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A simultaneous generalization of normality and paracompactness and some remarks on θ-regular spaces

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Abstract

We give new characterizations of normality, introduce a simultaneous generalization of normality and paracompactness and use it to give a necessary and sufficient condition for a regular space to be normal.

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1. Introduction and Preliminaries

It is well known that every regular space which is either paracompact or Lindelöf is normal ([1], [3]). Kohli and Das in [6] had introduced their concept of a θ-regular space and generalized the above results to the statements that every θ-regular space which is either paracompact or Lindelöf is normal (Theorems 1.3 and 1.4 below). These results also follow from our Theorems 2.5 and 2.9 below. Das [2] has used the notion of θ-regularity to give other simultaneous generalizations of regularity and normality. Among others, the following results are in [2].

Theorem 1.1. [c.f., Theorems 2 and 4 [2]] A space is T_3 if and only if it is θ-regular and $T_{1.}$.

Theorem 1.2. [c.f., Theorems 1 and 3 [2]] A θ-regular space is T_2 if and only if it is T_1.

Theorem 1.3. [Theorem 7 [2]] Every paracompact θ-regular space is normal.

Theorem 1.4. [Theorem 8 [2]] Every Lindelöf θ-regular space is normal.

On the other hand in order to find the most general conditions in which particular kinds of compact sets are closed in normal and regular spaces ([4], [5]), this author introduced the concepts of normal regularity and normally S_2 in [7] which are yet another simultaneous generalizations of normal and regular spaces. The following results are pertinent:

Theorem 1.5. [c.f., Lemma 2.1 [7]] Every normal or a regular space is normally regular and every S_2 or a normally regular space is normally S_2.

The following result should be compared with Theorem 1.1 above. Whereas the latter gives us a decomposition of T_3 spaces, the following theorem gives the decomposition of regular spaces. It is however to be noted that for S_1 spaces the concepts of θ-regularity and normal regularity coincide (Theorem 2.2 below).

Theorem 1.6. [Theorem 2.6 [7]] A space X is regular if and only if X is normally regular and S_1.

Theorem 1.7. [Theorem 2.1 [7]] For a space X the following conditions are equivalent:

(a) X is normally regular,

(b) for each x $\in$ X and open neighborhood U of Cl{x}, there exists an open neighborhood V of x such that x $\in$ V $\subset$ Cl (V) $\subset$ U,

(c) for each x $\in$ X and closed set F such that $Cl\{x\} \cap F = \phi$, there is an open neighborhood V of x such that $Cl(V) \cap F = \phi$.

(d) for each closed set F and for any family $\{G_\alpha\}_\alpha$ of open sets covering F, there exists a family of open sets $\{H_\beta\}_\beta$ covering F such that for each β, $H_\beta \subset G_\alpha$ for some α and $\cup_\beta Cl(H_\beta) \subset \cup_\alpha G_\alpha$.

In this paper we (i) obtain the Theorems 1.1 to 1.4 above as Corollaries to more general results, (ii) obtain a new characterization of normal spaces (Theorem 2.9 below), (iii) introduce a simultaneous generalization of normality and paracompactness (Definition 2.1 below) and use it to (iv) obtain another decomposition of normality (Theorem 2.11 below), besides providing us with a necessary and sufficient condition for a regular space to be normal (Theorem 2.15 below).

By a space we shall mean a topological space in which no separation axiom is assumed unless mentioned explicitly. In a space X, Cl(A) and A^C will denote the closure and complement of the subset A of X respectively.

A space is said to be

(i) **S_2 (S_1)** [1] if for every pair of points x, y in X, whenever one of them has a neighborhood not containing the other, then x and y have disjoint neighborhoods (the other has a similar neighborhood as well). A T_2 or a regular space is S_2 and an S_2 or a T_1 space is S_1 [1],

(ii) **S_4** [1] if it is normal and S_1. Every S_4 (T_4) space is regular (T_3) [1],

(iii) **normally regular** [7] if for any point x and any closed set F in X, $Cl\{x\} \cap F = \phi$ implies that x and F can be separated by disjoint open sets in X,

(iv) **normally S_2** [7] if any two points of X having disjoint closures can be separated by disjoint open sets in X. Every normally regular space is normally S_2 [7],

(v) **θ-regular** [6] if for each closed set F and for each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$, where a subset A of a space is **θ-open** if for each x in A there is an open set U such that $x \in U \subset Cl(U) \subset A$ [Lemma 2.5 [6]].

2. Results

We begin with some

A. Some Remarks on θ-regular spaces

Theorem 2.1. Every θ-regular space is normally regular.

