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Asymptotic Theory For a Model of Dendritic Solidification with Effect of an Oscillatory Source

Tat Leung Yee
Department of Mathematics and Information Technology,
The Hong Kong Institute of Education,
Tai Po, New Territories, Hong Kong
E-mail: tlyee@ied.edu.hk

Abstract

The paper is devoted to the asymptotic theory for a mathematical model of dendritic crystal solidification. A single needle dendrite is considered from an under-cooled pure melt with temperature T_∞ and it is supposed to grow under the effect of convection motion induced by an oscillating external source with a small magnitude U_∞. By assuming that the Reynolds number Re is small, and using the analytical method of matched asymptotic expansions, we can generate the globally valid asymptotic expansion solutions of the flow field in the whole physical domain. This enables us to explore the effect of the externally applied convection motion on the crystal growth and its pattern formation.

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1. Introduction

Dendritic solidification of axisymmetric needle crystal is one of the fundamental subjects in the areas of condensed matter physics and modern material science in the past several decades [1]–[10]. This challenging subject has raised significant issues that involve not only the experimental results in physics but also the analytical methods for solving problems in mathematics [5]–[10].

The first important result in dendritic growth was Ivantsov's exact similarity solution published in 1947 [1]. It describes the steady, isothermal, paraboloidal axisymmetric, needle-like crystal growth with zero surface tension on the interface between the solid phase and liquid phase. The mathematical solution contributes a fundamental basis for further research works such as the perturbed dendritic growth with nonzero surface tension. The next significant contribution was the experimental results by Schaefer, Glicksman and Ayers (1975) [4]. These researchers made extensive, detailed experiments and correctly defined the pattern selection problem for realistic dendritic growth: the growth velocity of the dendrite-tip is a uniquely determined function of the growth condition and the properties of material.

This paper is an investigation dealing with the interaction of external convection and dendritic growth. The experimental observations have shown that convective motion in melt may have a significant effect on the instability mechanisms, and consequently, affects the micro-structure formation at the interface in dendritic solidification. Convective motion in melt can be induced by a variety of sources such as the density change during phase transition; the buoyancy effect due to the presence of gravity field; an applied external flow or other sources.

With the presence of convection, the governing mathematical model becomes more complicated and difficult to solve in the sense that the hydrodynamics must be introduced into the system. In the literature, the steady dendritic growth in an external flow was studied by a number of researchers, such as Ananth & Gill (1991) [11], Dash & Gill (1984) [12], Saville & Beaghton (1988) [13] numerically and analytically. These researchers considered the special case of zero surface tension, and obtained the similarity solutions that based on some simplified models of Navier-Stokes equations, including Stokes flow model, and Oseen flow model, etc. As long as the Navier-Stokes equations are adopted, they are only approximate solutions. However, as we know that the Stokes model is only a good approximation to the Navier-Stokes model in the near field, whereas the Oseen model a good approximation in the far field. Therefore, their solutions cannot be considered as good approximations in the whole physical domain, as far as the Navier-Stokes model is concerned. Moreover, the approaches adopted by these researchers neither allow the generation of the next-order approximations, nor give an estimation of error between their solutions and the exact solutions.

An uniformly valid asymptotic expansion solution, based on the Navier-Stokes model, for large Prandtl number, was first introduced by Xu [7]–[8]. On the basis of Xu's work, we attempt to generate an uniformly valid asymptotic solution for the problem that involves a small Reynolds number under the effect of convection motion induced by an oscillatory external flow. Our results reveal that even for the case of zero surface tension, the system no longer allows an exact similarity solution. When we further consider a small Reynolds number, the system under the effect of convective motion allows a nearly similarity solution which involves

(1) the perturbed term dependent of the under-cooling temperature T_∞ and the convection magnitude U_∞;

(2) an error of $\mathcal{O}\left(\dfrac{\mathrm{Re}}{\ln(1/\mathrm{Re})}\right)$.

2. Mathematical Formulation

The dendritic evolution of needle crystal during solidification has been under intensively study by physicists and material scientists for several decades. We shall confine our attention to the simpler case of a single needle dendrite from an under-cooled pure melt in the negative Z-axis direction with a constant average tip velocity V. The major transport process in a pure melt is heat conduction. The under-cooling temperature of the melt is T_∞. The melt is considered as an incompressible Newtonian fluid and is assumed to be infinite in extent. The dendrite is supposed to grow in an oscillating external flow, along the Z-axis in the far field ahead of the tip with a small amplitude U_∞, with zero surface tension on the interface between liquid and solid states. Assume that the thermal diffusivity κ_T and the heat capacity c_p of the liquid state are the same as those of the solid state, the mass density of liquid state is ρ and the mass density of solid state is ρ_s. Both the tip growth velocity V and the flow velocity U_∞ are measured in laboratory frame. Let $\mathbf{U}$ be the absolute velocity field of the fluid and T be the temperature field in the liquid melt.

We first present the general mathematical formulation of the needle crystal growth with convection and in the next section we further reduce our system and shall consider the effect of the oscillatory external source. The governing equations consisting of the continuity equation, the Navier-Stokes equations and the heat conduction equations are as follows:

Mass conservation equation

$$\nabla \cdot \mathbf{U} = 0.$$

Momentum equations Applying the Boussinesq approximation, the Navier-Stokes equations become

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\, \nabla^2 \mathbf{U} + \beta(T - T_\infty)\, g\mathbf{e}_Z.$$

The first term in the R.H.S. of the above equation is the pressure term, the second term is the viscous stress term and the third term is the buoyancy force term. Taking the curl on both sides of the equation we obtain

$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times \mathbf{U}) = \nu \nabla^2 \Omega + \nabla \times [\beta(T - T_\infty)\, g\mathbf{e}_Z], \quad \Omega = \nabla \times \mathbf{U}.$$

Energy equation for the liquid state

$$\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T = \kappa_T \nabla^2 T.$$

Energy equation for the solid state

$$\frac{\partial T_S}{\partial t} = \kappa_T \nabla^2 T_S.$$

We denote by $\mathbf{U}$ the absolute velocity field of the fluid, T the temperature field in the liquid melt, T_S the temperature field in the solid state, ν the kinematic viscosity, β the thermal expansion coefficient, P the reduced pressure, g the gravitational acceleration, $\mathbf{e}_Z$ the unit vector along the Z-axis.

