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# A Note on Right Ideals of Ordered Semigroups

**Jaya Lalitha. G[1], Dr. Y. Sarala[2] and Dr. Madhusudhana Rao[3]**

[1]*Research Scholar, Department of Mathematics, KL University, A.P, India*
[2]*Faculty of Mathematics, Department of Mathematics, KL University, A.P, India*
[3]*Faculty of Mathematics, Department of Mathematics, VSR & NVR College, Tenali, AP*
*saralayella1970@gmail.com*

**Abstract**

The idea of maximal rightideals is initiated from ordered semigroups. Characterizations of maximal rightideals are discussed here. For an ordered semigroup Z, one and only one of the adopting four conditions is fulfilled. (1) Z is left simple. (2) R ($l$) ≠ Z, ∀ $l$ ∈ Z. (3) ∃ $l$ ∈ Z ∋ Z = R ($l$), $l$ ∉ ($l$Z], $l^2$ ∈ Ʊ = Z\{$l$, and Ʊ is the distinct maximal rightideal of Z. (4) Z\Ʊ = {$y$ ∈ Z / ($y$Z] = Z, Z\Ʊ is a subsemigroup of Z and Ʊ is the distinct maximal rightideal of Z.

**Keywords:** Maximal rightideal, right-simple and right-simple semigroup.

## 1. Preliminaries

An Ordered semigroup is an ordered set Z & at the same time a semigroup ∋ $l$ ≤ $m$ ⇒ $yl$ ≤ $ym$ and $ly$ ≤ $my$ ∀ $y$ ∈ $Z$.

In this paper, Z indicates an arbitrary ordered semigroup. Assume J be a nonempty subset of Z. Then the set {$y$ ∈ Z / $y$ ≤ $j$ for some $j$ ∈ J} is denoted by the notation (J]. For J = {$l$} we write ($l$] instead of ({$l$}]. We have J ⊆ (J] and A ⊆ B ⇒ (A]⊆ (B], for any non empty subset A, B of Z. [1], [2].

A nonempty subset R of Z is a rightideal of Z, if RZ ⊆ R and (R]⊆ R, [1]. The Intersection of all rightideals of Z accommodating a nonempty subset A of Z is the rightideal of Z generated by A. For A = {$l$, we denote by R($l$) the rightideal of Z generated by $l$ ($l$ ∈ Z). A rightideal R of Z is known to be proper if R ≠ Z. An ordered semigroup Z is known to be right simple if it does not accommodate proper rightideals [3].

Assume Z be an ordered semigroup with a zero element (i.e ∃ 0 ∈ Z : 0$y$ = $y$0 = 0, 0 ≤ $y$ ∀ $y$ ∈ Z). If Z = {0}, then Z is a right simple. If ∃ $l$ ∈ Z, $l$ ≠ 0, then {0} is a rightideal of Z, {0} ⊂ Z. Thus, an ordered semigroup Z with a zero element ∋ Z ≠

{0} is not right simple.

**Definition 1.1:** Assume (Z,., $\leq$) be an ordered semigroup. A subsemigroup K of Z is known to be right simple, if the ordered semigroup (K,., $\leq$) is right-simple. Assume R be a rightideal of Z. As $R^2 \subseteq RZ \subseteq R$, R is a subsemigroup of Z.

**Definition 1.2:** Assume (Z,., $\leq$) be an ordered semigroup. A rightideal R of Z is known to be right simple, if the ordered semigroup (R,., $\leq$) is right simple.

**Lemma 1.3:** For any $l \in Z$, $R(l) = (l \cup lZ] = (l] \cup (lZ]$. [1]

**Lemma 1.4:** R is right simple $\Leftrightarrow$ $(lR] = R$ for all $l \in Z$. [3]

**Lemma 1.5:** Assume $\{R_x : x \in A\}$ be a family of rightideals of Z. then $\cup_{x \in A}^{Rx}$ is a rightideal of Z and $\cap_{x \in A}^{Rx}$ is also a rightideals of Z if $\cap_{x \in A}^{Rx} \neq \emptyset$.

The Proof a Lemma 1.5 is a easy.

**Lemma 1.6:** Assume R be a rightideal of Z and K is a right simple subsemigroup of Z if $K \cap R \neq \emptyset$. Then $K \subseteq R$.

***Proof*:** Let $l \in K \cap R$. since K is right simple, $l \in K$, by Lemma 1.4, we have $(lK] = K$. Then we have $K = (lK] \subseteq (RK] \subseteq (RZ] \subseteq (R] = R$.

## 2. Maximal RightIdeals

**Definition 2.1:** Assume R be a proper rightideal of Z. R is known to be a maximal if T is a rightideal of Z $\ni R \subset T$, then T = Z.

