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Comparison of the Adomian Decomposion Method and Laplace Transform on a System of Cauchy PDEs

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Abstract

In this paper, the Adomian decomposition method (ADM) and Laplace transform are used to construct the solution of a linear system of partial differential equations.

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Introduction

Many problems are governed by partial differential equations, or by systems of partial differential equations. It is difficult to find their exact solutions. In this paper, we use the Laplace transform [9-11] and the Adomian decomposition method [1-8] permitted us to find the exact solution of the problem (1).

About solution of linear system of partial differential equations (PDEs)

Let us consider the following initial value problem of a linear system partial differential equations Cauchy kind:

$$(S):\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \lambda u(x,t) - \beta v(x,t); -\infty < x < +\infty, t \geq 0 \\ \dfrac{\partial v(x,t)}{\partial t} = \beta u(x,t) - \lambda v(x,t) \\ u(x,0) = f(x) \\ v(x,0) = g(x) \end{cases} \tag{1}$$

with $0 < \beta << 1$ and $0 < \lambda << 1$, where $u(x,t)$ and $v(x,t)$ the unknown functions, f and g are continu on $\mathbb{R}$.

The Adomian decomposition Method

General properties of the ADM and its applications can be found in [1-8]. Suppose that we need to solve the following equation:

$$Au = f \tag{2}$$

In a real Hilbert space H, where $A: H \to H$ is a linear or nonlinear operator, $f \in H$ and u is the unknown function. The principle of the ADM is based on the decomposition of the nonlinear operator A in the following form:

$$A = L + R + N \tag{3}$$

where $L + R$ is linear, N nonlinear, L invertible with L^{-1} as inverse. Using that decomposition, equation (2) is equivalent to:

$$u = \theta + L^{-1} f - L^{-1} Ru - L^{-1} Nu \tag{4}$$

where θ verifies $L\theta = 0$, (4) is called the Adomian fundamental equation or Adomian canonical form. We look for the solution of (2) in a series expansion form $u = \sum_{n=0}^{+\infty} u_n$ and we consider that $Nu = \sum_{n=0}^{+\infty} A_n$, where A_n are special polynomials of variables $u_0, u_1, ..., u_n$ called Adomian polynomials and defined by:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{+\infty} \lambda^i u_i \right) \right]_{\lambda=0} \qquad n = 0, 1, 2, ...$$

where λ is a parameter used by “convenience”. Thus (4) can be rewritten as follows:

$$\sum_{n=0}^{+\infty} u_n = \theta + L^{-1} f - L^{-1} R\left(\sum_{n=0}^{+\infty} u_n \right) - L^{-1}\left(\sum_{n=0}^{+\infty} A_n \right) \tag{5}$$

We suppose that the series $\left(\sum_{n=0}^{+\infty} u_n \right)$ and $\left(\sum_{n=0}^{+\infty} A_n \right)$ are convergent, and obtain by identification the following Adomian algorithm:

$$\begin{cases} u_0 = \theta + L^{-1} f \\ u_1 = -L^{-1}(Ru_0) - L^{-1} A_0 \\ . \\ . \\ . \\ u_{n+1} = -L^{-1}(Ru_n) - L^{-1} A_n \; ; \; n \geq 0 \end{cases} \tag{6}$$

In practice, it is often difficult to calculate all the terms of an Adomian series, so we approach the series solution by the truncated series: $u = \sum_{i=0}^{n} u_i$, where the choice of

n depends on error requirements. If this series converges, the solution of (2) is:

$$u = \lim_{n \to +\infty} \sum_{i=0}^{n} u_i$$

According to the Adomian decomposition method, we suppose that the solution (u,v) of (1) has the following form: $u = \sum_{i=0}^{n} u_i$ and $v = \sum_{i=0}^{n} v_i$. From (1), we have:

$$\begin{cases} u(x,t) = u(x,0) + \lambda \int_0^t u(x,s)\,ds - \beta \int_0^t v(x,s)\,ds \\ v(x,t) = v(x,0) + \beta \int_0^t u(x,s)\,ds - \lambda \int_0^t v(x,s)\,ds \end{cases} \tag{7}$$

