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Magic Squares as a Field

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Abstract:

Some advanced mathematical properties of semi magic squares are discussed in this paper.

Key Words: Magic Square, Magic Constant, Abelian Group, Commutative Ring, Field

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1. *Introduction*:

A magic square of order 'n' is an n^{th} order matrix such that the sum of elements in every row/column/diagonal remains the same. The common sum is known as 'magic constant' or 'magic number'. 'Cornelius Agrippa' (1486 B.C. to 1535 B.C.) of China is believed to be the first to take up construction of magic squares. There it was called 'Loh Shu'. Interest in magic squares spread from China to Japan, India and the middle East. They were introduced to Europe in Byzantine times. The first magic square of order 4 in the first century was introduced in India by a mathematician named 'Nagarjuna'. In real life situations, some problems relating to division of objects equal in numbers and value can be easily solved by constructing a magic square in accordance with the given conditions.

Apart from the recreational aspect of magic squares, it is found that they posses several advanced mathematical properties. A few among them are discussed here. The main aim is to construct a field from a proper subset of the set of all magic squares.

2. *Notations and Mathematical Preliminaries*

(Most of the concepts in this section are taken from **[1]**, **[3]**)

2.1 *Magic Square*:

A magic square of order 'n' is an n^{th} order matrix $[a_{ij}]$ such that

$\sum_{j=1}^{n} a_{ij} = k$, for $i = 1,2,3,\dots\dots\dots,n \longrightarrow$ (1)

$\sum_{j=1}^{n} a_{ji} = k$, for $i = 1,2,3,\dots\dots\dots n \longrightarrow$ (2)

$\sum_{i=1}^{n} a_{ii} = k$ and $\sum_{i=1}^{n} a_{i,n-i+1} = k, \longrightarrow$ (3)

Where $'k'$ is a constant and the above mentioned a_{ij}'s and a_{ji}'s are the row and column elements and a_{ii}'s & $a_{i,n-i+1}$'s are the left and right diagonal elements of the magic square respectively.

2.2 *Magic Constant*:

The constant $'k'$ in the above definition is known as the magic constant or magic number. Magic constant of the magic square A is denoted as $\rho(A)$.

For example, the below given magic squares A *and* A' *are of order* 3 *and* $\rho(A) = 21$ & $\rho(A') = 15$

$A =$

4	9	8
11	7	3
6	5	10

$A' =$

4	3	8
9	5	1
2	7	6

Here sum of elements of each row/column = 30. Sum of the left diagonal elements = 8 + 2 + 14 = 24 and sum of the right diagonal elements = 12.

2.3 *Group*:

A nonempty set G together with an operation * is known as a group if it satisfy the following properties

(i) G is closed with respect to *. i.e., $a * b \in G, \forall\, a, b \in G$.$(ii)$ * is associative in G. i.e., $a * (b * c) = (a * b) * c\ \forall\, a, b, c \in G$. (iii) $\exists\, e \in G$, such that $e * a = a * e = a, \forall\, a \in G$. Here e is called the 'identity element' in G with respect to *.

(iv) $\forall\, a \in G, \exists\, b \in G$ such that $a * b = b * a = e$, where $'e'$ is the identity element. Here b is called the 'inverse of a and similarly vice versa. The inverse of the element a is denoted as a^{-1}.

Note: If G is a group with respect to *, it is denoted as $\langle G,*\rangle$

2.4 *Ring*:

A non – empty set R together with two binary operations called 'addition' and 'multiplication' denoted by '+' and '**.**' respectively is called a Ring, if the following postulates are satisfied.

(i) $\langle R, +\rangle$ is an abelian group.(If $\langle G,*\rangle$ is an abelian group, then $a * b = b * a, \forall a, b \in G$).

(ii) Multiplication is associative, i.e., $a.(b.c) = (a.b).c\ \forall a, b, c \in R$. (iii) Multiplication is distributive with respect to addition, i.e., $\forall a, b, c \in R$, $a.(b + c) =$

$a.b + a.c$ (Left distributive law) and $(b + c).a = b.a + c.a$ (Right distributive law).

2.5 *Ring with unity*:
If in a ring R, $\exists$ an element denoted by 1, such that $1.a = a.1 = a, \forall a \in R$, then R is called a ring with unit element. Here $'1'$ is called the unit element of the ring and is obviously the multiplicative identity. If a ring possesses multiplicative identity, then it is called a ring with unity.

2.6 *Commutative ring*:
If in a ring R, the multiplication is also commutative, i.e., if $a.b = b.a, \forall a, b \in R$, then R is called a commutative ring.

2.7 *Field*:
A ring R with at least two elements is called a field if it, (i) is commutative (ii) has unity (iii) is such that each non zero element possesses multiplicative inverse.