**Theorem 2.2:** Assume R be a proper rightideal of Z. Then R is maximal $\Leftrightarrow$ (1) $Z \backslash R = \{l\}$ and $l^2 \in R$ or (2) $Z \backslash R$ accommodating $(lZ]$, $\forall\, l \in Z \backslash R$.

***Proof*:** Assume R be a maximal rightideal of Z. we consider the two cases.

(i) $\exists\, l \in Z \backslash R \ni (lZ] \subseteq R$. In this case, we prove that the property (1) holds. In fact $l^2 = l\, l \in lZ \subseteq (lZ] \subseteq R$. By $(lZ] \subseteq R$ and Lemma 1.3 we have $R \cup (l] = (R \cup (lZ]) \cup (l] = R \cup ((Z] \cup (l]) = R \cup (l \cup lZ] = R \cup R(l)$ Then $R \cup (l]$is a rightideal of Z, by Lemma 1.5 ; on the another hand, as $l \in Z \backslash R$. We have $R \subset R \cup (l]$, as $R \cup (l]$is a rightideal of Z and R is a maximal right ideal of Z, we have $R \cup (l] = Z$. Thus $Z \backslash R \subseteq (l]$. Assume $y \in Z \backslash R$. Then $y \leq l$ and so $(yZ] \subseteq (lZ] \subseteq R$. From $(yZ] \subseteq R$. $y \in Z \backslash R$, a similar argument shows that $Z \backslash R \subseteq (y]$. Consequently $l \in (y]$. i.e $l \leq y$. Hence $l = y$. Thus $Z \backslash R = \{l\}$.

(ii) $(lZ] \nsubseteq R\ \forall\, l \in Z \backslash R$. In this case, the property (2) holds. Indeed **:** Assume $l \in Z \backslash R$. as $(lZ]$is a rightideal of Z, by Lemma 1.5 we have that $R \cup (lZ]$is also a rightideal of Z. on the another hand, as $(lZ] \nsubseteq R$, we have $R \subset R \cup (lZ]$. As $R \cup (lZ]$is a rightideal of Z and R is a maximal ideal of Z we have $R \cup (lZ] = Z$. Thus $Z\backslash R \subseteq (lZ] \forall\, l \in Z\backslash R$. Assume R be a proper rightideal of Z and T be a rightideal of Z such that $R \subset T$.

(1) Assume $Z\backslash R = \{l\}$ and $l^2 \in R$. Then $Z = R \cup \{l\}$. as $R \subset T$. We have $T\backslash R \neq \emptyset$. On the another hand $T\backslash R \subseteq Z\backslash R = \{l\}$. Then we have $T\backslash R = \{l\}$ and $T = R \cup \{l\} = Z$. Thus R is maximal.

(2) Assume $Z\backslash R \subseteq (lZ] \forall\, l \in Z\backslash R$. Assume $t \in T\backslash R$. as $t \in Z\backslash R$, by hypothesis we have $Z\backslash R \subseteq (tZ] \subseteq (TZ] \subseteq (T] = T$. Hence $Z = L \cup (Z\backslash R) \subseteq T \cup T = T$ and so $T = Z$. Thus R is a maximal rightideal of Z. we indicate by $\mho$ the union of all rightideals of Z. In other words if $A = \{I \,/\, I$ is a proper rightideal of Z$\}$ then $\mho = U\{I \,/\, I \in R\}$.

**Remark:** $Z = \mho \iff Z \neq R\,(l)\ \forall\, l \in Z$. In fact, suppose that $\exists\, l \in Z \ni Z = R\,(l)$. As $l \in S = \mho = U\{I \,/\, I \in R\}$, it follows that $l \in I$ for some proper rightideal I of Z and $Z = R\,(l) \subseteq I$. Impossible (As I is a proper rightideal of Z). Assume $l \in Z$, by hypothesis, $R\,(l) \neq Z$. Then $R\,(l)$ is a proper rightideal of S such that $l$ belongs to $R\,(l)$. As $R(l) \in \mathbb{R}$, $l \in R(l)$, we have $l \in \mho$. Thus $Z = \mho$.

**Theorem 3.3:** For an ordered semigroup Z, one and only one of the adopting four conditions is fulfilled. (1) Z is left simple. (2) $R\,(l) \neq Z$, $\forall\, l \in Z$. (3) $\exists\, l \in Z \ni Z = R\,(l)$, $l \notin (lZ]$, $l^2 \in \mho = Z\backslash\{l\}$, and $\mho$ is the one and only maximal rightideal of Z. (4) $Z\backslash\mho = \{y \in Z \,/\, (yZ] = Z\}$, $Z\backslash\mho$ is a subsemigroup of Z and $\mho$ is the distinct maximal rightideal of Z.