And we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x,t) = u(x,0) \\ v_0(x,t) = v(x,0) \\ u_n(x,t) = \lambda \int_0^t u_{n-1}(x,s)\,ds - \beta \int_0^t v_{n-1}(x,s)\,ds \ ; \ n \geq 1 \\ v_n(x,t) = \beta \int_0^t u_{n-1}(x,s)\,ds - \lambda \int_0^t v_{n-1}(x,s)\,ds \ ; \ n \geq 1 \end{cases} \tag{8}$$

Finally, we get:

$$\begin{cases} u_0(x,t) = f(x) \\ u_1(x,t) = \left[\lambda f(x) - \beta g(x)\right] t \\ u_2(x,t) = \left(\lambda^2 - \beta^2\right) f(x) \dfrac{t^2}{2!} \\ u_3(x,t) = \left(\lambda^2 - \beta^2\right)\left[\lambda f(x) - \beta g(x)\right] \dfrac{t^3}{3!} \\ \ldots \\ u_{2n}(x,t) = \left(\lambda^2 - \beta^2\right)^n f(x) \dfrac{t^{2n}}{(2n)!} \\ u_{2n+1}(x,t) = \left(\lambda^2 - \beta^2\right)^n \left[\lambda f(x) - \beta g(x)\right] \dfrac{t^{2n+1}}{(2n+1)!} \end{cases} \tag{9}$$

$$\begin{cases} v_0(x,t) = g(x) \\ v_1(x,t) = \left[\beta f(x) - \lambda g(x)\right] t \\ v_2(x,t) = \left(\lambda^2 - \beta^2\right) g(x) \dfrac{t^2}{2!} \\ v_3(x,t) = \left(\lambda^2 - \beta^2\right)\left[\beta f(x) - \lambda g(x)\right] \dfrac{t^3}{3!} \\ \dots \\ v_{2n}(x,t) = \left(\lambda^2 - \beta^2\right)^n g(x) \dfrac{t^{2n}}{(2n)!} \\ v_{2n+1}(x,t) = \left(\lambda^2 - \beta^2\right)^n \left[\beta f(x) - \lambda g(x)\right] \dfrac{t^{2n+1}}{(2n+1)!} \end{cases} \tag{10}$$

Let us put

$$\begin{cases} \varphi_n^1(x,t) = \sum_{k=0}^{n} u_{2k}(x,t) \\ \varphi_n^2(x,t) = \sum_{k=0}^{n} u_{2k+1}(x,t) \\ \phi_n^1(x,t) = \sum_{k=0}^{n} v_{2k}(x,t) \\ \phi_n^2(x,t) = \sum_{k=0}^{n} v_{2k+1}(x,t) \end{cases} \tag{11}$$

Thus

$$\begin{cases} u(x,t) = \lim_{n\to+\infty} \varphi_n^1(x,t) + \lim_{n\to+\infty} \varphi_n^2(x,t) \\ v(x,t) = \lim_{n\to+\infty} \phi_n^1(x,t) + \lim_{n\to+\infty} \phi_n^2(x,t) \end{cases} \tag{12}$$

First case : $0 < \beta < \lambda << 1$

From (9), (10) and (11), we get:

$$\begin{cases} \varphi_n^1(x,t) = f(x) \sum_{k=0}^{n} \dfrac{\left(t\sqrt{\lambda^2 - \beta^2}\right)^{2k}}{(2k)!} \\ \varphi_n^2(x,t) = \dfrac{\lambda f(x) - \beta g(x)}{\sqrt{\lambda^2 - \beta^2}} \sum_{k=0}^{n} \dfrac{\left(t\sqrt{\lambda^2 - \beta^2}\right)^{2k+1}}{(2k+1)!} \\ \phi_n^1(x,t) = g(x) \sum_{k=0}^{n} \dfrac{\left(t\sqrt{\lambda^2 - \beta^2}\right)^{2k}}{(2k)!} \\ \phi_n^2(x,t) = \dfrac{\beta f(x) - \lambda g(x)}{\sqrt{\lambda^2 - \beta^2}} \sum_{k=0}^{n} \dfrac{\left(t\sqrt{\lambda^2 - \beta^2}\right)^{2k+1}}{(2k+1)!} \end{cases} \tag{13}$$