Proof. Let X be a θ-regular space. Let U be any open set containing $Cl\{x\}$ for a point x in X. Then by θ-regularity there is a θ-open set V such that $Cl\{x\} \subset V \subset U$. Since V is θ-open there is an open set G such that $x \in G \subset Cl(G) \subset V$. This implies the normal regularity of X by the equivalence of (a) and (b) in Theorem 1.7 above. ■

We have now another characterization of θ-regularity of S_1 (and therefore of T_1) spaces:

Theorem 2.2. An S_1 (and therefore a T_1) space is θ-regular if and only if it is normally regular.

Proof. Since every regular space θ-regular the result follows from Theorems 1.6 and 2.1 above. ■

Theorem 1.1 immediately follows from this theorem and from the characterization of regular spaces in Theorem 1.6:

Corollary 2.3. Theorem 1.1 above.

It is important to observe that Theorem 1.2 is merely a very special case of Theorem 1.1 and unlike the latter it does not give a decomposition of T_2 spaces, since a T_2 space need not be θ-regular. It begs us to give a necessary and sufficient condition for a T_1 space to be T_2. Since a T_2 space need not be θ-regular (take any T_2 space which is not regular) but since every θ-regular space is normally regular and therefore normally S_2, the following theorem is a significant improvement on Theorem 1.2 above:

Theorem 2.4. [c.f., Theorem 2.7 [7]] A space X is $S_2(T_2)$ if and only if X is normally S_2 and $S_1(T_1)$.

Since every T_1 space is S_1 and every θ-regular space is normally regular and therefore normally S_2, we have

Corollary 2.5. Theorem 1.2 above.

We now state the following

Theorem 2.6. [Theorem 2.2 [7]]. A Lindelöf space X is normal if and only if X is normally regular.

As immediate Corollaries of this result we have the following

Corollary 2.7. Theorem 1.4 above.

Corollary 2.8. A Lindelöf space is θ-regular if and only if it is normally regular.

As our final remark we note that all the conditions obtained for the closedness for a compact (countably compact) sets in normal, regular or normally regular space in [4], [5] or [7] are valid for θ-regular spaces.

B. Characterizations of normality and a simultaneous generalization of normality and paracompactness.

In A above we notice that by weakening θ-regularity in Theorem 1.2 to normally S_2 we are able to obtain decomposition of T_2 spaces in Theorem 2.4 above. We now seek to weaken paracompactness in Theorem 1.3 to obtain similarly a decomposition of normality by an axiom weaker than both paracompactness and normality. We achieve this decomposition in Theorem 2.11 below with the help of the concept of paranormality introduced in Definition 2.1 below. Towards this purpose we are motivated by the equivalence of (a) and (d) in Theorem 1.7. It is to be observed that in the condition (d) of Theorem 1.7, the refinement $\{H_\beta\}_\beta$ of $\{G_\alpha\}_\alpha$ **cannot** be taken as a precise refinement, since then the condition will imply normality which is in fact is not true since every regular space is normally regular. In case of normality however we should expect such a precision. This is in fact true as shown by the following:

Theorem 2.9. For a space X the following conditions are equivalent:

(a) X is normal.

(b) if F is any closed subset of X and $\{G_\alpha : \alpha \in \mathcal{A}\}$ is any family of open sets containing F, then there exists an open refinement $\{H_\beta : \beta \in \mathcal{B}\}$ of $\{G_\alpha : \alpha \in \mathcal{A}\}$ such that $F \subset \cup_\beta H_\beta \subset Cl(\cup_\beta H_\beta) \subset \cup_\alpha G_\alpha$.

(c) if F is any closed subset of X and $\{G_\alpha : \alpha \in \mathcal{A}\}$ is any family of open sets containing F, then there exists a precise open refinement $\{H_\alpha : \alpha \in \mathcal{A}\}$ of $\{G_\alpha : \alpha \in \mathcal{A}\}$ such that $F \subset \cup_\alpha H_\alpha \subset Cl(\cup_\alpha H_\alpha) \subset \cup_\alpha G_\alpha$.

Proof. (a) ⇒ (b): By taking $\cup_\alpha G_\alpha = U$ to be an open subset of X containing F, since X is normal, there exists an open subset H of X containing F such that $F \subset H \subset Cl(H) \subset U$. Let $H_\alpha = H \cap G_\alpha$. Then $\{H_\alpha\}_\alpha$ is an open refinement of $\{G_\alpha\}_\alpha$ such that $F \subset H = \cup_\alpha H_\alpha \subset Cl(\cup_\alpha H_\alpha) = Cl(H) \subset U = \cup_\alpha G_\alpha$ and so (b) holds.

