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Uniform Continuity Characterization of Compact or Discrete Topological Groups

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Abstract

In this paper we shall show that if G is a metrizable topological group, then the space of bounded real valued continuous functions on G is equal to the space of left uniformly continuous functions on G if and only if G is totally bounded or discrete.

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1. Introduction

Let $C(G)$ be the space of bounded real valued continuous functions on G and $WAP(G)$ be the space of weakly almost periodic functions on G. For a locally compact topological

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group G, [1] proved that $C(G) = WAP(G)$ if and only if G is compact and [2] improved the result to $UC(G) = WAP(G)$ if and only if G is compact, where $UC(G)$ is the space of uniformly continuous functions on G. In [4] it was shown that $LUC(G) = C(G)$ if and only if G is compact or discrete, where $LUC(G)$ is the space of left uniformly continuous functions on G.

Definition 1.1. Let G be a group and also a topological space. Suppose that

1. the mapping $(x, y) \longrightarrow xy$ of $G \times G$ onto G be continuous and
2. the mapping $x \longrightarrow x^{-1}$ of G onto G be continuous,

then G is called a topological group.

Note 1.2. The above conditions can be replaced by requiring that $(x, y) \longrightarrow xy^{-1}$ be continuous.

Definition 1.3. A function $f : G \longrightarrow \mathbb{C}$ is left uniformly continuous, if for all $\epsilon > 0$, there is W a neighbourhood of e, the identity of G, such that

$$x^{-1}y \in W \Rightarrow |f(x) - f(y)| < \epsilon, \quad \forall x, y \in G.$$

Definition 1.4. A topological group G is said to be totally bounded if for every neighbourhood V of the identity $e \in G$, there exists a finite set

$$F = \{x_1, x_2, \ldots, x_n\} \subset G$$

such that

$$G \subset \bigcup_{k=1}^{n} V x_k.$$

Definition 1.5. A topological group is paracompact if every open covering $\mathbb{A}$ of G has a locally finite refinement $\mathcal{B}$ that covers G.

Theorem 1.6. [3] Let G be a T_0 topological group. Then G is metrizable if and only if there is a countable open basis at e.

Theorem 1.7. If G is a metrizable topological group. Then $C(G) = LUC(G)$ if and only if G is totally bounded or discrete.

Proof. If G is discrete, the neighbourhood of the identity $V = \{e\}$. So $yx^{-1} \in V \Longleftrightarrow x = y$.

Thus $|f(x) - f(y)| < \epsilon$ for all $x, y \in G$. If G is totally bounded. Let $x \in G$ then $f(x) \in \mathbb{C}$, let $B_x := B(f(x), \epsilon/2)$. By the continuity of f, $f^{-1}(B_x) = V_x$ is open in G and is a neighbourhood of x. Then $x^{-1}V_x$ is a neighbourhood of the identity $e \in G$. Since G is totally bounded there exists $F = \{x_1, x_2, \ldots, x_n\} \subset G$ such that

$$G \subset \bigcup_{k=1}^{n} x^{-1} V_x x_k.$$

Take W a neighbourhood of e such that $Wx \subset V_x \subset x^{-1}V_xV_m$ for some $m : 1 \leq m \leq n$.

Then $f(x), f(y) \in B_x$for all $x, y : yx^{-1} \in W$. Hence $|f(x) - f(y)| < \epsilon$. Conversely suppose G is not totally bounded then there is V a neighbourhood of e with $V = V^{-1}$ such that no finite number of translates of V cover G.

Choose $\{x_n\} \subset G$ such that $x_n \notin V^{-1}(V^2x_1 \cup \ldots \cup V^2x_{n-1})$, that is, $V^2X_n \cap V^2x_m = \emptyset$ if and only if $m \neq n$. There exists a sequence $\{V_n\}$ of neighbourhoods of e such that $V_n \subset V_{n-1} \subset \ldots \subset V$ and $\bigcap_{n=1}^{\infty} V_n = \{e\}$ with $V_n^3 \subset V$ and $V_nx_n \cap V_mx_m = \emptyset$ if $m \neq n$, since G is metrizable.

There is a continuous function $f_n : G \longrightarrow \mathbb{R} : f_n(G) \subset [0, 1]$ and $f(x_n) = 1$, $f_n(G \setminus V_nx_n) = 0$.

Consider $f := \sum_{n=1}^{\infty} f_n$. Fix $x \in G$ and note that Vx is a neighbourhood of x. If $Vx \cap Vx_k \neq \emptyset$ for all $k \in \mathbb{N}$, then $f = 0$ on Vx and f is continuous at x.

Suppose $Vx \cap Vx_k \neq \emptyset$ for some $k \in \mathbb{N}$. Then $x \in V^{-1}Vx_k = V^2x_k$. Hence $x \notin V^2x_n$, for all $n \neq k$. Therefore $Vx \cap Vx_n = \emptyset$, for all n with $n \neq k$.

Thus $f = f_k$ on Vx. Hence f continuous at x and $f(x) \in [0, 1]$. So $f \in LUC(G)$, that is, there exists W a neighbourhood of e such that $yx^{-1} \Longrightarrow |f(x) - f(y)| < 1/2$ and $W \subset V$.

Fix $w \in W$ and note that $(wx_n)x_n^{-1} = w \in W$ and so $1/2 > |f(x_n) - f(wx_n)| = |1 - f_n(wx_n)|$, hence $f(wx_n) > 1/2$. Therefore $wx_n \in V_nx_n$, so $w \in V_n$.

Thus $W \subset \bigcap_{n=1}^{\infty} V_n = \{e\}$. Thus $W = \{e\}$ hence G is discrete. If G is not discrete then $W = \{e\}$ is a contradiction, hence G totally bounded. ■

Remark 1.8. The concept of total boundedness is very different from compactness.

Let us consider the following example.

Example 1.9. Let $W = \{z \in : |z| = 1\}$ with multiplication. If we take $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, θ being rational, the group is totally bounded but not compact.

Since every metrizable space is paracompact [5], it follows that the above theorem can be rewritten as follows: If a topological group G is paracompact then $C(G) = LUC(G)$ if and only if G is totally bounded or discrete.

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