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A Note on Inverse Problem with Boundary Element Method for Inhomogeneous Anisotropic Materials

Jeffry Kusuma

Dept. of Mathematics, Fac. of Mathematics and Natural Sciences,
Hasanuddin University, Jalan Perintis Kemerdekaan, 90245 Makassar, Indonesia
jeffry.kusuma@gmail.com

ABSTRACT

Ground amplification effect due to incident plane surface horizontal waves through anisotropic elastic materials is studied. Mathematical formulation over the modeled earth's and deposit materials buried down in the earth are transformed into integral equations. These integral equations altogether with the boundary condition and continuity equation are then simulated and solved numerically. The solution of these amplification effects over the simulation half circular inhomogeneous materials deposit are shown graphically.

KEY WORDS: ground amplification effect, surface horizontal wave, inhomogeneous anisotropic materials.

INTRODUCTION

The ground amplification of seismic wave on alluvial valleys have been studied by numerous authors. Bravo et al, Trifunac, Sánchez-Sesma and Esquivel, Wong and Jennings, Wong and Trifunac, Wong et al. Integral equation formulations have been found to be particularly useful in obtaining numerical solutions to problems of this type. In particular, Wong and Jennings have used singular integral equations to solve the problem of scattering and diffraction of incident surface horizontaly (SH) waves by canyons of arbitrary cross section. Also Bravo extended the method by considering stratified alluvial deposits. Clements and Larsson extending these integral formulation techniques by including the case of homogeneous anisotropic materials. More recently Kusuma extending further these integral formulation techniques by including the case of inhomogeneous anisotropic materials. Since the flexibility on the integral formulation method can be applied not only on the alluvial valleys at the surface can also be applied for the materials buried deep down from the surface.

This paper studied the ground amplification of the seismic wave through inhomogeneous anisotropic materials buried down the earth surface. Some terminology, backgound and construction of the model are discused in section 2. In section 3 the problem formulation of integral equation are developed. Section 4 provides the numerical results for the simulation and test problem.

GROUND MOTION ABOVE INHOMOGENEOUS ANISOTROPIC DEPOSIT MATERIALS

Using Cartesian coordinates system $Ox_1x_2x_3$, let's consider an anisotropic elastic half space occupying the region $x_2 > 0$ as illustrated in figure 1. The half space here is divided into two regions in which the first region contains a homogeneous isotropic material with shear moduli $\mu_{ij}^{(1)} = \lambda_{ij}^{(1)}$ and the second region contains an inhomogeneous anisotropic deposit material with the shear moduli $\mu_{ij}^{(2)} = \lambda_{ij}^{(2)}\left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2$. Both materials are assumed to adhere rigidly to each other so that the displacement and stress are continuous across the interface boundary between the first and the second regions and the constants in the shear moduli satisfy the symmetry conditions $\lambda_{ij}^{(1)} = \lambda_{ji}^{(1)}$ and $\lambda_{ij}^{(2)} = \lambda_{ji}^{(2)}$.

Let $u^{(1)}$ and $u^{(2)}$ be the displacement in the x_3 direction in the half space of earth and the deposit materials respectively. For the propagation of horizontally polarizes *SH* waves, the displacement satisfies the equations of motion

$$\lambda_{ij}^{(1)} \frac{\partial^2 u^{(1)}}{\partial x_i \partial x_j} = \rho_0^{(1)} \frac{\partial^2 u^{(1)}}{\partial t^2}, \tag{1}$$

for the region 1 and

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}^{(2)} \left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2 \frac{\partial u^{(2)}}{\partial x_j}\right] = \rho_0^{(2)} \left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2 \frac{\partial^2 u^{(2)}}{\partial t^2}, \tag{2}$$

for region 2. Here $\rho_0^{(1)}$ and $\rho_0^{(2)}$ are constants and denote the density of the materials in region 1 and 2 respectively, t denotes the time and repeated Latin subscripts denote summation from 1 to 2.

