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Composition operators on Bergman space of the upper half plane

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Abstract

Let $\Pi_+ = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ be the upper half plane in the complex plane $\mathbb{C}$. In this paper we study the composition operators on Bergman spaces $A^p(\Pi_+)$ and prove that when $1 < p < \infty$, there are no weakly compact composition operators on Bergman space $A^p(\Pi_+)$.

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1. Introduction

Let $\Pi_+ = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ be the upper half plane in the complex plane $\mathbb{C}$, $H(\Pi_+)$ the space of all analytic functions on Π_+ and $dA(z)$ the area measure on Π_+. For $1 \le p < \infty$ the Bergman space $A^p(\Pi_+)$ consists of all $f \in H(\Pi_+)$ such that

$$\|f\|^p_{A^p(\Pi_+)} = \int_{\Pi_+} |f(z)|^p dA(z) < \infty.$$

The Bergman space $A^p(\Pi_+)$ with the norm $\|\cdot\|_{A^p(\Pi_+)}$ is a Banach space.

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Let $\varphi : \Pi_+ \to \Pi_+$ be an analytic self-map of Π_+. For $f \in H(\Pi_+)$, the composition operator C_φ is defined by

$$C_\varphi f(z) = f(\varphi(z)), \quad z \in \Pi_+.$$

For some discussions of Bergman spaces of the upper half plane and some operators on them see, e.g., [3, 5, 10, 11].

It is well-known that every composition operator is bounded on Hardy spaces and weighted Bergman spaces of the unit disc $\mathbb{D}$. However, if we consider the Hardy space and the Bergman space of the upper half plane, the situation is entirely different. Matache [8] proved that there didn't exist compact composition operators on Hardy spaces of the upper half plane. Shapiro and Smith [11] also showed that there were no compact composition operators on Bergman spaces of the upper half plane. For some recent discussions of composition operators see [6, 8, 9, 10, 12, 13].

Once boundedness and compactness have been established, the next natural question one can ask about any composition operator on Bergman space of the upper half plane is: Is it weakly compact? Let X and Y be Banach spaces, $T : X \to Y$ be a bounded linear operator. Recall that $T : X \to Y$ is *weakly compact* if it maps bounded sets into relatively weakly compact sets. For some results in this topic see [2] and [4]. In this paper, we study the composition operators on Bergman spaces $A^p(\Pi_+)$ and prove that when $1 < p < \infty$, there are no weakly compact composition operators on Bergman space $A^p(\Pi_+)$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. Auxiliary results

To deal with the weak compactness of composition operator, we need the *pseudo-hyperbolic metric* on Π_+. Recall that for $w, z \in \Pi_+$ the pseudo-hyperbolic metric on Π_+ is defined by

$$\rho(z, w) = \frac{|z - w|}{|z - \overline{w}|}.$$

For $0 < r < 1$ and $w = x + iy \in \Pi_+$, let $D(w, r) = \{z \in \mathbb{C} : \rho(z, w) < r\}$ denote the pseudo-hyperbolic metric disk with center w and radius r. It is easy to see that $z \in D(w, r)$ if and only if $z \in B\left(\left(x, \frac{1 + r^2}{1 - r^2} y\right), \frac{2ry}{1 - r^2}\right)$, where $B(w, r)$ is the Euclidean disc. For pseudo-hyperbolic metric on Π_+ and the following lemma, see [7].

Lemma 2.1. For $r \in (0, 1)$, there is a sequence $(z_n)_{n\in\mathbb{N}}$ in Π_+ such that $\cup_{n=1}^{\infty} D(z_n, r) = \Pi_+$ and there is a natural number N such that each $z \in \Pi_+$ belongs to at most N of the sets $D(z_n, r)$.

The following lemma should be folklore but we give a proof for the completeness.

Lemma 2.2. Let $p > 1$ and $0 < r < 1$. Then for any positive Borel measure μ on Π_+, the following conditions are equivalent.

1. There is a positive constant C_1 such that

$$\int_{\Pi_+} |f(z)|^p d\mu(z) \leq C_1 \int_{\Pi_+} |f(z)|^p dA(z) \text{ for all } f \in A^p(\Pi_+);$$

2. There is a positive constant C_2 such that

$$\mu(D(z,r)) \leq C_2(Imz)^2 \text{ for all } z \in \Pi_+.$$

Proof. Assume that condition (a) holds. For $w \in \Pi_+$, setting

$$f_w(z) = \frac{1}{(z-\overline{w})^{4/p}}, \quad z \in \Pi_+,$$

we have

$$\int_{\Pi_+} |f_w(z)|^p dA(z) = \frac{\pi}{4(Imw)^2}.$$

By an easy calculation, we have

$$\begin{aligned} \frac{C_1\pi}{4(Imw)^2} &= C_1 \int_{\Pi_+} |f_w(z)|^p dA(z) \geq \int_{\Pi_+} |f_w(z)|^p d\mu(z) \geq \int_{D(w,r)} |f_w(z)|^p d\mu(z) \\ &\geq \inf\{|f_w(z)| : z \in D(w,r)\}\mu(D(w,r)) = \frac{(1-r)^4}{16(Imw)^4}\mu(D(w,r)). \end{aligned}$$

Then $\mu(D(w,r)) \leq C_2(Imw)^2$ and this shows that condition (b) holds.

