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Mathematical Modeling of Nutrient Uptake by Growing Root Considered as Moving Boundary Value Problem

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Abstract

Nutrient like potassium and phosphate does not flow with water quickly. Uptake of such nutrient is possible only when root surface come contact with root surface, i.e., diffusion of nutrient with moving boundary. Goal of present work is to evaluate solution for moving boundary model for the nutrient uptake by re-scaling and by using simple method described in [1], which takes into account a growing root system.

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Key words and Phrases:- Plant root, nutrient uptake by plant, moving boundary value problem

1 Introduction

The nutrient uptake has been evaluated through models that solve differential equations of transport in soils (diffusive and mass flows) coupled with absorption kinetics. These equations are solved on fixed domains (Cushman, 1979, Barber et al., 1984) through which the nutrient concentration at root-soil interface and the resulting nutrient uptake are estimated. The model of nutrient uptake by growing root is studied as the moving boundary value problem and solved by integral Balance method in [4]. Also the moving boundary value problem is solved by numerical method in [3].It is considered to be a one-dimensional model, i.e., a single cylindrical root in the soil for which the conditions of humidity, light and temperature are assumed to be controlled. We use the nutrient uptake model developed as moving boundary value problem and we split the differential into set of simple differential equations and solved using simple methods of ordinary differential equations. These models estimate the nutrient concentration in the root-soil interface and consequently the resultant nutrient uptake.

2 Moving Boundary Value Problem

One-dimensional root growth model through a moving boundary value problem to one phase in cylindrical co-ordinates is proposed in [2][3][4][5]

$$D\frac{\partial^2 C}{\partial r^2} + D(1+\varepsilon_0)\frac{1}{r}\frac{\partial C}{\partial r} = \frac{\partial C}{\partial t}, s_0 < r < R(t), \qquad 0 < t < T \tag{2.1}$$

$$C(r,0) = \varphi(r), \qquad s_0 < r < R_0 \tag{2.2}$$

$$-Db\frac{\partial}{\partial r}C(R(t),t) + v_0 C(R(t),t) = 0, \;\; 0 < t < T \tag{2.3}$$

$$Db\frac{\partial}{\partial r}C(s_o,t) + v_0 C(s_0,t) = \frac{k_a[C(s_0,t) - C_u]}{1 + \frac{k_a[C(s_0,t) - C_u]}{j_m}}, 0 < t < T \tag{2.4}$$

$$R(t) = R_0\sqrt{\frac{l_0}{l(t)}}, \qquad 0 < t < T \tag{2.5}$$

Where C is total diffusible ion concentration, r is the radial distance from the root axis [cm] and t is the time [sec]. The parameter v_0 is given by $\varepsilon_0 = \frac{v_0 s_0}{Db}$ where D is the diffusion coefficient, $v_0$is the effective velocity of flux solution on the root, s0 is the root initial radius [cm] and b is the buffer powder. $R(t)$ is the half distance between roots axis, T is the maximum time for which there is solution [s], R_0 is the initial half distance between root axis[cm] and $\varphi(r)$ is the initial concentration profile[mol/cm^3]. Let $k_a = \frac{j_m}{K_m}$, where j_m is the maximum in flux [mol cm^{-1}] and K_m is the concentration for which the influx is $\frac{j_m}{2}$ [mol cm^{-1}]. $C_u$is the minimum concentration under which growth is stopped [mol/cm^3]. $l(t)$ is the root length as a function of time [cm] and l_0 is the initial root length [cm].

Re-scaling equations (2.1) to (2.5) with $r = (1+\varepsilon_0)r'$, we get

$$\frac{\partial^2 C}{\partial r'^2} + \frac{1}{r'}\frac{\partial C}{\partial r'} = \left(\frac{(1+\varepsilon_0)}{D}\right)\frac{\partial C}{\partial t}, \frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)}, \qquad 0 < t < T \tag{2.6}$$

$$C((1+\varepsilon_0)r',0) = \varphi((1+\varepsilon_0)r'), \frac{s_0}{(1+\varepsilon_0)} < r' \frac{R_0}{(1+\varepsilon_0)}, \tag{2.7}$$

$$-Db\frac{\partial}{\partial r'}C\left(\frac{R(t)}{(1+\varepsilon_0)},t\right) + (1+\varepsilon_0)v_0 C\left(\frac{R(t)}{(1+\varepsilon_0)},t\right) = 0, \;\; 0 < t < T \tag{2.8}$$