2.8:
We use $(i)\mathfrak{R}$ to denote the set of all real numbers. (ii) M_S to denote the set of all n^{th} order magic squares.(iii) S_{J_a} to denote the set of all n^{th} order magic squares of the form $[a_{ij}]$, where $a_{ij} = a$, for $i, j = 1,2,3, \dots \dots, n, a \in \mathfrak{R}. \rho([a_{ij}]) = na$. For convenience, $A \in S_{J_a}$ is represented as $[a]$. (iv) I_n to denote the identity matrix of order n.

3. Propositions and Theorems

***Theorem* 3.1**: *S_{J_a} forms an abelian group with respect to matrix addition.*

Proof: We know that M_S is an abelian group with respect to matrix addition (by [2]) and $S_{J_a} \subset M_S$, we need to show that S_{J_a} is a subgroup of M_S. For that it is enough to show that $A - B \in S_{J_a}$, whenever $A, B \in S_{J_a}$. Let $A = [a]$ and $B = [b] \in S_{J_a}$. Then, by simple calculations it can be shown that $A - B = [a - b]$. Hence $A - B \in S_{J_a}$. This completes the proof.

***Proposition* 3.1:** *If $A, B \in S_{J_a}$, then $A.B \in S_{J_a}$ and $A.B = B.A$; where $'.'$ denotes matrix multiplication.*

Proof: Since $A, B \in S_{J_a}$, they will be of the form $A = [a]$ & $B = [b]$ $a, b \in \mathfrak{R}$. Then it can be shown that $A.B = [nab] = [nba] = B.A$. Hence $A.B \in S_{J_a}$ and matrix multiplication is commutative in S_{J_a}.

***Theorem* 3.2:** *$\langle S_{J_a}, +, . \rangle$ forms a commutative ring with unity, where $' + '$ and $'.'$ denote the addition and multiplication of matrices respectively.*

Proof: Multiplication of matrices is associative and distributive over addition. Usually for square matrices I_n will act as the unity element under matrix multiplication. But unfortunately here, $I_n \notin S_{J_a}$. So we have to find another option. We need an element E in S_{J_a} such that $A.E = E.A = A, \forall A = [a] \in S_{J_a}, a \in \Re$. It can be shown that $E = \left[\frac{1}{n}\right] \in S_{J_a}$ and $A.E = [a].\left[\frac{1}{n}\right] = \left[\frac{1}{n}\right].[a] = E.A = A, \forall A \in S_{J_a}$. i.e., $E = \left[\frac{1}{n}\right] \in S_{J_a}$ will act as the unity element. The rest of the proof will follow from Theorem 3.1 and Proposition 3.1.

***Proposition* 3.2**: *If $A \neq 0 \in S_{J_a}$, then A has a multiplicative inverse in S_{J_a}.*

Proof: $A \neq 0 \in S_{J_a}$. Let $A = [a]$. We have to show that A has a multiplicative inverse. i.e., we need an element B in S_{J_a} such that $A.B = B.A = E = \left[\frac{1}{n}\right]$. Let $B = \left[\frac{1}{n^2a}\right]$. Then it can be verified that $B \in S_{J_a}$ and $A.B = [a].\left[\frac{1}{n^2a}\right] = \left[\frac{1}{n^2a}\right].[a] = B.A = \left[\frac{1}{n}\right] = E$. This completes the proof.

***Theorem* 3.3:** *$\langle S_{J_a}, +, . \rangle$ forms a field, where $'+'$ and $'.'$ denote the addition and multiplication of matrices respectively.*

Proof: It immediately follows from Theorem 3.2 and Proposition 3.2. Proposition 3.2 shows that $\forall A \in \boldsymbol{S_{J_a}}$, $\exists$ an inverse under matrix multiplication.

4. Conclusion

Here we have seen some important properties of the subset S_{J_a} of M_S under the operations of matrix addition and multiplication. It is proved that S_{J_a} is an abelian group under matrix addition. S_{J_a} is a commutative ring with unity under matrix addition and multiplication. Finally, it is showed that S_{J_a} is a field under the same operations.

References

[1] W.S. Andrews, *Magic Squares and Cubes* – second edition, Dover Publications, Inc. 180 Varick Street, New York 14, N.Y. (1960)

[2] Sreeranjini K.S, V.Madhukar Mallayya, “Some Properties of Magic Squares”, “*International Journal of Algebra and Statistics*” ISSN: 2314 - 4556, Vol.1, No.2, 2012, pp. 63 – 67, Modern Science Publishers, India

[3] A. R. Vasishtha and A. K. Vasishtha, - *Modern Algebra* - fiftieth edition, Krishna Prakashan Media (P) Ltd., H.O. – 11, Shivaji Road, Meerut - 250 001 (U.P.), 2006