In order to generate a wave amplification, it is necessary assuming the displacement take the form $u^{(1)}(x_1, x_2) = v^{(1)}(x_1, x_2)\exp(\iota\omega t)$ and $u^{(2)}(x_1, x_2) = v^{(2)}(x_1, x_2)\exp(\iota\omega t)$. Equation (1) and (2) will be reduced to

$$\lambda_{ij}^{(1)} \frac{\partial^2 v^{(1)}}{\partial x_i \partial x_j} + \rho_0^{(1)}\omega^2 v^{(1)} = 0, \tag{3}$$

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}^{(2)}\left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2 \frac{\partial v^{(2)}}{\partial x_j}\right] + \rho_0^{(2)}\omega^2\left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2 v^{(2)} = 0. \tag{4}$$

Our interest is a plane wave of unit amplitude which propagates toward the surface of the elastic half space

$$v_I^{(1)} = \exp\iota\omega\left(t + \frac{x_1}{c_1} + \frac{x_2}{c_2}\right), \tag{5}$$

where $c_1 = \beta^{(1)} / \sin\gamma_I$, $c_2 = \beta^{(1)} / \cos\gamma_I$, $\beta^{(1)}$ denotes the velocity of the incident waves and γ_I denotes the angle of the incident wave. Since $v_I^{(1)}$ in (5) propagates in the first material, it must satisfy equation (3) so that

$$\left[\beta^{(1)}\right]^2 = \frac{\lambda_{11}^{(1)}\sin^2\gamma_I + 2\lambda_{12}^{(1)}\sin\gamma_I\cos\gamma_I + \lambda_{22}^{(1)}\cos^2\gamma_I}{\rho_0^{(1)}}. \tag{6}$$

Furthermore, in the case when region 1 and 2 are occupied by the same materials. The traction free condition on $x_2 = 0$ should be hold. These mean, there are reflected wave $v_R^{(1)}$ which take the form

$$v_R^{(1)} = \exp\iota\omega\left(t + \frac{x_1}{d_1} - \frac{x_2}{d_2}\right). \tag{7}$$

Thus if there are no irregularities, the free field solution of the displacement can be written as

$$v_O^{(1)} = v_I^{(1)} + v_R^{(1)}. \tag{8}$$

The stresses are given by

$$\sigma_{i3}^{(1)} = \lambda_{ij}^{(1)}\frac{\partial v^{(1)}}{\partial x_j}, \tag{9}$$

so that the stress $\sigma_{23}^{(1)}$ on $x_2 = 0$ is

$$\sigma_{23}^{(1)} = \left(\frac{\lambda_{21}^{(1)}}{c_1} + \frac{\lambda_{22}^{(1)}}{c_2}\right)\exp\left[\iota\omega\left(t + \frac{x_1}{c_1}\right)\right] + \left(\frac{\lambda_{21}^{(1)}}{d_1} - \frac{\lambda_{22}^{(1)}}{d_2}\right)\exp\left[\iota\omega\left(t + \frac{x_1}{d_1}\right)\right]. \tag{10}$$

This stress will be zero for all times *t* if

$$d_1 = c_1, \tag{11}$$

$$\frac{1}{d_2} = \frac{1}{c_2} + \frac{2\lambda_{21}^{(1)}}{\lambda_{22}^{(1)}c_1}. \tag{12}$$

These equations serves to provide d_2 in terms of the unkwown quantities c_2, c_1, $\lambda_{21}^{(1)}$ and $\lambda_{22}^{(1)}$. Note that if (7) is substituted into (3) then since it represents a solution to (3) it follows that

$$\frac{\lambda_{11}^{(1)}}{c_1^2} - \frac{2\lambda_{12}^{(1)}}{c_1 d_2} + \frac{\lambda_{22}^{(1)}}{d_2^2} = \rho_0^{(1)}, \tag{13}$$

and if (11) is used to substitute for $1/d_2$ in (13) and then into (6) so that (12) ensures (7) is a solution to (3) on the assumption that (5) is also solution to (3).

Let $d_1 = \beta'/\sin\gamma_R$ and $d_2 = \beta'/\cos\gamma_R$ where γ_R is the angle of the reflection, then

$$\tan(\gamma_R) = \frac{d_2}{d_1} = \frac{\tan(\gamma_I)}{1 + 2\left(\lambda_{12}^{(1)}/\lambda_{22}^{(1)}\right)\tan(\gamma_I)}, \tag{14}$$

and once γ_R has been determined from this equation, the wave speed β' of the reflected wave may be readily determined from equation $\beta' = d_1\sin(\gamma_R)$.

To include the influence of the inhomogeneous anisotropic deposit materials in region 2, the solution for the exterior of the deposit is put in the form

$$v^{(1)} = v_O^{(1)} + v_D^{(1)}, \tag{15}$$

in which $v_D^{(1)}$ is the displacement due to the diffracted waves. In this region 2, the displacement $v^{(2)} = v_R^{(2)}$ will be caused by the refracted waves.

