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Exact Traveling Wave Solutions to Camassa-Holm Equation

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Abstract

In this paper, the Camassa-Holm equation was transformed into a solvable indefinite integral form by the trial equation method, and then its exact traveling wave solutions were obtained, including triangular functional periodic solution and hyperbolic functional periodic solution.

Key words: Trial equation method, Camassa-Holm equation, Exact solution, Traveling wave solution

0 Introduction

Many methods have been put forward to solve nonlinear evolution equations, such as Backlund transformation method[1], hyperbolic function method[2], Adomian decomposition method[3], homogeneous balance method[4], Jacobi elliptic function expansion method[5] and so on. And a lot of exact solutions to nonlinear evolution equations have been obtained through these above methods. Camassa and Holm[6] raised a dispersive shallow water equation, and this equation was widely used and studied in many fields[7-14]. Recently, Liu Cheng-shi put forward a new kind of method of exact traveling wave solutions to nonlinear evolution equation, namely trial equation method[15-16]. Through this trial equation, the initial equation can be transformed into elementary integral form, and then its exact solution can be obtained. Trial equation method can also be applied in nonlinear evolution equations with variable coefficient. In this paper, CH-r equation was transformed into an elementary indefinite integral form by the trial equation method, and then its exact traveling wave solutions were obtained.

1 The main steps of trial equation method

Step 1. Select a nonlinear equation:

$$N(u,u_t,u_x,u_{xx},\cdots)=0 \tag{1}$$

Make wave transformation:

$$u = u(\xi)\,, \quad \xi = kx + \omega t \tag{2}$$

Where, k, ω are traveling wave parameters.

Plug (2) into (1), and then nonlinear ordinary differential equation is obtained:

$$M(u, u', u'', \cdots) = 0 \tag{3}$$

Step 2. Select a trial equation:

$$u'' = \sum_{i=0}^{m} a_i u^i \tag{4}$$

Where, $a_i (i = 0, 1, \cdots m)$ is constant. Multiply both sides of equation (4) by $2u'$, and then integrate it. The following can be gotten:

$$(u')^2 = F(u) = \sum_{i=0}^{m} \frac{2a_i}{i+1} u^{i+1} + d \tag{5}$$

Where, d is integral constant. Plug u and its derivatives into equation (3), and a polynomial $G(u)$ is gotten. The value of m is determined according to the principle of balance. If the values of the coefficients of $G(u)$ are all set 0, algebraic equations can be gotten, and the values of $a_0, a_1, \cdots a_m$ and d can be determined through solving this algebraic equations.

Step 3. Transform equation (5) into elementary integral form:

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}u}{\sqrt{F(u)}} \tag{6}$$

Classify the roots of the equation $F(u) = 0$ using complete discrimination system of the m-order polynomial, solve equation (6), and then the exact travelling wave solutions of equation (1) are gotten.

2 Solutions of CH-r equation

This is CH-r equation:

$$u_t + c_0 u_x + 3uu_x - \alpha^2 (u_{xxt} + uu_{xxx} + 2u_x u_{xx}) + ru_{xxx} = 0 \tag{7}$$

Where, $c_0, \alpha \neq 0, r$ are all arbitrary constants.

The multiple peakon solutions of equation (7) were discussed in references [9-12]. To seek its travelling wave solutions, make the wave transformation:

$$u = u(\xi)\,, \quad \xi = lx + \omega t \tag{8}$$

Plug (8) into (7) and integrate it. The following can be gotten:

$$l^2[(\alpha^2 lu - rl + \alpha^2 \omega)u'' + \frac{1}{2}\alpha^2 l(u')^2] = \frac{3}{2} lu^2 + (\omega + c_0 l)u + c \tag{9}$$

Equation (9) can not be transformed into elementary integral form directly, but it can do through the trial equation method.

Ihe trial equation is:

$$u'' = Au + B \tag{10}$$

Integrate equation (10), and then the following can be gotten:

$$(u')^2 = Au^2 + 2Bu + C \tag{11}$$

Plug equation (10) and (11) into (9):

$$\frac{3}{2}(\alpha^2 l^2 A - 1)lu^2 + [(\alpha^2\omega - rl)l^2 A + 2\alpha^2 l^3 B - (\omega + c_0 l)]u + \frac{\alpha^2 l^3}{2}C + Bl^2(\alpha^2\omega - rl) - c = 0 \tag{12}$$

In order to determine the values of parameters A, B and C, the coefficients of equation (12) are all set 0, and then algebraic system of equations is gotten:

$$\begin{cases} \dfrac{3}{2}(\alpha^2 l^2 A - 1)l = 0 \\ (\alpha^2\omega - rl)l^2 A + 2\alpha^2 l^3 B - (\omega + c_0 l) = 0 \\ \dfrac{\alpha^2 l^3}{2}C + Bl^2(\alpha^2\omega - rl) - c = 0 \end{cases} \tag{13}$$

Through solving equation (13), the values of parameters A, B and C are determined.

$$\begin{cases} A = \dfrac{1}{\alpha^2 l^2} \\ B = \dfrac{c_0\alpha^2 + r}{2\alpha^4 l^2} \\ C = \dfrac{(2c - c_0\omega)\alpha^4 + (c_0 l - 1)r\alpha^2 + r^2 l}{\alpha^6 l^3} \end{cases} \tag{14}$$

Solve equation (11) using the formula (14), and transform equation (11) into the same form as equation (6). Where, $F(u) = Au^2 + 2Bu + C$. Its discriminant $\Delta = B^2 - AC$ is discussed in three cases now.