The boundary conditions are as follows.

(1) In the far field: $\mathbf{U} = U_\infty \mathbf{e}_Z$, and $T = T_\infty$.

(2) On the interface, the system is assumed in the local thermodynamical equilibrium state. Thus the system have to satisfy

 (a) Thermodynamical equilibrium condition;
 (b) Gibbs-Thomson condition (This condition is that the temperature at each point on the solid-liquid interface equals the local equilibrium freezing temperature, which depends on the local interface curvature.);
 (c) Enthalpy conservation condition;
 (d) Total Mass conservation condition;
 (e) Continuity condition of tangential component of velocity.

We shall non-dimensionalize the governing equations and the boundary conditions in order to reduce the complexity of the formulation and next introduce the paraboloidal coordinate system (ξ, η, θ) for (x, y, z):

$$x = \eta_0^2 \xi \eta \cos\theta, \quad y = \eta_0^2 \xi \eta \sin\theta, \quad z = \frac{1}{2}\eta_0^2 (\xi^2 - \eta^2),$$

where the parameter η_0^2 is to be determined. It will be seen that this parameter is needed to normalize the interface shape function, so that the zero-order inner solution has the interface shape $\eta_* = 1$ for any given under-cooling T_∞.

The primary focus of this paper is to investigate the effect of convective motion induced only by the oscillatory external flow on needle dendritic growth, in the far field ahead of the tip with a small magnitude U_∞. We assume that gravity is taken to be negligible, the surface tension is assumed to be zero, so the dendrite should be axisymmetric and convection is only induced by the oscillatory external source. The governing equations are consisting of the continuity equation, the Navier-Stokes equations and the heat conduction equations. We express below the complete system in terms of the paraboloidal coordinates.

Kinematic equation

$$\mathrm{D}_1^2 \Psi = -\eta_0^4 (\xi^2 + \eta^2)\, \zeta, \tag{2.1}$$

Vorticity equation

$$\frac{1}{\mathrm{Re}} \mathrm{D}_1^2 \zeta = \eta_0^4 (\xi^2 + \eta^2) \frac{\partial \zeta}{\partial t} + \frac{2\zeta}{\eta_0^2 \xi^2 \eta^2} \frac{\partial(\Psi, \xi\eta)}{\partial(\xi, \eta)} - \frac{1}{\eta_0^2 \xi\eta} \frac{\partial(\Psi, \zeta)}{\partial(\xi, \eta)}, \tag{2.2}$$

Energy equation for the liquid state

$$\nabla_1^2 T = \eta_0^4 (\xi^2 + \eta^2) \frac{\partial T}{\partial t} + \frac{1}{\eta_0^2 \xi \eta} \left(\frac{\partial \Psi}{\partial \eta} \frac{\partial T}{\partial \xi} - \frac{\partial \Psi}{\partial \xi} \frac{\partial T}{\partial \eta} \right), \tag{2.3}$$

where the differential operators are defined as:

$$\mathrm{D}_1^2 := \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{\eta} \frac{\partial}{\partial \eta},$$

and

$$\nabla_1^2 := \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi}, + \frac{1}{\eta} \frac{\partial}{\partial \eta}.$$

The boundary conditions are:

1. The far-field conditions: as $\eta \to \infty$,

$$\Psi \to \frac{1}{2} \eta_0^4 \big(1 + U_\infty \exp(i\omega t)\big) \xi^2 \eta^2, \qquad \zeta \to 0, \qquad T \to T_\infty. \tag{2.4}$$

2. The smooth tip condition: at $\xi = 0$,

$$\eta_S'(0, t) = 0, \qquad \eta_S(0, 0) = 1. \tag{2.5}$$

3. On the interface $\eta = \eta_S(\xi, t)$, we must have

   (a) thermodynamical equilibrium condition:

$$T = T_S = 0, \tag{2.6}$$

   (b) enthalpy conservation condition:

$$\left(\frac{\partial T}{\partial \eta} - \eta_S' \frac{\partial T}{\partial \xi} \right) + \eta_0^2 (\xi \eta_S)' + \eta_0^4 (\xi^2 + \eta_S^2) \frac{\partial \eta_S}{\partial t} = 0, \tag{2.7}$$

   (c) mass conservation condition:

$$\left(\frac{\partial \Psi}{\partial \xi} + \eta_S' \frac{\partial \Psi}{\partial \eta} \right) = \eta_0^4 (\xi \eta_S)(\xi \eta_S)', \tag{2.8}$$

   (d) continuity condition of tangential component of velocity:

$$\left(\frac{\partial \Psi}{\partial \eta} - \eta_S' \frac{\partial \Psi}{\partial \xi} \right) + \eta_0^4 (\xi \eta_S) (\eta_S \eta_S' - \xi) = 0. \tag{2.9}$$

3. Matched Asymptotic Expansions

To study the asymptotic expansion solution of flow field for the small Reynolds number $\mathrm{Re} = Vl_{\mathrm{T}}/\nu$ which is equivalent to the case, when the Prandtl number $\mathrm{Pr} = \nu/\kappa_{\mathrm{T}}$ is large. The entire physical space is divided into two regions, the inner region near the dendrite and the outer region apart from the dendrite. The asymptotic solution which is valid in the inner region is called the inner solution, while which is valid in the outer region is called the outer solution. It will be seen that the inner solution has a different asymptotic expansion from the outer solution, therefore they should be solved independently and then be matched in an intermediate region. It is believed that we could find the solution, therefore we assume there exists such intermediate region and see how the solution looks like. The matched solution should be a globally valid asymptotic solution in the whole physical domain.

