)e_1 - (\sin a)e_2 - (\sinh a)e_3 \in G \,|\, (\sin A), -(a), -(\sin a), -(\sinh a) \in R\}$$

Magnitude

The magnitude of $\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$ is

$$|\Psi| = \sqrt{(\sin A)^2 + (a)^2 + (\sin a)^2 + (\sinh a)^2}$$

Multiplicative Inverse

The multiplicative inverse of $\Psi = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$ is

$$\Psi^{-1} = \frac{1}{\Psi}\ , \Psi \neq 0 \text{ and } \Psi^{-1} = \frac{\overline{\Psi}}{\Psi\overline{\Psi}}$$

$$\Psi^{-1} = \frac{(\sin A) - (a)e_1 - (\sin a)e_2 - (\sinh a)e_3}{((\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3)((\sin A) - (a)e_1 - (\sin a)e_2 - (\sinh a)e_3)}$$

$$\Psi^{-1} = \frac{(\sin A) - (a)e_1 - (\sin a)e_2 - (\sinh a)e_3}{(\sin A)^2 + (a)^2 + (\sin a)^2 + (\sinh a)^2}$$

Division

Let

$$\Psi_1 = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$$

$$\Psi_2 = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3 \text{ and } \Psi_2 \neq 0$$

$$\frac{\Psi_1}{\Psi_2} = (\sin C) + (c)e_1 + (\sin c)e_2 + (\sinh c)e_3$$

$$\xi = \{\Psi = (\sin C) + (c)e_1 + (\sin c)e_2 + (\sinh c)e_3 \in G \,|\, (\sin C), (c), (\sin c), (\sinh c) \in R\}$$

The conjugates of $\Psi_1 = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$ and $\Psi_2 = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3$ are

$$\overline{\Psi_1} = (\sin A) - (a)e_1 - (\sin a)e_2 - (\sinh a)e_3 \text{ and}$$

$$\overline{\Psi_2} = (\sin B) - (b)e_1 - (\sin b)e_2 - (\sinh b)e_3$$

$$\frac{\Psi_1}{\Psi_2} = \frac{((\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3)((\sin B) - (b)e_1 - (\sin b)e_2 - (\sinh b)e_3)}{((\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3)((\sin B) - (b)e_1 - (\sin b)e_2 - (\sinh b)e_3)}$$

$$\frac{\Psi_1}{\Psi_2}=\frac{((\sin A)+(a)e_1+(\sin a)e_2+(\sinh a)e_3)((\sin B)-(b)e_1-(\sin b)e_2-(\sinh b)e_3)}{((\sin B)^2+(b)^2+(\sin b)^2+(\sinh b)^2)}$$

Polar Notation

I developed these polar forms.

Let

$$\Psi=(\sin A)+(a)e_1+(\sin a)e_2+(\sinh a)e_3$$

$$\xi=\{\Psi=(\sin A)+(a)e_1+(\sin a)e_2+(\sinh a)e_3\in G|(\sin A),(a),(\sin a),(\sinh a)\in R\}$$

The magnitude of $\Psi=(\sin A)+(a)e_1+(\sin a)e_2+(\sinh a)e_3$ is

$$|\Psi|=\sqrt{(\sin A)^2+(a)^2+(\sin a)^2+(\sinh a)^2}$$

$Arg(T)=\{\theta_1+2\pi k,\theta_2+2\pi k,\theta_3+2\pi k\}$ and

$$\theta=\{0\le\theta_1<360^0,0\le\theta_2<360^0,0\le\theta_3<360^0\mid\theta_1,\theta_2,\theta_3\in R\}$$

The radius set is

$$r=\left\{r_1=\sqrt{\sin^2 A+a^2},r_2=\sqrt{\sin^2 A+\sin^2 a},r_3=\sqrt{\sin^2 A+\sinh^2 a}\mid r_1,r_2,r_3\in R\right\}$$

The polar notation is $\Psi=r_1(\cos\theta_1+e_1\sin\theta_1)r_2(\cos\theta_2+e_2\sin\theta_2)r_3(\cos\theta_3+e_3\sin\theta_3)$

$$\cos\theta=\left\{\begin{array}{l}\cos\theta_1=\frac{\sin A}{\sqrt{\sin^2 A+a^2}},\cos\theta_2=\frac{\sin A}{\sqrt{\sin^2 A+\sin^2 a}},\cos\theta_3=\frac{\sin A}{\sqrt{\sin^2 A+\sinh^2 a}}\mid\\ \cos\theta_1,\cos\theta_2,\cos\theta_3\in R\end{array}\right\}$$

$$\sin\theta=\left\{\begin{array}{l}\sin\theta_1=\frac{a}{\sqrt{\sin^2 A+a^2}},\sin\theta_2=\frac{\sin a}{\sqrt{\sin^2 A+\sin^2 a}},\sin\theta_3=\frac{\sinh a}{\sqrt{\sin^2 A+\sinh^2 a}}\mid\\ \sin\theta_1,\sin\theta_2,\sin\theta_3\in R\end{array}\right\}$$

The polar notation is

$$\Psi=r_1(\cos\theta_1+e_1\sin\theta_1)r_2(\cos\theta_2+e_2\sin\theta_2)r_3(\cos\theta_3+e_3\sin\theta_3)$$

Its conjugate is

$$\overline{\Psi}=r_1(\cos\theta_1-e_1\sin\theta_1)r_2(\cos\theta_2-e_2\sin\theta_2)r_3(\cos\theta_3-e_3\sin\theta_3)$$

Exponential Form

I developed these exponential forms.