INTEGRAL EQUATION

Proceeding further as in Clements and Larsson for the region $\mathcal{R}_1$ with boundary $\mathcal{C}_1$ and outward pointing normal components n_1 and n_2, the integral equation corresponding to (3) is

$$\tau v^{(1)}(a,b) = \int_{C_1} \left[\lambda_{ij}^{(1)} \frac{\partial V^{(1)}}{\partial x_j} n_i v^{(1)} - \lambda_{ij}^{(1)} \frac{\partial v^{(1)}}{\partial x_j} n_i V^{(1)}\right] dS, \tag{16}$$

where $\tau = 1$ if $(a,b) \in \mathcal{R}_1$ and $0 < \tau < 1$ if $(a,b) \in \mathcal{C}_1$. The fundamental solution of $V^{(1)}$ is given by

$$V^{(1)} = \frac{\iota}{4} K^{(1)} \left[H_0^2\left(\bar{v}^{(1)} R^{(1)}\right) + H_0^2\left(\bar{v}^{(1)}{}_{\bar{R}}{}^{(1)}\right)\right], \tag{17}$$

where

$$R^{(1)} = \left[(x_1 - a)^2 + \frac{\lambda_{11}^{(1)}}{\lambda_{22}^{(1)}}(x_2 - b)^2 - \frac{2\lambda_{12}^{(1)}}{\lambda_{22}^{(1)}}(x_1 - a)(x_2 - b)\right]^{\frac{1}{2}}, \tag{18}$$

$${}_{\bar{R}}{}^{(1)} = \left[(x_1 - a)^2 + \left(\frac{\lambda_{12}^{(1)}}{\lambda_{22}^{(1)}}\right)^2 (x_2 - b)^2 - \frac{2\lambda_{12}^{(1)}}{\lambda_{22}^{(1)}}(x_1 - a)(x_2 - b) + \right.$$

$$\left. (x_2 + b)^2 \left(\frac{\lambda_{11}^{(1)}\lambda_{22}^{(1)} - \lambda_{12}^{(1)^2}}{\lambda_{22}^{(1)^2}}\right)\right]^{\frac{1}{2}}, \tag{19}$$

$$K^{(1)} = \frac{\lambda_{22}^{(1)}}{\lambda_{11}^{(1)}\lambda_{22}^{(1)} - \lambda_{12}^{(1)^2}}, \tag{20}$$

$$\bar{v}^{(1)} = \left[\rho_0^{(1)} \omega^2 K^{(1)}\right]^{\frac{1}{2}}, \tag{21}$$

and H_0^2 denotes the Hankel function of the second kind of order zero.

Furthermore, for the region $\mathcal{R}_2$ with boundary C_2 and outward pointing normal components n_1 and n_2, the integral equation corresponding to (6) is

$$K^{(2)} v^{(2)}(a,b) = \int_{C_2} \left[\lambda_{ij}^{(2)}\left(\alpha_1^{(2)} x_1 + \alpha_2^{(2)} x_2 + \alpha_3^{(2)}\right)^2 \frac{\partial V^{(2)}}{\partial x_j} n_i v^{(2)} - \right.$$

$$\left. \lambda_{ij}^{(2)} (\alpha_1^{(2)} x_1 + \alpha_2^{(2)} x_2 + \alpha_3^{(2)})^2 \frac{\partial v^{(2)}}{\partial x_j} n_i V^{(2)}\right] dS, \tag{22}$$

where

$$V^{(2)} = \frac{\iota}{4}\left(\alpha_1^{(2)} x_1 + \alpha_2^{(2)} x_2 + \alpha_3^{(2)}\right)^{-1} H_0^2\left(\bar{v}^{(2)} R^{(2)}\right), \tag{23}$$

$$R^{(2)} = \left[(x_1 - a)^2 + \frac{\lambda_{11}^{(2)}}{\lambda_{22}^{(2)}}(x_2 - b)^2 - \frac{2\lambda_{12}^{(2)}}{\lambda_{22}^{(2)}}(x_1 - a)(x_2 - b)\right]^{\frac{1}{2}}, \tag{24}$$

$$\bar{v}^{(2)} = \left[\frac{\rho_0^{(2)}\omega^2\lambda_{22}^{(2)}}{\lambda_{11}^{(2)}\lambda_{22}^{(2)} - \lambda_{12}^{(2)^2}}\right]^{\frac{1}{2}}. \quad (25)$$