Zero-order Inner Solution of Flow Field

The well-known Ivantsov solution represents that the dendrite is isothermal and its interface shape is paraboloidal. It should be noted that this similarity solution with zero surface tension is a particular solution of eqs. (2.1)–(2.3):

$$T_* = T_\infty + \frac{1}{2}\eta_0^2 e^{\frac{1}{2}\eta_0^2} \mathrm{E}_1\Big(\frac{1}{2}\eta_0^2 \eta^2\Big), \quad T_{\mathrm{S}*} = 0, \quad \eta_* = 1, \quad \zeta_* = 0, \quad \Psi_* = \frac{1}{2}\eta_0^4 \xi^2 \eta^2, \tag{3.1}$$

where $\mathrm{E}_1(x)$ is an exponential integral defined by $\mathrm{E}_1(x) := \int_x^\infty e^{-t}/t\, dt$. The parameter η_0^2 is determined by the under-cooling T_∞ because $-T_\infty = b\, e^b\, \mathrm{E}_1(b)$, where $b = \eta_0^2/2$. To hunt for the inner region asymptotic expansions around the Ivantsov solution in the limit as $\mathrm{Re} \to 0$ we assume that

$$\begin{cases} \Psi - \Psi_* &= \varepsilon_0(\mathrm{Re})\, \Psi_0(\xi, \eta, t) + \varepsilon_1(\mathrm{Re})\, \Psi_1(\xi, \eta, t) + \cdots, \\ \zeta - \zeta_* &= \varepsilon_0(\mathrm{Re})\, \zeta_0(\xi, \eta, t) + \varepsilon_1(\mathrm{Re})\, \zeta_1(\xi, \eta, t) + \cdots, \\ T - T_* &= \varepsilon_0(\mathrm{Re})\, T_0(\xi, \eta, t) + \varepsilon_1(\mathrm{Re})\, T_1(\xi, \eta, t) + \cdots, \\ \eta_{\mathrm{S}} - 1 &= \varepsilon_0(\mathrm{Re})\, h_0(\xi, t) + \varepsilon_1(\mathrm{Re})\, h_1(\xi, t) + \cdots, \end{cases} \tag{3.2}$$

where the inner region asymptotic sequence, $\{\varepsilon_0(\mathrm{Re}),\ \varepsilon_1(\mathrm{Re}),\ \varepsilon_2(\mathrm{Re}), \cdots\}$, are to be determined. In order to obtain the inner region solution, we substitute the expansions (3.2) into the eqs. (2.1)–(2.3). Then we are able to derive each order of the inner region expansions successively. We shall first find the velocity field. Accordingly the temperature field and the interface shape could be obtained subsequently.

The zero-order inner solution of flow field is subject to the system of governing equations:

$$\begin{cases} \mathrm{D}_1^2 \Psi_0 &= -\eta_0^4 (\xi^2 + \eta^2)\, \zeta_0, \\ \mathrm{D}_1^2 \zeta_0 &= 0, \end{cases} \tag{3.3}$$

which satisfies the interface boundary conditions at $\eta = 1$:

$$\Psi_0 \;=\; 0, \qquad \frac{\partial \Psi_0}{\partial \eta} \;\approx\; 0. \tag{3.4}$$

By solving the system (3.3)–(3.4) we utilize the intermediate variables (σ, τ):

$$\sigma = \frac{1}{2}\,\eta_0^2\,\xi^2, \qquad \tau = \frac{1}{2}\,\eta_0^2\,\eta^2. \tag{3.5}$$

Using the new variables induces the new differential operator:

$$\mathrm{L}^2 \;=\; \frac{1}{2\eta_0^2}\,\mathrm{D}_1^2 \;=\; \sigma\frac{\partial^2}{\partial \sigma^2} + \tau\frac{\partial^2}{\partial \tau^2}.$$

The equations in (3.3) become

$$\begin{cases} \mathrm{L}^2\Psi_0 \;=\; -(\sigma+\tau)\,\zeta_0, \\ \mathrm{L}^2\zeta_0 \;=\; 0, \end{cases} \tag{3.6}$$

respectively. It is known that the equations in (3.6) may allow the following forms of solutions: $\zeta_0 = f_0(\tau, t) + \sigma\, f_1(\tau, t)$ and $\Psi_0 = g_0(\tau, t) + \sigma\, g_1(\tau, t)$. By substituting the form of solutions into (3.6) we can deduce the solutions

$$\begin{aligned} \zeta_0 \;&=\; a_0 + b_0\,\tau, \\ \Psi_0 \;&=\; \Big(d_0 + d_1\,\tau - \frac{a_0}{2}\,\tau^2 - \frac{b_0}{6}\,\tau^3\Big) + \sigma\,\Big[e_0 + (e_1 + a_0)\,\tau - a_0\,\tau\ln\tau - \frac{b_0}{2}\,\tau^2\Big], \end{aligned}$$

where a_0, b_0, d_0, d_1, e_0, e_1 are functions of t to be determined. In fact we are able to find the functions b_0, d_0, d_1, e_0, e_1 in terms of a_0, by applying the boundary conditions in (3.4). Consequently we could obtain the zero-order inner region solutions

$$\begin{aligned} \zeta_0 &= a_0(t), \\ \Psi_0 &= a_0(t)\,\psi, \end{aligned} \tag{3.7}$$

where

$$\psi(\sigma,\tau) \;=\; \Big(-\frac{1}{8}\eta_0^4 + \frac{1}{2}\eta_0^2\,\tau - \frac{1}{2}\tau^2\Big) + \sigma\Big[-\frac{1}{2}\eta_0^2 + \Big(1 + \ln\big(\frac{1}{2}\eta_0^2\big)\Big)\,\tau - \tau\,\ln\tau\Big].$$

Using the transformations (3.5) we could also write the solution $\Psi_0(\sigma, \tau)$ into the variables (ξ, η) such that $\Psi_0 = a_0(t)\,\psi(\xi, \eta)$, where

$$\psi(\xi,\eta) \;=\; -\frac{1}{4}\eta_0^4\Big\{\Big(\frac{1}{2}\eta^4 - \eta^2 + \frac{1}{2}\Big) - \xi^2\Big[-\eta^2\,\ln\eta^2 + \eta^2 - 1\Big]\Big\}. \tag{3.8}$$

The zero-order inner solutions ζ_0 and Ψ_0 contain an arbitrary function $a_0(t)$. The solutions satisfy the boundary conditions in (3.4) on the interface, but fail to satisfy the far field conditions at $\eta \to \infty$. The inner asymptotic expansion solution is not valid in the far field. In fact, we should consider a different asymptotic expansion solution which could satisfy all the far field boundary conditions. This solution is called the outer expansion solution. The inner and the outer expansion solutions would be matched in an intermediate region and $a_0(t)$ could then be determined by matching. We will show this step by step in the followings.