Nextly, we assume that condition (b) holds. For each $f \in A^p(\Pi_+)$, by Lemma 2.1 and $|D(z,r)| \asymp (Imz)^2$ it follows that

$$\begin{aligned} \int_{\Pi_+} |f(z)|^p d\mu(z) &\leq \sum_{n=1}^{\infty} \int_{D(z_n,r)} |f(z)|^p d\mu(z) \\ &\leq \sum_{n=1}^{\infty} \sup\{|f(z)|^p : z \in D(z_n,r)\}\mu(D(z_n,r)) \\ &\leq C \sum_{n=1}^{\infty} \frac{\mu(D(z_n,r))}{|D(z_n,r)|} \int_{D(z_n,\frac{2r+1}{3})} |f(z)|^p dA(z) \\ &\leq CN \sup\left\{\frac{\mu(D(z_n,r))}{|D(z_n,r)|} : z \in \Pi\right\} \int_{\Pi_+} |f(z)|^p dA(z) \\ &\leq C_1 \int_{\Pi_+} |f(z)|^p dA(z), \end{aligned}$$

from which condition (a) holds. ∎

For each $f \in A^p(\Pi_+)$, we have

$$\|C_\varphi f\|^p_{A^p(\Pi_+)} = \int_{\Pi_+} |f(\varphi(z))|^p dA(z) = \int_{\Pi_+} |f(z)|^p dA \circ \varphi^{-1}(z),$$

and denote $A \circ \varphi^{-1}$ by μ_φ. We can similarly prove the following lemma, which is the little oh version of Lemma 2.2.

Lemma 2.3. Suppose $p \geq 1$ and $0 < r < 1$, then for any positive Borel measure μ on Π_+, the following conditions are equivalent.

1. The identity map is compact from $A^p(\Pi_+)$ to $L^p(\Pi_+, d\mu)$;
2. The limit
$$\lim_{Imz \to 0} \frac{\mu(D(z,r))}{(Imz)^2} = 0.$$

To prove the main results, we also need the following important lemma.

Lemma 2.4. ([2, Proposition 1]) Let X, Y, Z be Banach spaces and let $T : X \to Y$, $S : X \to Z$ be bounded operators such that $\|Sx\| \leq \|Tx\|$. Suppose that there are two linear topologies τ_1 on X and τ_2 on Y such that T is $\tau_1 - \tau_2$ continuous, (B_X, τ_1) is metrizable and compact and the weak topology of Y is finer than τ_2. If T is weakly compact, then so is S.

3. Main results

Here we formulate and prove the main results of this paper.

Theoremthm3.1.Let $p > 1$ and the operator $C_\varphi : A^p(\Pi_+) \to A^p(\Pi_+)$ be bounded. Then the following statements are equivalent:

1. $C_\varphi : A^p(\Pi_+) \to A^p(\Pi_+)$ is compact;
2. $C_\varphi : A^p(\Pi_+) \to A^p(\Pi_+)$ is weakly compact;
3. $\lim_{Imz \to 0} \frac{\mu_\varphi(D(z,r))}{(Imz)^2} = 0.$

Proof. It is obvious that (i) implies (ii). Now we want to prove that (ii) implies (iii). Suppose that the operator $C_\varphi : A^p(\Pi_+) \to A^p(\Pi_+)$ is weakly compact.

Let τ_1 the topology of uniform convergence on compact subsets of Π_+, τ_2 the topology of the pointwise convergence, $X = Y = A^p(\Pi_+)$, $Z = L^p(\Pi_+, \mu_\varphi)$ and $S : A^p(\Pi_+) \to L^p(\Pi_+, \mu_\varphi)$ given by $f \mapsto f$. Then by Lemma 2.4 it follows that $S : A^p(\Pi_+) \to L^p(\Pi_+, \mu_\varphi)$ is weakly compact.