Here, the value of $K^{(2)}$ may be determined by the help of solution of (4)

$$w^{(2)} = \frac{\iota}{4}\left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^{-1} H_0^2(\bar{v}^{(2)}S^{(2)}), \quad (26)$$

where

$$S^{(2)} = \left[x_1{}^2 + \frac{\lambda_{11}^{(2)}}{\lambda_{22}^{(2)}}x_2{}^2 - \frac{2\lambda_{12}^{(2)}}{\lambda_{22}^{(2)}}x_1x_2\right]^{\frac{1}{2}}. \quad (27)$$

Thus using (26) in (22) in we obtain

$$K^{(2)} = \left[w^{(2)}(a,b)\right]^{-1}\int_{C_2}\left[\lambda_{ij}^{(2)}(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)})^2\frac{\partial V^{(2)}}{\partial x_j}n_i w^{(2)} - \lambda_{ij}^{(2)}(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)})^2\frac{\partial w^{(2)}}{\partial x_j}n_i V^{(2)}\right] dS, \quad (28)$$

By applying equation (16) and its fundamental solution (17) in the region 1 (outside of the deposited materials) then the only non-zero integral is the integral over the deposited materials interface boundary (Sommerfeld radiation condition). If we denote this interface boundary as curve C_1 and deposited materials free boundary as C_F and specifying the normal components n_1 and n_2 pointing outward of the deposited materials boundary, thus (16) and (22) on the deposited materials boundary are

$$\frac{1}{2}v^{(1)}(a,b) = \int_{C_1}\left[\lambda_{ij}^{(1)}\frac{\partial V^{(1)}}{\partial x_j}n_i v^{(1)} - \lambda_{ij}^{(1)}\frac{\partial v^{(1)}}{\partial x_j}n_i V^{(1)}\right] dS, \quad (29)$$

and

$$K^{(2)}\left[v^{(2)}(a,b)\right] = \int_{C_1+C_F}\left[\lambda_{ij}^{(2)}(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)})^2\frac{\partial V^{(2)}}{\partial x_j}n_i v^{(2)} - \lambda_{ij}^{(2)}(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)})^2\frac{\partial v^{(2)}}{\partial x_j}n_i V^{(2)}\right] dS. \quad (30)$$

The equation (29) and (30) together with the continuity equations

$$v^{(1)} = v^{(2)}, \quad (31)$$

$$\lambda_{ij}^{(1)}\frac{\partial v^{(1)}}{\partial x_j}n_i = \lambda_{ij}^{(2)}\left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2\frac{\partial v^{(2)}}{\partial x_j}n_i, \quad (32)$$

and traction free boundary condition on $x_2 = 0$

$$\lambda_{ij}^{(2)}\left(\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \alpha_3^{(2)}\right)^2\frac{\partial v^{(2)}}{\partial x_j}n_i = 0, \quad (33)$$

may be used to solved for displacement and stress over the interface boundary $\boldsymbol{C_1}$ and displacement along traction free surface $\boldsymbol{x_2} = \mathbf{0}$. Once this has been done, we can obtain the value of the displacement $\boldsymbol{v}^{(1)}$ and/or $\boldsymbol{v}^{(2)}$ at all points $(\mathbf{a},\mathbf{b})$ in the half space $\boldsymbol{x_2} > 0$ through equation (16) and (22).

NUMERICAL RESULTS

In order to verify the acuracy of the numerical method, we consider a semi circular deposit with a unit radius. The material deposit here is defined by non-dimensional quantities which satisfy equation $(x_1 - 2)^2 + (x_2 + 1)^2 \leq 1$, and $x_2 \leq -1$ (Figure 2). Furthermore, suppose that the materials properties are non dimensional quantities $\lambda_{ij}^{(\Omega)}, \rho_0^{(\Omega)}, \alpha_1^{(\Omega)}, \alpha_2^{(\Omega)}, \alpha_3^{(\Omega)}$, $\Omega = 1{,}2$ and using the normalised frequency

$$\eta = \frac{\omega}{\pi \beta^{(1)}}, \tag{34}$$

where $\beta^{(1)}$ is given by (6), then the numerical results are ready to be calculated.

Using the boundary element method by using 80 segments on semi circular deposit boundary and 70 segments on the free surface boundary, we obtain the numerical results as shown in figure 3 up to figure 5.