Zero-order Outer Solution

The zero-order inner solution given by (3.7) contains an undetermined function $a_0(t)$. In fact, the function $a_0(t)$ and the zero-order inner expansion coefficient $\varepsilon_0(\mathrm{Re})$ in (3.2) could be determined by asymptotically matching the zero-order inner solution with the zero-order outer solution. Next we shall determine the zero-order outer solution which is supposed to be valid in the far field.

In the far field when the variables (ξ, η) are large enough, we postulate that $\xi = \mathcal{O}\left(\frac{1}{\sqrt{\mathrm{Re}}}\right)$, $\eta = \mathcal{O}\left(\frac{1}{\sqrt{\mathrm{Re}}}\right)$ when Re is small. It is more convenient to introduce the so-called outer variables (ξ_*, η_*):

$$\xi_* = \sqrt{\mathrm{Re}}\,\xi, \qquad \eta_* = \sqrt{\mathrm{Re}}\,\eta. \tag{3.9}$$

The outer region asymptotic expansions as $\mathrm{Re} \to 0$ are written as

$$\left\{\begin{aligned} \hat{\Psi}(\xi_*, \eta_*, t) - \Psi_*(\xi_*, \eta_*) &= \delta_0(\mathrm{Re})\,\hat{\Psi}_0(\xi_*, \eta_*, t) + \delta_1(\mathrm{Re})\,\hat{\Psi}_1(\xi_*, \eta_*, t) + \cdots, \\ \frac{1}{\mathrm{Re}^2}\left[\hat{\zeta}(\xi_*, \eta_*, t) - \zeta_*(\xi_*, \eta_*)\right] &= \delta_0(\mathrm{Re})\,\hat{\zeta}_0(\xi_*, \eta_*, t) + \delta_1(\mathrm{Re})\,\hat{\zeta}_1(\xi_*, \eta_*, t) + \cdots, \\ \hat{T}(\xi_*, \eta_*, t) - T_*(\xi_*, \eta_*) &= \delta_0(\mathrm{Re})\,\hat{T}_0(\xi_*, \eta_*, t) + \delta_1(\mathrm{Re})\,\hat{T}_1(\xi_*, \eta_*, t) + \cdots, \end{aligned}\right. \tag{3.10}$$

where the outer region asymptotic sequence, $\{\delta_0(\mathrm{Re}),\ \delta_1(\mathrm{Re}),\ \delta_2(\mathrm{Re}),\ \cdots\}$, are to be determined. Using the outer variables in (3.9) induces the new differential operator:

$$\mathrm{D}_*^2 = \frac{1}{\mathrm{Re}}\,\mathrm{D}_1^2 = \frac{\partial^2}{\partial \xi_*^2} + \frac{\partial^2}{\partial \eta_*^2} - \frac{1}{\xi_*}\frac{\partial}{\partial \xi_*} - \frac{1}{\eta_*}\frac{\partial}{\partial \eta_*}. \tag{3.11}$$

The governing equations (2.1) and (2.2) in terms of the outer variables become

$$\mathrm{D}_*^2 \hat{\Psi} = -\frac{1}{\mathrm{Re}^2}\,\eta_0^4\,(\xi_*^2 + \eta_*^2)\,\hat{\zeta}, \tag{3.12}$$

$$\begin{aligned} \frac{1}{\mathrm{Re}^2}\,\mathrm{D}_*^2 \hat{\zeta} &= \frac{1}{\mathrm{Re}^3}\,\eta_0^4\,(\xi_*^2 + \eta_*^2)\,\frac{\partial \hat{\zeta}}{\partial t} + \frac{2\hat{\zeta}}{\eta_0^2\,\xi_*^2 \eta_*^2}\left(\xi_*\,\frac{\partial \hat{\Psi}}{\partial \xi_*} - \eta_*\,\frac{\partial \hat{\Psi}}{\partial \eta_*}\right) \\ &\quad - \frac{1}{\eta_0^2\,\xi_* \eta_*}\left(\frac{\partial \hat{\Psi}}{\partial \xi_*}\,\frac{\partial \hat{\zeta}}{\partial \eta_*} - \frac{\partial \hat{\Psi}}{\partial \eta_*}\,\frac{\partial \hat{\zeta}}{\partial \xi_*}\right), \end{aligned} \tag{3.13}$$

which satisfy the boundary condition (2.4) as $\eta_* \to \infty$

$$\hat{\Psi} \approx \frac{1}{\mathrm{Re}^2}\frac{1}{2}\eta_0^4\big(1+U_\infty \exp(i\omega t)\big)\,\xi_*^2\eta_*^2 = \Psi_*(\xi_*,\eta_*)+\frac{1}{\mathrm{Re}^2}\frac{U_\infty}{2}\eta_0^4 \exp(i\omega t)\,\xi_*^2\eta_*^2. \tag{3.14}$$

Comparing (3.14) with the expansion of $\hat{\Psi}$ in (3.10) induces

$$\delta_0(\mathrm{Re}) = \frac{1}{\mathrm{Re}^2} \tag{3.15}$$

and

$$\hat{\Psi}_0(\xi_*,\eta_*,t) = \frac{U_\infty}{2}\eta_0^4\,\xi_*^2\eta_*^2 \exp(i\omega t). \tag{3.16}$$