(b) ⇒ (c): Let F be any closed subset of X and $\{G_\alpha : \alpha \in \mathcal{A}\}$ be any open covering of F. By (b), there exists an open refinement $\{V_\beta : \beta \in \mathcal{B}\}$ of $\{G_\alpha : \alpha \in \mathcal{A}\}$ such that $F \subset \cup_\beta V_\beta \subset Cl(\cup_\beta V_\beta) \subset \cup_\alpha G_\alpha$. Let $H_\alpha = \cup_\beta\{V_\beta : V_\beta \subset G_\alpha\}$. Then for each α, $H_\alpha \subset G_\alpha$ and $F \subset \cup_\beta V_\beta = \cup_\alpha H_\alpha \subset Cl(\cup_\alpha H_\alpha) = Cl(\cup_\beta V_\beta) \subset \cup_\alpha G_\alpha$. Hence $\{H_\alpha\}_\alpha$ is a precise open refinement of $\{G_\alpha\}_\alpha$ satisfying the condition (c).

(c) ⇒ (a): obvious. ■

This Theorem motivates the following

Definitions 2.10. (a) In a topological space X, a family of sets $\{B_\beta : \beta \in \mathcal{B}\}$ will be called a **paranormal refinement** of a family of sets $\{A_\alpha : \alpha \in \mathcal{A}\}$ covering a set S in X if for each β, $B_\beta \subset A_\alpha$ for some α and $S \subset \cup_\beta B_\beta \subset Cl(\cup_\beta B_\beta) \subset \cup_\alpha Cl(A_\alpha)$. If each set B_β is open in X the refinement will be called an open paranormal refinement.

(b) A topological space X will be called a **paranormal space** if for each closed set F, every family of open sets covering F has an open paranormal refinement.

Lemma 2.11. If an open cover of a closed set F in a space X has an open paranormal refinement then it has a precise open paranormal refinement.

Proof. The proof is similar to that of (b) ⇒ (c) in Theorem 2.9. ■

Remark 2.12. In view of the above lemma, in a paranormal space an open paranormal refinement of a family of open sets covering a closed set can always be taken to be a precise open paranormal refinement.

The following lemma is obvious from Definitions 2.1 and Theorem 2.9.

Remark 2.13. Every normal space is paranormal.

Following Theorem 2.10 justifies the name paranormal:

Theorem 2.14. Every paracompact space is paranormal.

Proof. Let X be a paracompact space and $\{U_\alpha : \alpha \in \mathcal{A}\}$ be any open cover of a closed subset F of X. Then the family $\{U_\alpha, F^C\}_{\alpha\in\mathcal{A}}$, being an open cover of the paracompact space X, has a precise locally finite open refinement $\{V_\alpha, V\}_{\alpha\in\mathcal{A}}$. Therefore, $V_\alpha \subset U_\alpha$, $V \subset F^C$ and so $F \subset \cup_\alpha V_\alpha \subset \mathrm{Cl}\,(\cup_\alpha V_\alpha) = (\cup_\alpha \mathrm{Cl}(V_\alpha)) \subset \cup_\alpha \mathrm{Cl}\,(U_\alpha)$. Thus $\{V_\alpha : \alpha \in \mathcal{A}\}$ is an open paranormal refinement of the open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of the closed set F of X and hence X is paranormal. ■

We now give a decomposition of normal spaces in terms of the weaker axioms of normal regularity and paranormality.

Theorem 2.15. For a space X the following conditions are equivalent:

(a) X is normal.

(b) X is paranormal and normally regular.

Proof. Only (b) implies (a) require proof. That proof which is straightforward from the equivalence of (a) and (c) in Theorem 2.9 and from the equivalence of (a) and (d) in Theorem 1.7 is omitted. ■

Since every paracompact space is paranormal and every θ-regular space is normally regular we have the following strengthening of Theorem 1.3 of Section 1.

Corollary 2.16. For a paranormal, in particular a paracompact space X the following are equivalent:

(a) X is θ-regular.

(b) X is normally regular.

(c) X is normal.

Since every S_4 (T_4) space is regular (T_3), we have the following decomposition of S_4 (T_4) spaces.

Corollary 2.17. For a space X, the following conditions are equivalent:

(a) X is S_4 (T_4).

(b) X is paranormal and regular (T_3).

From Theorem 2.2 we also have

Corollary 2.18. An S_1 (T_1) space is normal (T_4) if and only if it is paranormal and θ-regular.

We have thus obtained below a necessary and sufficient condition for a regular space to be normal:

Theorem 2.19. A regular space is normal if and only if it is paranormal.

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