Figure 3 shows the ground amplification obtained using materials specification are $\lambda_{11}^{(1)} = 0.12$, $\lambda_{12}^{(1)} = 0.00$, $\lambda_{22}^{(1)} = 0.12$, $\rho_0^{(1)} = 3.00$, $\lambda_{11}^{(2)} = 0.02$, $\lambda_{12}^{(2)} = 0.00$, $\lambda_{22}^{(2)} = 0.02$, $\rho_0{}^{(2)} = 2.00$, $\alpha_1^{(2)} = 0.00$, $\alpha_2^{(2)} = 0.00$, $\alpha_3^{(2)} = 1.00$ and incident wave angle $\gamma_I = 0^0$.

Figure 4 shows the ground amplification obtained using materials specification are $\lambda_{11}^{(1)} = 0.12$, $\lambda_{12}^{(1)} = 0.00$, $\lambda_{22}^{(1)} = 0.12$, $\rho_0^{(1)} = 3.00$, $\lambda_{11}^{(2)} = 0.02$, $\lambda_{12}^{(2)} = 0.00$, $\lambda_{22}^{(2)} = 0.02$, $\rho_0{}^{(2)} = 2.00$, $\alpha_1^{(2)} = 0.50$, $\alpha_2^{(2)} = 0.00$, $\alpha_3^{(2)} = 0.00$ and incident wave angle $\gamma_I = 0^0$.

Figure 5 shows the ground amplification obtained using materials specification are $\lambda_{11}^{(1)} = 0.12$, $\lambda_{12}^{(1)} = 0.00$, $\lambda_{22}^{(1)} = 0.12$, $\rho_0^{(1)} = 3.00$, $\lambda_{11}^{(2)} = 0.02$, $\lambda_{12}^{(2)} = 0.00$, $\lambda_{22}^{(2)} = 0.02$, $\rho_0{}^{(2)} = 2.00$, $\alpha_1^{(2)} = 0.50$, $\alpha_2^{(2)} = 0.00$, $\alpha_3^{(2)} = 0.00$ and incident wave angle $\gamma_I = 30^0$.

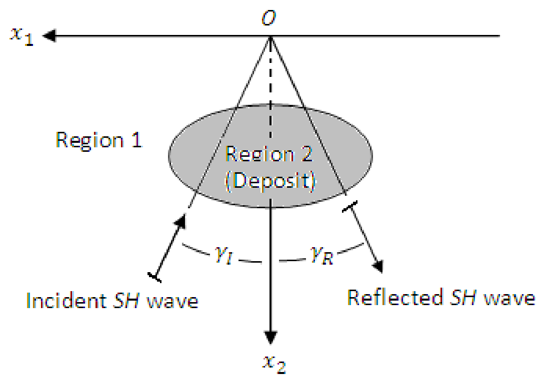

Figure 1: Incident and reflected waves on deposit materials and surrounding half-space

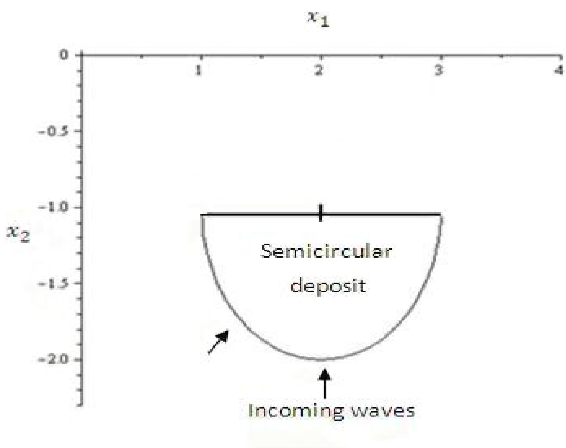

Figure 2: Effect of inhomogeneous deposit materials with $\gamma_I = 0^\circ$ and $\gamma_I = 30^\circ$