$$Db\frac{\partial}{\partial r'}C\left(\frac{s_o}{(1+\varepsilon_0)},t\right) + (1+\varepsilon_0)v_0 C\left(\frac{s_0}{(1+\varepsilon_0)},t\right) = (1+\varepsilon_0)\frac{k_a[C\left(\frac{s_0}{(1+\varepsilon_0)},t\right) - C_u]}{1 + \frac{k_a[C\left(\frac{s_0}{(1+\varepsilon_0)},t\right) - C_u]}{j_m}}, \;\; 0 < t < T \tag{2.9}$$

$$R(t) = R_0\sqrt{\frac{l_0}{l(t)}}, 0 < t < T \tag{2.10}$$

Equation (2.9) may be written as

$$Db\frac{\partial}{\partial r'}C\left(\frac{s_o}{(1+\varepsilon_0)},t\right) + (1+\varepsilon_0)v_0 C\left(\frac{s_0}{(1+\varepsilon_0)},t\right) = (1+\varepsilon_0)v_0\frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)},t\right) - C_u\right]}{v_0\left[1 + \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)},t\right) - C_u\right]}{j_m}\right]}, \;\; 0 < t < T \tag{2.11}$$

Let $f(t)$ is ambient concentration.
Where

$$f(t) = \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) - C_u\right]}{v_0\left[1 + \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) - C_u\right]}{j_m}\right]} \tag{2.12}$$

We assume that the concentration $C(r', t)$ can be split up into two components as

$$C(r', t) = \theta\,(r', t) + \emptyset(r')f(t) \tag{2.13}$$

Where the dimensionless functions $\emptyset(r')$ is the solution of the following steady-state problem.

$$\frac{\partial}{\partial r'}\left(r' \frac{\partial \emptyset}{\partial r'}\right) = 0, \qquad \frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)} \tag{2.14}$$

$$-k\frac{\partial \emptyset}{\partial r'} + (1+\varepsilon_0)v_0\emptyset = 0, \qquad r' = \frac{s_0}{(1+\varepsilon_0)} \tag{2.15}$$

$$k\frac{\partial \emptyset}{\partial r'} + (1+\varepsilon_0)v_0\emptyset = (1+\varepsilon_0)v_0, \qquad r' = \frac{R(t)}{(1+\varepsilon_0)} \tag{2.16}$$

Solution of equation (2.14) is given by

$$\emptyset = \frac{(1+\varepsilon_0)v_0 r'}{2k} log r' + \left[\frac{r'}{2k} - (1+\varepsilon_0)v_0\, r'\, log r'\right]\frac{r'}{2k} \tag{2.17}$$

Then it can shown that the function $\theta(r', t)$ is the solution of the following one dimensional concentration with homogeneous convection boundary conditions and modified initial condition function is given by $F^*(r')$

$$\frac{\partial^2\theta}{\partial r'^2} + \frac{1}{r'}\frac{\partial\theta}{\partial r'} = \left(\frac{(1+\varepsilon_0)}{D}\right)\frac{\partial\theta}{\partial t}, \qquad \frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)}, \qquad 0 < t < T \tag{2.18}$$

$$-Db\frac{\partial}{\partial r'}\theta\left(\frac{R(t)}{(1+\varepsilon_0)}, t\right) + (1+\varepsilon_0)v_0\theta\left(\frac{R(t)}{(1+\varepsilon_0)}, t\right) = 0, \quad 0 < t < T \tag{2.19}$$

$$Db\frac{\partial}{\partial r'}\theta\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) + (1+\varepsilon_0)v_0\theta\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) = 0, \quad 0 < t < T \tag{2.20}$$

$$\theta = F^*(r'), \quad \frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)} \tag{2.21}$$

We use method of separation of variables for solution of equation (2.18). Let

$$\theta = U(r')T(t) \tag{2.22}$$

$$T(t)\left[\frac{\partial^2 U}{\partial r'^2} + \frac{1}{r'}\frac{\partial U}{\partial r'}\right] = U(r')\left[\left(\frac{(1+\varepsilon_0)}{D}\right)\frac{\partial T}{\partial t}\right], \frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)}, \qquad 0 < t < T \tag{2.23}$$

corresponding boundary condition will be transformed as

$$-Db\frac{\partial}{\partial r'}U\left(\frac{R(t)}{(1+\varepsilon_0)}, t\right) + (1+\varepsilon_0)v_0 U\left(\frac{R(t)}{(1+\varepsilon_0)}, t\right) = 0, \quad 0 < t < T \tag{2.24}$$

$$Db\frac{\partial}{\partial r'}U\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) + (1+\varepsilon_0)v_0 U\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) = 0, \quad 0 < t < T \tag{2.25}$$

$$T(t) = F^{**}(r') \quad \frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)} \tag{2.26}$$

let

$$\frac{1}{U(r')}\left[\frac{\partial^2 U}{\partial r'^2}+\frac{1}{r'}\frac{\partial U}{\partial r'}\right]=\frac{1}{T(t)}\left[\left(\frac{(1+\varepsilon_0)}{D}\right)\frac{\partial T}{\partial t}\right]=-\beta^2,\quad \frac{s_0}{(1+\varepsilon_0)}<r'<\frac{R(t)}{(1+\varepsilon_0)}\quad 0<t<T \tag{2.27}$$