It also follows that

$$\hat{\zeta}_0(\xi_*,\eta_*,t) = 0, \tag{3.17}$$

which satisfies the far field boundary condition in (2.4). Combining (3.15), (3.16) and (3.17) forms the zero-order outer expansion solution. As we mentioned above we need this zero-order outer solution because we want to match it with the zero-order inner solution in order to determine the unknown function $a_0(t)$ and the zero-order inner expansion coefficient $\varepsilon_0(\mathrm{Re})$. Again, just like what we did in (3.5), it is more convenient to use the utilized outer variables $(\sigma_*, \tau_*) = (\eta_0^2\,\xi_*^2/2,\ \eta_0^2\,\eta_*^2/2)$. With this set of new variables we rewrite the zero-order outer solution $\delta_0\,\hat{\Psi}$ as

$$\delta_0(\mathrm{Re})\,\hat{\Psi}_0(\sigma_*,\tau_*,t) = \frac{1}{\mathrm{Re}^2}\,2U_\infty\,\sigma_*\tau_* \exp(i\omega t). \tag{3.18}$$

Our purpose is to look for the function $a_0(t)$ in (3.7) and the zero-order inner expansion coefficient $\varepsilon_0(\mathrm{Re})$ in (3.2). This can be done by matching the zero-order outer solution (3.18) with the zero-order inner solution $\varepsilon_0\,\Psi_0$ given by (3.7), (3.8) in terms of the utilized outer variables:

$$\begin{aligned}\varepsilon_0(\mathrm{Re})\,\Psi_0 = {} & -a_0(t)\left[\varepsilon_0(\mathrm{Re})\frac{1}{\mathrm{Re}^2}\ln\frac{1}{\mathrm{Re}}\right]\sigma_*\tau_* \\ & +a_0(t)\left[\varepsilon_0(\mathrm{Re})\frac{1}{\mathrm{Re}^2}\right]\left\{-\frac{1}{2}\tau_*^2-\sigma_*\tau_*\ln\tau_*+\sigma_*\tau_*\big(1+\ln(\tfrac{1}{2}\eta_0^2)\big)\right\} \\ & +a_0(t)\left[\varepsilon_0(\mathrm{Re})\frac{1}{\mathrm{Re}}\right]\left\{\frac{1}{2}\eta_0^2\,(\tau_*-\sigma_*)\right\} \\ & -\frac{a_0(t)}{8}\,\eta_0^4\left[\varepsilon_0(\mathrm{Re})\right].\end{aligned} \tag{3.19}$$

The above zero-order inner solution (3.19) contains four terms in descending order of Re in which Re is chosen to be small. Now we focus on the right hand side of (3.18) and that of (3.19) and will find that the zero-order outer solution (3.18) could only match with the first term of the zero-order inner solution (3.19). We set

$$a_0(t) = -2U_\infty \exp(i\omega t), \tag{3.20}$$

and

$$\varepsilon_0(\mathrm{Re}) = \frac{1}{\ln(1/\mathrm{Re})}. \tag{3.21}$$

The second, the third and the fourth term of the zero-order inner solution in (3.19) remain unmatched. In fact we need some higher-order outer solutions to match with them. First we need the first-order outer solution $\delta_1(\mathrm{Re})\,\hat{\Psi}_1$ to match with the second term of the zero-order inner solution in (3.19). We set

$$\delta_1(\mathrm{Re}) = \varepsilon_0(\mathrm{Re})\,\frac{1}{\mathrm{Re}^2} = \frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}. \tag{3.22}$$

Next we need the second-order outer solution $\delta_2(\mathrm{Re})\,\hat{\Psi}_2$ to match with the third term of the zero-order inner solution in (3.19). We set

$$\delta_2(\mathrm{Re}) = \varepsilon_0(\mathrm{Re})\,\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re}\,\ln(1/\mathrm{Re})}. \tag{3.23}$$

Similarly, we can match the fourth term of (3.19) with $\delta_3(\mathrm{Re})\,\hat{\Psi}_3$ in order to find

$$\delta_3(\mathrm{Re}) = \varepsilon_0(\mathrm{Re}) = \frac{1}{\ln(1/\mathrm{Re})}. \tag{3.24}$$

In the above we have exhausted all four terms of the zero-order inner solution and have found the outer expansion coefficients: $\delta_1(\mathrm{Re})$, $\delta_2(\mathrm{Re})$ and $\delta_3(\mathrm{Re})$. Since we are also interested in the explicit form of the first-order outer solution $\delta_1(\mathrm{Re})\,\hat{\Psi}_1$ now we go back to derive the outer solution $\hat{\Psi}_1$.

First-order Outer Solution

Here we hunt for the first-order outer solution $\delta_1(\mathrm{Re})\,\hat{\Psi}_1$ by asymptotically matching with some inner solutions. However we find that the first-order outer solution $\hat{\Psi}_1$ cannot be completely determined solely by the zero-order inner solution that we have just found it before. In fact we will see that first-order outer solution $\hat{\Psi}_1$ contains three terms in total and one term of which will match with the second term of the zero-order inner solution in (3.19). The remaining two terms of the first-order outer solution need some higher-order inner solution for matching. Now we substitute the outer region asymptotic expansions of $\hat{\Psi}$ and $\hat{\zeta}$ in (3.10), into the governing equations (3.12)–(3.13). Comparing the order of $\delta_1(\mathrm{Re})$ introduces the governing equations for the first-order outer solution:

$$\begin{cases} \mathrm{D}_*^2\hat{\Psi}_1 = -\eta_0^4\,(\xi_*^2+\eta_*^2)\,\hat{\zeta}_1, \\ \mathrm{D}_*^2\hat{\zeta}_1 = \eta_0^2\,(1+U_\infty)\left(\xi_*\dfrac{\partial\hat{\zeta}_1}{\partial\xi_*} - \eta_*\dfrac{\partial\hat{\zeta}_1}{\partial\eta_*}\right), \end{cases} \tag{3.25}$$