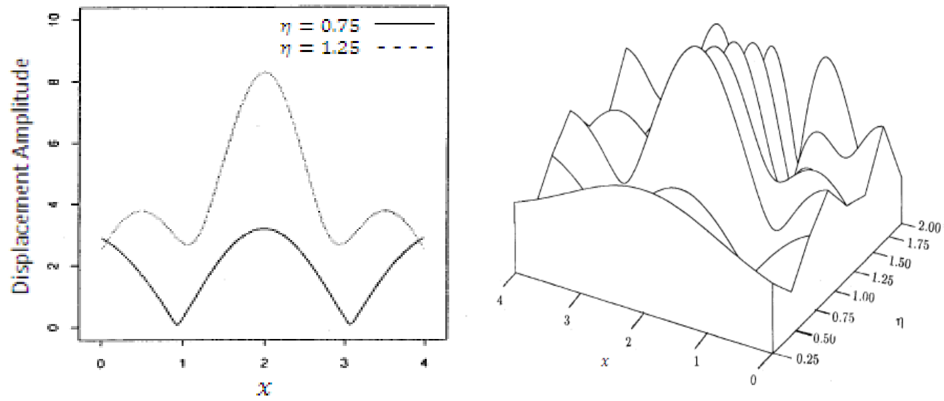

Figure 3: Effect of inhomogeneous semi-circular deposit materials with $\gamma_I = 0^\circ$

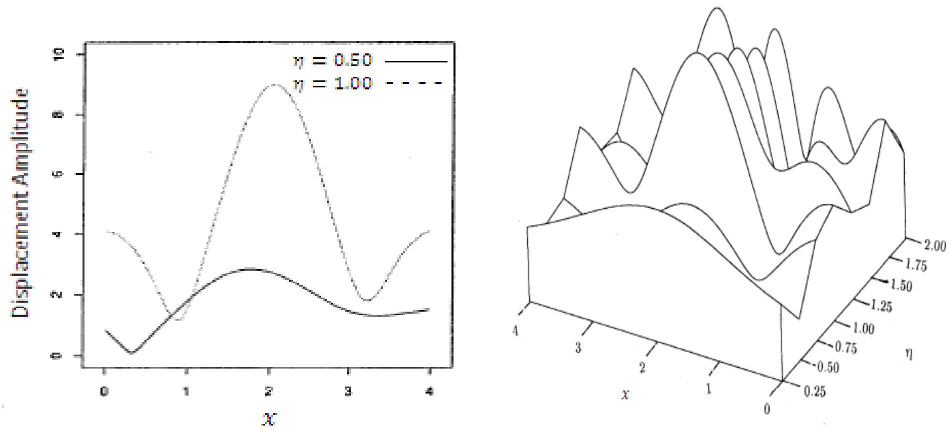

Figure 4: Effect of inhomogeneous semi-circular deposit materials with $\gamma_I = 0^\circ$

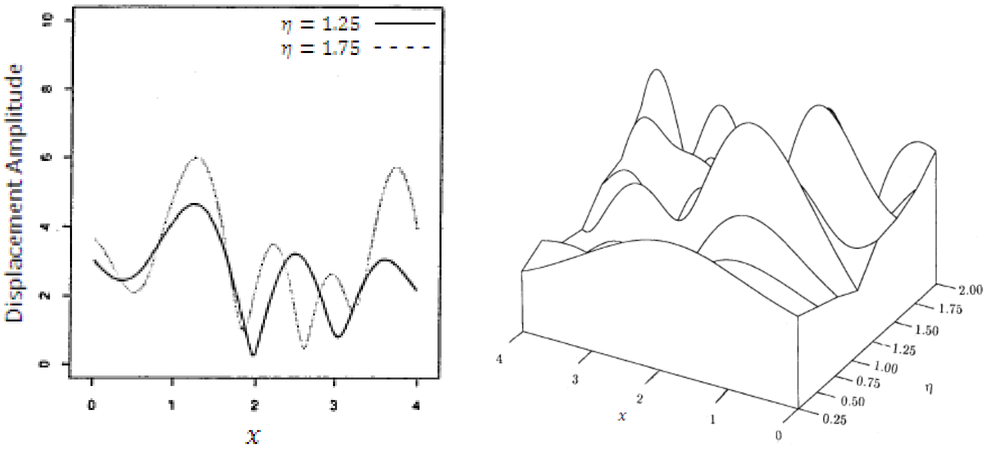

Figure 5: Effect of inhomogeneous semi-circular deposit materials with $\gamma_I = 30^{\circ}$

CONCLUSION

The results in figure 3 up to figure 5 show that amplification effects are significantly influenced by the materials specification and the angle of the incident waves. The flexibility of the boundary integral element methods is an advantage in dealing with irregularities of the deposit materials. For such kind of deposit materials buried down inside the earth, the method described above can be directly used to obtain the amplification effect.

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