Then we have Bessel's equtaion

$$\frac{\partial^2 U}{\partial r'^2}+\frac{1}{r'}\frac{\partial U}{\partial r'}+\beta^2 U(r')=0,\qquad 0<t<T \tag{2.28}$$

with boundary condition

$$-Db\frac{\partial}{\partial r'}U\left(\frac{R(t)}{(1+\varepsilon_0)},t\right)+(1+\varepsilon_0)v_0U\left(\frac{R(t)}{(1+\varepsilon_0)},t\right)=0,\qquad 0<t<T \tag{2.29}$$

$$Db\frac{\partial}{\partial r'}U\left(\frac{s_0}{(1+\varepsilon_0)},t\right)+(1+\varepsilon_0)v_0U\left(\frac{s_0}{(1+\varepsilon_0)},t\right)=0,\qquad 0<t<T \tag{2.30}$$

and

$$\left(\frac{(1+\varepsilon_0)}{D}\right)\frac{\partial T}{\partial t}+\beta^2 T(t)=0,\qquad \frac{s_0}{(1+\varepsilon_0)}<r'<\frac{R(t)}{(1+\varepsilon_0)} \tag{2.31}$$

with boundary condition

$$T(t)=F^{**}(r'),\quad \frac{s_0}{(1+\varepsilon_0)}<r'<\frac{R(t)}{(1+\varepsilon_0)} \tag{2.32}$$

boundary condition (2.29) and (2.30) are written as

$$-\frac{\partial}{\partial r'}U\left(\frac{R(t)}{(1+\varepsilon_0)},t\right)+\frac{(1+\varepsilon_0)v_0}{Db}U\left(\frac{R(t)}{(1+\varepsilon_0)},t\right)=0,\qquad 0<t<T \tag{2.33}$$

$$\frac{\partial}{\partial r'}U\left(\frac{s_0}{(1+\varepsilon_0)},t\right)+\frac{(1+\varepsilon_0)v_0}{Db}U\left(\frac{s_0}{(1+\varepsilon_0)},t\right)=0,\qquad 0<t<T \tag{2.34}$$

Equation (2.28) with boundary condition (2.33) and (2.34) is a special case of the Sturm-Liouville problem, hence the eigen functions $U(\beta_m,r')$have the following orthogonality property

$$\int_{\frac{s_0}{(1+\varepsilon_0)}}^{\frac{R(t)}{(1+\varepsilon_0)}} r'U(\beta_m,r')\,U(\beta_n,r')dr'=0,\qquad for\ m\neq n, and\ N(\beta_m)\ \ for\ m=n \tag{2.35}$$

where

$$N(\beta_m)=\int_{\frac{s_0}{(1+\varepsilon_0)}}^{\frac{R(t)}{(1+\varepsilon_0)}} r'U^2(\beta_m,r')\,dr' \tag{2.36}$$

Now we consider the representation of an arbitrary function $F(r')$ defined in the interval $\frac{s_0}{(1+\varepsilon_0)}<r'<\frac{R(t)}{(1+\varepsilon_0)}$ in terms of the eigen functions $U(\beta_m,r')$, of the above eigen value problem in the form

$$F(r')=\sum_{m=1}^{\infty}c_m\,U(\beta_m,r'),in\ \ \frac{s_0}{(1+\varepsilon_0)}<r'<\frac{R(t)}{(1+\varepsilon_0)} \tag{2.37}$$

The unknown coefficients c_m are determined and (2.37) is written as

$$F(r') = \sum_{m=1}^{\infty} \frac{1}{N(\beta_m)} U(\beta_m, r') \int_{\frac{s_0}{(1+\varepsilon_0)}}^{\frac{R(t)}{(1+\varepsilon_0)}} r'' U(\beta_m, r'')\, F(r'') dr'', in$$

$$\frac{s_0}{(1+\varepsilon_0)} < r' < \frac{R(t)}{(1+\varepsilon_0)} \qquad (2.38)$$

where $U(\beta_m, r')$ and $N(\beta_m)$ is given by

$$U(\beta_m, r') = J_0(\beta_m r') \left[\beta Y_1\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right) + \frac{(1+\varepsilon_0)v_0}{Db} Y_0\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right]$$

$$-Y_0(\beta_m r') \left[\beta_m J_1\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right) + \frac{(1+\varepsilon_0)v_0}{Db} J_0\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right] \qquad 2.39)$$