***Proof***: If Z is not right simple, then $\exists$ a proper rightideal I of Z. As $I \neq \emptyset$, we have $\mho \neq \emptyset$. Then by Lemma 1.5, $\mho$ is a rightideal of Z.

(i) Assume $\mho = Z$. Then the condition (2) is satisfied.

(ii) Assume $\mho \neq Z$. In this case, either condition (3) or condition (4) is satisfied.

In fact, first of all, $\mho$ is the distinct maximal rightideal of Z. Indeed**:** Assume T be a rightideal of S such that $T \supset \mho$. Suppose that $T \neq Z$. Then T is a proper rightideal of Z and so $T \subseteq U\,\{I \,/\, I \in \mathbb{R}\} = \mho$, Impossible. Thus U is a maximal rightideal of Z. Assume K be a maximal rightideal of Z. Suppose that $K \neq \mho$.

($\alpha$) Assume $K \backslash \mho = \emptyset$. Then $K \subset \mho$ and so $K \subset \mho$. $\because$ $\mho$ is a rightideal of Z and K is a maximal rightideal of Z. we have U = Z Impossible.

($\beta$) Assume $K \backslash \mho \neq \emptyset$. Assume $l \in K \backslash \mho$. $\because$ K is proper rightideal of Z, we have $K \subseteq U\,\{I \,/\, I \in \mathbb{R}\} = \mho$. Then $l \in K \subset \mho$ impossible. Hence $K = \mho$. Thus $\mho$ is the distinct maximal rightideal of Z. As $\mho$ is a maximal rightideal of Z, by Theorem 3.2 one and only one of the adopting two conditions is fulfilled.

(A) $Z\backslash\mho = \{l\}$ and $l^2 \in \mho$. (B) $Z\backslash\mho \subseteq (lZ] \forall\, l \in Z\backslash\mho$.

(A) Assume $Z\backslash\mho = \{l\}$ and $l^2 \in \mho$. In this case condition (3) is satisfied. In fact, we have ($\alpha$) $R\,(l) = Z$. Indeed**:** Assume $R\,(l) \neq Z$. Then $R\,(l)$ is a proper rightideal of Z and so $R\,(l) \subseteq \mho$. Thus $l \in \mho$ Impossible. ($\beta$) $l \notin (lZ]$. Indeed**:** Assume $l \in (lZ]$.

Then $(l] \subseteq ((lZ]] = (lZ]$ and $Z = R(l) = (l] \cup (lZ] = (lZ]$. We have $l \leq lz$ for some $z \in Z = (lZ]$ and $z \leq lt$ for some $t \in Z$. Thus we have $l \leq lz \leq l(lt) = l^2t \in \mho Z \subseteq \mho$ and so $l \in \mho$ Impossible. $(\gamma)$ $l^2 \in \mho = Z \setminus \{l\}$. Indeed**:** By hypothesis $l^2 \in \mho$. As $Z \setminus \mho = \{l\}$. We have $\mho = Z \setminus \{l\}$. (B) Assume that $Z \setminus \mho \subseteq (lZ]$, $\forall\, l \in Z \setminus \mho$. Then condition (4) is satisfied. In fact, we have $(\alpha)$ $Z \setminus \mho = \{y \in Z \,/\, (yZ] = Z\}$. Indeed**:** Assume $y \in Z \setminus \mho$. By hypothesis $y \in Z \setminus \mho \subseteq (yZ]$. Then $(y] \subseteq ((yZ]] = (yZ]$ and $R(y) = (y] \cup (yZ] = (yZ]$. On the another hand $R(y) = Z$. Indeed**:** Assume $R(y) \subset Z$. As $R(y)$ is a proper rightideal of Z, we have $y \in R(y) \subseteq \mho$. Impossible. Thus $(yZ] = R(y) = Z$. Assume $y \in Z$, $(yZ] = Z$. Suppose that $y \in \mho$. As $\mho$ is a rightideal of Z and $y \in \mho$, we have $R(y) \subseteq \mho$. On the another hand $R(y) = (y] \cup (yZ] = (y] \cup Z = Z$. since $\mho \neq Z$. We have $U \subset Z = R(y)$ which is impossible. Thus $y \notin \mho$ and $y \in Z \setminus \mho$.$(\beta)$ $Z \setminus \mho$ is a subsemigroup of Z. Indeed**:** $U \neq Z$. So $U \subset Z$ and $Z \setminus \mho \neq \emptyset$. Assume $a, b \in Z \setminus \mho$ By $(\alpha)$, we have $(aZ] = Z$, $(bZ] = Z$. Then $(abZ] = ((a(bZ]] = (aZ] = Z$ and so $ab \in Z \setminus \mho$.

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