Suppose to the contrary that (iii) is not true. Then there exists $\varepsilon_0 > 0$ and $(z_n)_{n\in\mathbb{N}} \subseteq \Pi_+$ with $Imz_n \to 0$ such that $\mu_\varphi(D(z_n, r)) \geq \varepsilon_0(Imz)^2$. For each $n \in \mathbb{N}$, defining

$$f_n(z) = \frac{1}{(z - \overline{z}_n)^{\frac{4}{p}}},$$

we have $f_n \in A^p(\Pi_+)$ and $\|f_n\|_{A^p(\Pi_+)} = \pi/4(\mathrm{Im}z_n)^2$. Taking $g_n = f_n/\|f_n\|_{A^p(\Pi_+)}$, $n \in \mathbb{N}$, we need prove that for each subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ the sequence $(Sg_{n_k})_{k\in\mathbb{N}}$ is not weakly convergent in $L^p(\Pi_+, \mu_\varphi)$. By [1, p.137], it will be enough to show that $(Sg_{n_k})_{k\in\mathbb{N}}$ is not uniformly integrable, i.e., there exists $\varepsilon > 0$ such that for every $\eta > 0$ there is a measurable subset A of Π_+ such that $\mu_\varphi(A) \leq \eta$ and $\int_A |g_{n_k}|^p d\mu_\varphi \geq \varepsilon$. Take $\varepsilon = \varepsilon_0$ and fix an arbitrary η. Since μ_φ is a Carleson measure, there is a positive constant C such that $\mu_\varphi(D(z, r)) \leq C(Imz)^2$ for all $z \in \Pi_+$. Since $Imz_n \to 0$ as $n \to \infty$, we can choose $k \in \mathbb{N}$ such that $\mu_\varphi(D(z_{n_k}, r)) \leq \eta$.

On the other hand, for $z \in D(z_{n_k}, r)$, we have $|f_{n_k}(z)|^p \geq (Imz_{n_k})^{-4}$. Thus we get

$$\begin{aligned}\int_{D(z_{n_k},r)} |g_{n_k}(z)|^p d\mu_\varphi(z) &\geq \frac{(Imz_{n_k})^{-4}}{\|f_{n_k}\|_{A^p(\Pi_+)}}\mu_\varphi(D(z_{n_k}, r)) \geq \frac{(Imz_{n_k})^{-4}}{\|f_{n_k}\|_{A^p(\Pi_+)}}(Imz_{n_k})^2\varepsilon_0 \\ &= \frac{4}{\pi}\varepsilon_0.\end{aligned}$$

Nextly, we will prove that (iii) implies (i). Suppose that $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence by a positive constant M in $A^p(\Pi_+)$ and $f_n \to 0$ uniformly on every compact subset of Π_+ as $n \to \infty$. It is enough to show $\|C_\varphi f_n\|_{A^p(\Pi_+)} \to 0$ as $n \to \infty$. Fix a sequence $(z_n)_{n\in\mathbb{N}}$ in Lemma 2.1. Since $|D(z_n, r)| \asymp (Imz_n)^2$ and condition (iii) holds, then

$$\lim_{n\to\infty} \frac{\mu_\varphi(D(z_n, r))}{|D(z_n, r)|} = 0.$$

Given $\varepsilon > 0$, there is a positive integer N_0 such that for each $n \geq N_0$ it follows that

$$\lim_{n\to\infty} \frac{\mu_\varphi(D(z_n, r))}{|D(z_n, r)|} < \varepsilon.$$

By the proof of Lemma 2.2 there is a positive constant C such that for all $k \geq 1$ we have that

$$\begin{aligned}\sum_{n=N_0}^{\infty} \int_{D(z_n,r)} |f_k(z)|^p d\mu_\varphi(z) &\leq C \sum_{n=1}^{\infty} \frac{\mu(D(z_n, r))}{|D(z_n, r)|} \int_{D(z_n,\frac{2r+1}{3})} |f_k(z)|^p dA(z) \\ &\leq \varepsilon C \int_{D(z_n,\frac{2r+1}{3})} |f_k(z)|^p dA(z) \\ &\leq \varepsilon CN \int_{\Pi_+} |f_k(z)|^p dA(z) \\ &\leq \varepsilon CNM.\end{aligned}$$

Since

$$\lim_{k\to\infty}\sum_{n=1}^{N_0-1}\int_{D(z_n,r)}|f_k(z)|^p d\mu_\varphi(z)=0,$$

by uniform convergence we have

$$\begin{aligned}\limsup_{k\to\infty}\int_{\Pi_+}|f_k|^p d\mu_\varphi &\le \limsup_{k\to\infty}\Big(\sum_{n=1}^{N_0-1}\int_{D(z_n,r)}|f_k|^p d\mu_\varphi+\sum_{n=N_0}^{\infty}\int_{D(z_n,r)}|f_k|^p d\mu_\varphi\Big)\\ &\le \varepsilon CNM.\end{aligned}$$

Because ε is arbitrary, we have

$$\lim_{k\to\infty}\int_{\Pi_+}|f_k(z)|^p d\mu_\varphi(z)=0,$$

and hence $C_\varphi : A^p(\Pi_+) \to A^p(\Pi_+)$ is compact. ∎

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