β_m is the value of

$$J_0(\beta_m r') \left[\beta_m Y_1\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right) + H Y_0\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right]$$

$$-Y_0(\beta_m r') \left[\beta_m J_1\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right) + H J_0\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right] = 0 \qquad (2.40)$$

and

$$N(\beta_m)$$

$$= \left[\beta_m J_1\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right) + \frac{(1+\varepsilon_0)v_0}{Db} J_0\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right]^2$$

$$+ \left[\beta_m Y_1\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right.$$

$$\left. + \frac{(1+\varepsilon_0)v_0}{Db} Y_0\left(\beta_m \frac{R(t)}{(1+\varepsilon_0)}\right)\right]^2 \qquad (2.41)$$

Solution of equation (2.31) with boundary condition (2.32) is given by

$$T(t) = F^{**}(r') e^{-\frac{\beta^2 Dt}{1+\varepsilon_0}} \qquad (2.42)$$

Hence $\theta(r', t)$ is given by

$$\theta(r', t) = \sum_{m=1}^{\infty} \frac{1}{N(\beta_m)} U(\beta_m, r') F^{**}(r') e^{-\frac{\beta^2 Dt}{1+\varepsilon_0}} \int_{\frac{s_0}{(1+\varepsilon_0)}}^{\frac{R(t)}{(1+\varepsilon_0)}} r'' U(\beta_m, r'')\, F(r'') dr'' \qquad (2.43)$$

The general solution of equation (2.6) is $C(r', t)$ and given by

$$C(r', t) = \sum_{m=1}^{\infty} \frac{1}{N(\beta_m)} U(\beta_m, r') F^{**}(r') e^{-\frac{\beta^2 Dt}{1+\varepsilon_0}} \int_{\frac{s_0}{(1+\varepsilon_0)}}^{\frac{R(t)}{(1+\varepsilon_0)}} r'' U(\beta_m, r'')\, F(r'') dr''$$

$$+ \left[\frac{(1+\varepsilon_0)v_0 r'}{2k} log r' + \left(\frac{r'}{2k} - (1+\varepsilon_0)v_0\, r'\, log r'\right)\frac{r'}{2k}\right] \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) - C_u\right]}{v_0\left[1 + \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) - C_u\right]}{j_m}\right]}$$

(2.44)

Putting $r' = \frac{r}{1+\varepsilon_0}$ in the equation (2.44), we get the general solution of equation (2.1) as

$$C(r,t) = \sum_{m=1}^{\infty} \frac{1}{N(\beta_m)} U\left(\beta_m, \frac{r}{1+\varepsilon_0}\right) F^{**}\left(\frac{r}{1+\varepsilon_0}\right) e^{-\frac{\beta^2 Dt}{1+\varepsilon_0}} \int_{\frac{s_0}{(1+\varepsilon_0)}}^{\frac{R(t)}{(1+\varepsilon_0)}} \frac{r'}{1+\varepsilon_0} U\left(\beta_m, \frac{r'}{1+\varepsilon_0}\right) F\left(\frac{r'}{1+\varepsilon_0}\right) dr'$$
$$+ \frac{v_0 r}{2k} log \frac{r}{1+\varepsilon_0}$$
$$+ \left[\frac{r}{2(1+\varepsilon_0)k} - v_0 r log \frac{r}{1+\varepsilon_0}\right] \frac{r}{2(1+\varepsilon_0)k} \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) - C_u\right]}{v_0\left[1 + \frac{k_a\left[C\left(\frac{s_0}{(1+\varepsilon_0)}, t\right) - C_u\right]}{j_m}\right]} \quad (2.45)$$

3 Discussion

Due to the re-scaling equation (1) reduces into diffusion equation. The solution of equation (2.6) may be written as combination of ambient concentration with two solutions of two component differential, i.e., (2.14) and (2.18). Equation (2.14) is time independent equation with mixed boundary condition which is nohomogeneous and equation (2.18) is the advection diffusion equation with homogeneous boundary condition. Solution of equation is obtained by simple integration and solution of equation (2.18) is obtained by using method of separation of variables in term of Bessel's function.

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