Case 1: If $\Delta = 0$, there are two equal real roots in equation $F(u) = 0$:

$$F(u) = A(u + \frac{B}{A})^2 \tag{15}$$

If $A > 0$, integrate equation (6), and the exact solution of equation (7) is:

$$u = \pm\exp(\pm\sqrt{A}(\xi - \xi_0)) - \frac{B}{A} \tag{16}$$

Case 2: If $\Delta > 0$, there are two different real roots in the equation $F(u) = 0$:

$$F(u) = A(u + \frac{B}{A})^2 - \frac{B^2 - AC}{A} \tag{17}$$

Integrate equation (6), and the following can be gotten:

$$\pm\sqrt{A}(\xi - \xi_0) = \ln\left|Au + B + \sqrt{A}\sqrt{Au^2 + 2Bu + C}\right|,\ A > 0 \tag{18}$$

$$\pm\sqrt{-A}(\xi - \xi_0) = \arcsin\frac{-Au - B}{\sqrt{B^2 - AC}},\ A < 0 \tag{19}$$

Accordingly, the exact solution of equation (7) is:

$$u = \frac{1}{A}(\pm \sec h(\pm\sqrt{A}(\xi - \xi_0)) - B)\,,\ A > 0 \tag{20}$$

$$u = \frac{1}{A}(\pm\sqrt{B^2 - AC}\,\sin(\pm\sqrt{-A}(\xi - \xi_0)) - B)\,,\ A < 0 \tag{21}$$

Case 3: If $\Delta < 0$, there is no real roots in the equation $F(u) = 0$. When $A > 0$, integrate equation (6), and equation (18) can also be gotten.

So, solution (16)、solution (20) and solution (21) are exact travelling wave solutions to equation (7).

3 Conclusions

In this paper, Camassa-Holm equation was transformed into the elementary integral form by the trial equation method, and then its exact traveling wave solutions were obtained, including triangular functional periodic solution (formula (21)) and hyperbolic functional periodic solution (formula (20)). So this trial equation method is much more simple and powerful to get new solutions to various kinds of nonlinear evolution equations. This method will play an important role in finding exact solutions to mathematical physics equations.

References:

[1] C. Rogers, W. F. Shadwich Backlund transformation and their application, Academic Press, New York, 1982.

[2] Parkes E J, Duffy B R. Traveling solitary wave solutions to a compound Kdv-Burgers equation[J]. Phys Lett, 1997, 229A:217-220.

[3] Adomian G. Explicit solutions of nonlinear partial differential equations [J]. Appl Math Comput, 1997, 88:117-126.

[4] Lei Y. Exact solutions of nonlinear equations [J]. Phys Lett, 1999, 260A:55-59.

[5] Parkes E J, Duffy B R. The Jacobi elliptic function method for finding periodic wave solutions to nonlinear evolution equations [J]. Phys Lett, 2001, 295A:280-286.

[6] Camassa R, Holm D D. An integrable shallow water equation with peaked solitons[J]. Phys Rev Lett, 1993, 71 (11):1661-1664.

[7] Cooper F, Shepard H. Solitons in the Camassa-Holm shallow water equation[J]. Phys Lett A, 1994, 194 (4):246-250.

[8] Alber M S, Camassa R, Holm D D, et al. The geometry of peaked solitons and billiard solutions of a class of integrable PDEs[J]. Lett in Math Phys, 1994, 32 (2): 137-151.

[9] Boyd J P. Peakons and coshoidal waves: travelling wave solutions of the Camassa-Holm equation[J]. Appl Math Comput, 1997, 81 (2-3):173-187.

[10] Qian Tifei, Tang Minying. Peakons and periodic cusp waves in a generalized Camassa-Holm equation[J]. Choas, Solitons and Fractals, 2001, 12:1347-1360.

[11] Liu Zhengrong, QIAN Tifei. Peakons and their bifurcation in a generalized Camassa-Holm equation[J]. International Journal of Bi-furcation and Choas, 2001, 11 (3):781-792.

[12] Liu Zhengrong, Qian Ti-fei. Peakons of the Camassa-Holm equation[J]. Applied Mathematical Modeling, 2002, 26: 473-480.

[13] Constantin A, Strauss W A. Stability of the Cammassa-Holm solitons[J]. J Nonlinear Sci, 2002, 12:415-422.

[14] Reyes E G. Geometric integrability of the Camassa-Holm equation[J]. Lett Math Phys, 2002, 59 (2): 117-131.

[15] Liu Chengshi. Trial equation method and its applications on nonlinear evolution equations [J]. Chinese physics letters, 2005, 54 (6): 2 502-2 509.

[16] Liu Chengshi. Using trial equation method to solve exact solutions to variable coefficient nonlinear evolution equations [J]. Chinese physics letters, 2005, 54 (10): 4 506-4 510.