Let

$Arg(\Psi)=\{\theta_1+2\pi k,\theta_2+2\pi k,\theta_3+2\pi k\}$ and

$$\theta=\{0\le\theta_1<360^0,0\le\theta_2<360^0,0\le\theta_3<360^0\mid\theta_1,\theta_2,\theta_3\in R\},$$

The radius set is

$$r=\left\{r_1=\sqrt{\sin^2 A+a^2},r_2=\sqrt{\sin^2 A+\sin^2 a},r_3=\sqrt{\sin^2 A+\sinh^2 a}\mid r_1,r_2,r_3\in R\right\}$$

The polar form is $\Psi=r_1(\cos\theta_1+e_1\sin\theta_1)r_2(\cos\theta_2+e_2\sin\theta_2)r_3(\cos\theta_3+e_3\sin\theta_3)$

The exponential form is

$$e^{e_1\theta_1+e_2\theta_2+e_3\theta_3}=(\cos\theta_1+e_1\sin\theta_1)(\cos\theta_2+e_2\sin\theta_2)(\cos\theta_3+e_3\sin\theta_3)$$

Its conjugate is

$$e^{-e_1\theta_1-e_2\theta_2-e_3\theta_3}=(\cos\theta_1-e_1\sin\theta_1)(\cos\theta_2-e_2\sin\theta_2)(\cos\theta_3-e_3\sin\theta_3)$$

Power Form

I developed these power forms.

Let

$Arg(\Psi)=\{\theta_1+2\pi k,\theta_2+2\pi k,\theta_3+2\pi k\}$ and

$\theta=\left\{0\le\theta_1<360^0,0\le\theta_2<360^0,0\le\theta_3<360^0\mid\theta_1,\theta_2,\theta_3\in R\right\}$,

The radius set is

$$r=\left\{r_1=\sqrt{\sin^2 A+a^2},r_2=\sqrt{\sin^2 A+\sin^2 a},r_3=\sqrt{\sin^2 A+\sinh^2 a}\mid r_1,r_2,r_3\in R\right\}$$

The polar form is $\Psi=r_1(\cos\theta_1+e_1\sin\theta_1)r_2(\cos\theta_2+e_2\sin\theta_2)r_3(\cos\theta_3+e_3\sin\theta_3)$

The power form is from degree nth power and $n\in Z$

$$\Psi^n=r_1^n(\cos n\theta_1+e_1\sin n\theta_1)r_2^n(\cos n\theta_2+e_2\sin n\theta_2)r_3^n(\cos n\theta_3+e_3\sin n\theta_3)$$

Root Form

I developed these root forms.

Let

$Arg(\Psi)=\{\theta_1+2\pi k,\theta_2+2\pi k,\theta_3+2\pi k\}$ and

$\theta=\left\{0\le\theta_1<360^0,0\le\theta_2<360^0,0\le\theta_3<360^0\mid\theta_1,\theta_2,\theta_3\in R\right\}$,

The radius set is

$$r=\left\{r_1=\sqrt{\sin^2 A+a^2},r_2=\sqrt{\sin^2 A+\sin^2 a},r_3=\sqrt{\sin^2 A+\sinh^2 a}\mid r_1,r_2,r_3\in R\right\}$$

The root form is from degree nth root, $k=0,1,2,...,n-1$ and $k,n\in Z$

$$\Psi_k=\sqrt[n]{r_1}\left(\cos\frac{\theta_1+2k\pi}{n}+e_1\sin\frac{\theta_1+2k\pi}{n}\right)\sqrt[n]{r_2}\left(\cos\frac{\theta_2+2k\pi}{n}+e_2\sin\frac{\theta_2+2k\pi}{n}\right)\sqrt[n]{r_3}\left(\cos\frac{\theta_3+2k\pi}{n}+e_3\sin\frac{\theta_3+2k\pi}{n}\right)$$

Its roots are $\Psi_k=\{\Psi_0,\Psi_1,...,\Psi_{n-1}\}$

Addition

$$\Psi_1 = (\sin A) + (a)e_1 + (\sin a)e_2 + (\sinh a)e_3$$

$$\Psi_2 = (\sin B) + (b)e_1 + (\sin b)e_2 + (\sinh b)e_3$$

$$\Psi_1 + \Psi_2 = (\sin A + \sin B) + (a + b)e_1 + (\sin a + \sin b)e_2 + (\sinh a + \sinh b)e_3$$

$$\Psi_1 + \Psi_2 = (\sin D) + (d)e_1 + (\sin d)e_2 + (\sinh d)e_3$$

$$\xi = \{\Psi = (\sin D) + (d)e_1 + (\sin d)e_2 + (\sinh d)e_3 \in G \,|\, (\sin D), (d), (\sin d), (\sinh d) \in R\}$$

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