where D_*^2 is defined in (3.11). Using exactly the same technique when we solve (3.6) for the zero-order inner solution, the system (3.25) admits the solution in terms of the utilized outer variables (σ_*, τ_*):

$$\begin{cases} \hat{\zeta}_1(\sigma_*, \tau_*, t) = -2A_3 + C_1 \exp\big(-(1+U_\infty)\,\tau_*\big), \\ \hat{\Psi}_1(\sigma_*, \tau_*, t) = A_0 + A_1\sigma_* + A_2\tau_* + A_3\tau_*^2 + A_4\sigma_*\tau_* + 2A_3\sigma_*\tau_* \ln\tau_* \\ \qquad\qquad -\dfrac{C_1}{(1+U_\infty)^2}\exp\big(-(1+U_\infty)\,\tau_*\big) - \dfrac{C_1}{1+U_\infty}\sigma_*\,\mathrm{E}_2\big((1+U_\infty)\,\tau_*\big), \end{cases} \tag{3.26}$$

where C_1, A_0, A_1, A_2, A_3, A_4 are functions of t that can be determined by matching, the integral function $\mathrm{E}_2(x)$ is given by

$$\mathrm{E}_2(x) \;:=\; \int_x^\infty \int_y^\infty \frac{e^{-z}}{z}\,dz\,dy.$$

We present some details of finding the above mentioned six undetermined functions of t. In order for the outer solution $\hat{\Psi}_1$ in (3.26) to match with the inner solution (3.19), we expand the outer solution $\hat{\Psi}_1$ in the limit as $\tau_* \to 0$, using the following standard expansions as $x \to 0$,

$$e^{-x} \;=\; 1 - x + \frac{1}{2}x^2 + \mathcal{O}(x^3), \qquad \mathrm{E}_2(x) \;=\; 1 + (\gamma-1)\,x + x\,\ln x + \mathcal{O}(x^2),$$

where $\gamma = 0.57721\cdots$ is the Euler's constant. Applying these standard expansions in the limit as $\tau_* \to 0$, the first-order flow field solution $\hat{\Psi}_1$ in (3.26) can be written as

$$\begin{aligned} \delta_1(\mathrm{Re})\,\hat{\Psi}_1 \;=\; & \frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\Big\{C_1\big[-\frac{1}{2}\tau_*^2 - \sigma_*\tau_*\ln\tau_* + \sigma_*\tau_*\big(1+\ln(\frac{1}{2}\eta_0^2)\big)\big] \\ & +\big[A_0 - \frac{C_1}{(1+U_\infty)^2} + (A_1 - \frac{C_1}{1+U_\infty})\,\sigma_* + (A_2 + \frac{C_1}{1+U_\infty})\,\tau_* \\ & \quad + A_3\tau_*^2 + A_4\sigma_*\tau_* + 2A_3\sigma_*\tau_*\,\ln\tau_*\big]\Big\} \\ & +\frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\Big\{\,\mathcal{O}\big(\tau_*^3,\,\tau_*^4,\,\cdots,\,\sigma_*\tau_*^2,\,\sigma_*\tau_*^3,\,\cdots\big)\Big\} \\ & +\frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\,C_1\,\sigma_*\tau_*\Big\{-\gamma - \ln\big(\frac{1}{2}\eta_0^2\,(1+U_\infty)\big)\Big\}. \end{aligned} \tag{3.27}$$

The above first-order flow field solution contains three terms (in three curly brackets). In order for the first term in (3.27) to match with the second term in (3.19), we must set

$$C_1 \;=\; a_0(t) \;=\; -2U_\infty\,\exp(i\omega t), \qquad A_0 \;=\; -\frac{2U_\infty}{(1+U_\infty)^2}\,\exp(i\omega t),$$

$$A_1 \;=\; -\frac{2U_\infty}{1+U_\infty}\,\exp(i\omega t), \qquad A_2 \;=\; -A_1, \qquad A_3 \;=\; A_4 \;=\; 0.$$

Substituting the above into the original solution (3.26) introduces the first-order outer solutions:

$$\begin{cases} \hat{\zeta}_1(\sigma_*, \tau_*, t) = -2U_\infty \exp\big(i\omega t - (1+U_\infty)\,\tau_*\big), \\ \delta_1(\mathrm{Re})\,\hat{\Psi}_1(\sigma_*, \tau_*, t) = \dfrac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\,\hat{\phi}(\sigma_*, \tau_*;\, U_\infty)\,\exp(i\omega t), \end{cases} \tag{3.28}$$

where

$$\hat{\phi}(\sigma_*, \tau_*;\, U_\infty) := \frac{2U_\infty}{1+U_\infty}\left[\frac{\exp\big(-(1+U_\infty)\,\tau_*\big)-1}{1+U_\infty} + \tau_* - \sigma_* + \sigma_* \mathrm{E}_2\big((1+U_\infty)\,\tau_*\big)\right]. \tag{3.29}$$

Note that the remaining two terms of the first-order outer solution (3.27) remain unmatched. In fact we need some higher-order inner solutions to match with them. In view of that we shall find out what further inner solutions we need and see how could they match with these two terms in the first-order outer solution.

Higher-order Inner Solutions

Recall that the unmatched second term of the first-order outer solution (3.27) is given by $\delta_1(\mathrm{Re})\left\{\mathcal{O}\big(\tau_*^3,\, \tau_*^4,\, \cdots,\, \sigma_*\tau_*^2,\, \sigma_*\tau_*^3,\, \cdots\big)\right\}$. We intend to match this term with some higher-order inner solutions $\varepsilon_{j+1}(\mathrm{Re})\,\Psi_{j+1}$, $j = 0, 1, 2, \cdots$, which must satisfy the far field conditions:

$$\begin{aligned} \varepsilon_{j+1}(\mathrm{Re})\,\Psi_{j+1} &\approx \frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\,\mathcal{O}(\tau_*^{3+j},\, \sigma_*\tau_*^{2+j}) \\ &= \frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\cdot \mathrm{Re}^{3+j}\,\mathcal{O}(\tau^{3+j},\, \sigma\tau^{2+j}). \end{aligned}$$

We deduce the inner expansion coefficient of each order, by the above matching principle. We write

$$\varepsilon_{j+1}(\mathrm{Re}) = \frac{\mathrm{Re}^{j+1}}{\ln(1/\mathrm{Re})}, \qquad j = 0, 1, 2, \cdots. \tag{3.30}$$

Lastly the third term of the first-order outer solution in (3.27) still remains unmatched. We need a higher-order inner solution which we denote by

$$\varepsilon_0^{(1)}(\mathrm{Re})\,\Psi_0^{(1)}(\sigma, \tau, t)$$

to match with the unmatched third term. This higher-order inner solution must also satisfy the far field condition:

$$\varepsilon_0^{(1)}(\mathrm{Re})\,\Psi_0^{(1)}(\sigma_*, \tau_*, t) \approx \frac{1}{\mathrm{Re}^2\,\ln(1/\mathrm{Re})}\,2U_\infty\,\sigma_*\tau_*\,\lambda\,\exp(i\omega t),$$

where λ is a constant defined as

$$\lambda \;:=\; \gamma + \ln\left(\frac{1}{2}\,\eta_0^2\,(1+U_\infty)\right). \tag{3.31}$$

We note that $\varepsilon_0^{(1)}(\mathrm{Re})\,\Psi_0^{(1)}(\sigma,\tau,t)$ is only a term in the expansion of Ψ in (3.2). We substitute it into the governing equation (2.1) and compare the terms of order $\varepsilon_0^{(1)}(\mathrm{Re})$ on both sides. We obtain the governing equation for that higher-order inner solution and solve it for the flow field solution. To use the similar matching technique for finding the zero-order inner solution we finally deduce that

$$\varepsilon_0^{(1)}(\mathrm{Re})\,\Psi_0^{(1)}(\xi,\eta,t) \;=\; \frac{1}{\ln^2(1/\mathrm{Re})}\Big[-2U_\infty\,\psi(\xi,\eta)\,\lambda\,\exp(i\omega t)\Big], \tag{3.32}$$

where $\psi(\xi,\eta)$ is exactly the same as in (3.8).

Asymptotic Matching for Obtaining Global Solution

We present the summary of the asymptotic matching process so far in the following. We first derive the zero-order inner solution (3.19) which contains four terms of different orders of Re. The first term could match with the zero-order outer solution (3.18) completely. The second term matches with the first-order outer solution (3.27). But the third term of that outer solution needs a higher-order inner solution, which has the same form of solution as (3.19), to match with it. The second term of the induced higher-order inner solution also needs a higher-order outer solution to match with it. That outer solution would have the same form of solution in (3.27). Still, the third term of this higher-order outer solution needs another higher-order inner solution for matching. The above matching principle can be continued as a cycle.

The higher-order inner solution (3.32) induces a higher-order outer solution which is found to be

$$\delta_1^{(1)}(\mathrm{Re})\,\hat{\Psi}_1^{(1)}(\sigma_*,\tau_*,t) \;=\; \frac{1}{\mathrm{Re}^2\,\ln^2(1/\mathrm{Re})}\,\hat{\phi}(\sigma_*,\tau_*;\,U_\infty)\,\lambda\,\exp(i\omega t),$$

where $\hat{\phi}(\sigma_*,\tau_*;\,U_\infty)$ is given by (3.29).

Proceeding with this matching cycle generally induces more higher-order inner and outer solutions:

$$\begin{cases} \varepsilon_0^{(k)}(\mathrm{Re})\,\Psi_0^{(k)}(\xi,\eta,t) \;=\; \dfrac{1}{\ln^{k+1}(1/\mathrm{Re})}\Big[-2U_\infty\,\psi(\xi,\eta)\,\lambda^k\,\exp(i\omega t)\Big], \\ \delta_1^{(k)}(\mathrm{Re})\,\hat{\Psi}_1^{(k)}(\sigma_*,\tau_*,t) \;=\; \dfrac{1}{\mathrm{Re}^2\,\ln^{k+1}(1/\mathrm{Re})}\,\hat{\phi}(\sigma_*,\tau_*;\,U_\infty)\,\lambda^k\,\exp(i\omega t), \end{cases} \tag{3.33}$$

where $k = 1, 2, 3, \cdots$.

It, ultimately, follows from the results of (3.1), (3.7), (3.20)–(3.21), (3.30), (3.33) that the inner region matched asymptotic expansion solution for flow field is given by

$$\begin{aligned}
\Psi(\xi,\eta,t) &= \Psi_* + \varepsilon_0(\mathrm{Re})\,\Psi_0(\xi,\eta,t) \\
&\quad + \varepsilon_0^{(1)}(\mathrm{Re})\,\Psi_0^{(1)}(\xi,\eta,t) \\
&\quad + \varepsilon_0^{(2)}(\mathrm{Re})\,\Psi_0^{(2)}(\xi,\eta,t) + \cdots \\
&\quad + \varepsilon_1(\mathrm{Re})\,\Psi_1(\xi,\eta,t) + \varepsilon_2(\mathrm{Re})\,\Psi_2(\xi,\eta,t) + \cdots \\
&= \frac{1}{2}\eta_0^4\,\xi^2\eta^2 - \frac{2U_\infty}{\ln(1/\mathrm{Re})}\,\psi(\xi,\eta)\,\exp(i\omega t) \\
&\qquad \times\Big[1 + \frac{\lambda}{\ln(1/\mathrm{Re})} + \frac{\lambda^2}{\ln^2(1/\mathrm{Re})} + \cdots\Big] \\
&\quad + \frac{\mathrm{Re}}{\ln(1/\mathrm{Re})}\,\Psi_1(\xi,\eta,t) + \frac{\mathrm{Re}^2}{\ln(1/\mathrm{Re})}\,\Psi_2(\xi,\eta,t) + \cdots \\
&= \frac{1}{2}\eta_0^4\,\xi^2\eta^2 - \frac{2U_\infty}{\ln(1/\mathrm{Re}) - \lambda}\,\psi(\xi,\eta)\,\exp(i\omega t) \\
&\quad + \frac{\mathrm{Re}}{\ln(1/\mathrm{Re})}\,\Psi_1(\xi,\eta,t) + \frac{\mathrm{Re}^2}{\ln(1/\mathrm{Re})}\,\Psi_2(\xi,\eta,t) + \cdots,
\end{aligned} \tag{3.34}$$

where ψ and λ are respectively given by (3.8) and (3.31). With reference to (3.2) and (3.34), we may assume that the temperature field $T(\xi,\eta,t)$ and the interface shape function $\eta_S(\xi,t)$ will have the same asymptotic form of expansion solutions around the Ivantsov solution in the limit of $\mathrm{Re} \to 0$.

The outer region matched asymptotic expansion solution for the flow field, however, which based on the results of (3.1), (3.9), (3.15)–(3.16), (3.23)–(3.24), (3.28), (3.33) can be written as

$$\begin{aligned}
\hat{\Psi}(\xi_*,\eta_*,t) &= \Psi_* + \delta_0(\mathrm{Re})\,\hat{\Psi}_0(\xi_*,\eta_*,t) \\
&\quad + \delta_1(\mathrm{Re})\,\hat{\Psi}_1(\xi_*,\eta_*,t) + \delta_1^{(1)}(\mathrm{Re})\,\hat{\Psi}_1^{(1)}(\xi_*,\eta_*,t) \\
&\quad + \delta_1^{(2)}(\mathrm{Re})\,\hat{\Psi}_1^{(2)}(\xi_*,\eta_*,t) + \cdots \\
&\quad + \delta_2(\mathrm{Re})\,\hat{\Psi}_2(\xi_*,\eta_*,t) + \delta_3(\mathrm{Re})\,\hat{\Psi}_3(\xi_*,\eta_*,t) + \cdots \\
&= \frac{1}{\mathrm{Re}^2}\frac{1}{2}\eta_0^4\,\xi_*^2\eta_*^2 + \frac{1}{\mathrm{Re}^2}\frac{U_\infty}{2}\,\eta_0^4\,\xi_*^2\eta_*^2\,\exp(i\omega t) \\
&\quad + \frac{1}{\mathrm{Re}^2\big(\ln(1/\mathrm{Re}) - \lambda\big)}\,\hat{\phi}(\xi_*,\eta_*;\,U_\infty)\,\exp(i\omega t) \\
&\quad + \frac{1}{\mathrm{Re}\,\ln(1/\mathrm{Re})}\,\hat{\Psi}_2(\xi_*,\eta_*,t) + \frac{1}{\ln(1/\mathrm{Re})}\,\hat{\Psi}_3(\xi_*,\eta_*,t) + \cdots,
\end{aligned} \tag{3.35}$$

where $\hat{\phi}$ is given by (3.29).

4. Conclusion

In this paper we have introduced the method of matched asymptotic expansions and successfully applied it to the physical problem of dendritic crystal growth as the Reynolds number is considered to be small. Our results reveal that the asymptotic solution which is valid in the inner region has a different expansion from the asymptotic solution which is valid in the outer region and they should be solved independently and then be matched in an intermediate region. We are successful to find the matched asymptotic solution of the flow field which is a globally valid expansion solution in the whole physical domain. Further study and investigation on dendritic growth under the effect of convection induced by external flow may proceed as our analytical result provides a basis to resolve the yet unsolved problems of the selection of the tip velocity of dendrite and the formation of pattern on the interface.

References

[1] G. P. Ivantsov, Temperature field around a spheroidal, cylindrical and acicular crystal growing in a supercooled melt, *Dokl. Akad. Nauk SSSR* **58** (1947) 567.

[2] D. A. Kessler, J. Koplik and H. Levine, Pattern selection in fingered growth phenomena, *Advances in Physics* **37** (1988) 255–339.

[3] G. E. Nash and M. E. Glicksman, Capillarity-limited steady-state dendritic growth, I. Theoretical development, *Acta Metall.* **22** (1974) 1283–1299.

[4] R. J. Schaefer, M. E. Glicksman and J. D. Ayers, High-confidence measurement of solid/liquid surface energy in a pure material, *Phil. Mag.* **32** (1975) 725–743.

[5] M. D. Kruskal and H. Segur, Asymptotics beyond all orders in a Model of Crystal Growth, *Stud. Appl. Math.* **85** (1991) 129–181.

[6] P. Pelce and U. Pomeau, Dendrites in the small undercooling limit, *Stud. Appl. Math.* **74** (1986) 245–258.

[7] J. J. Xu, Uniformly Valid Asymptotic Solution for Steady Dendrite Growth in External Flow, *J. Crystal Growth* **128** (1993) 219–223.

[8] J. J. Xu, Dendritic Growth From Melt with External Flow: Uniformly Valid Asymptotic Solution of Steady State, *J. Fluid Mech.* **263** (1994) 227–243.

[9] J. J. Xu and D. S. Yu, Selection and Resonance of Dendritic Growth with Interference of Oscillatory External Sources, *J. Crystal Growth* **226** (2001) 378–392.

[10] J. J. Xu and J. Schimizu, Asymptotic Theory of Disc-like Crystal Growth: (II). Interfacial Instability and Pattern Formation at Early Stage of Growth, *Comm. on Pure and Applied Analysis* **3** (2004) 527–543.

[11] R. Ananth and W. N. Gill, Self-consistent theory of dendritic growth with convection, *J. Crystal Growth* **108** (1991) 173–189.

[12] S. K. Dash and W. G. Gill, Forced-convection heat and momentum-transfer to dendritic structures (parabolic cylinders and paraboloids of revolution), *Int. J. Heat Mass Transfer* **27** (8) (1984) 1345–1356.

[13] D. A. Saville and P. J. Beaghton, Growth of needle-shaped crystals in the presence of convection, *Phys. Rev. A* **37** (9) (1988) 3423–3430.