So we have:

$$\begin{cases} u(x,t) = f(x)ch\left(t\sqrt{\lambda^2-\beta^2}\right) + \dfrac{\lambda f(x)-\beta g(x)}{\sqrt{\lambda^2-\beta^2}} sh\left(t\sqrt{\lambda^2-\beta^2}\right) \\ v(x,t) = g(x)ch\left(t\sqrt{\lambda^2-\beta^2}\right) + \dfrac{\beta f(x)-\lambda g(x)}{\sqrt{\lambda^2-\beta^2}} sh\left(t\sqrt{\lambda^2-\beta^2}\right) \end{cases} \tag{14}$$

Second case : $0 < \lambda < \beta << 1$

$$\begin{cases} \varphi_n^1(x,t) = f(x)\sum_{k=0}^{n}(-1)^k \dfrac{\left(t\sqrt{\beta^2-\lambda^2}\right)^{2k}}{(2k)!} \\ \varphi_n^2(x,t) = \dfrac{\lambda f(x)-\beta g(x)}{\sqrt{\beta^2-\lambda^2}}\sum_{k=0}^{n}(-1)^k \dfrac{\left(t\sqrt{\beta^2-\lambda^2}\right)^{2k+1}}{(2k+1)!} \\ \phi_n^1(x,t) = g(x)\sum_{k=0}^{n}(-1)^k \dfrac{\left(t\sqrt{\beta^2-\lambda^2}\right)^{2k}}{(2k)!} \\ \phi_n^2(x,t) = \dfrac{\beta f(x)-\lambda g(x)}{\sqrt{\beta^2-\lambda^2}}\sum_{k=0}^{n}(-1)^k \dfrac{\left(t\sqrt{\beta^2-\lambda^2}\right)^{2k+1}}{(2k+1)!} \end{cases} \tag{15}$$

So we have:

$$\begin{cases} u(x,t) = f(x)\cos\left(t\sqrt{\beta^2-\lambda^2}\right) + \dfrac{\lambda f(x)-\beta g(x)}{\sqrt{\beta^2-\lambda^2}} \sin\left(t\sqrt{\beta^2-\lambda^2}\right) \\ v(x,t) = g(x)\cos\left(t\sqrt{\beta^2-\lambda^2}\right) + \dfrac{\beta f(x)-\lambda g(x)}{\sqrt{\beta^2-\lambda^2}} \sin\left(t\sqrt{\beta^2-\lambda^2}\right) \end{cases} \tag{16}$$

Third case: $\lambda = \beta$

From (1), we have:

$$\begin{cases} u(x,t) = u(x,0) + \beta\int_0^t \left(u(x,s)-v(x,s)\right)ds \\ v(x,t) = v(x,0) + \beta\int_0^t \left(u(x,s)-v(x,s)\right)ds \end{cases} \tag{17}$$

And we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x,t) = u(x,0) \\ v_0(x,t) = v(x,0) \\ u_n(x,t) = \beta\int_0^t \left(u_{n-1}(x,s)-v_{n-1}(x,s)\right)ds \ ; \ n \geq 1 \\ v_n(x,t) = \beta\int_0^t \left(u_{n-1}(x,s)-v_{n-1}(x,s)\right)ds \ ; \ n \geq 1 \end{cases} \tag{18}$$

We get:

$$\begin{cases} u_0(x,t) = f(x) \\ v_0(x,t) = g(x) \\ u_1(x,t) = \beta\left[f(x) - g(x)\right]t \\ v_1(x,t) = \beta\left[f(x) - g(x)\right]t \\ u_n(x,t) = v_n(x,t) = 0\ ;\ n \geq 2 \end{cases} \tag{19}$$

Thus:

$$\begin{cases} u(x,t) = u_0(x,t) + u_1(x,t) \\ v(x,t) = v_0(x,t) + v_1(x,t) \end{cases}$$

Finally, we get:

$$\begin{cases} u(x,t) = f(x) + \beta\left[f(x) - g(x)\right]t \\ v(x,t) = g(x) + \beta\left[f(x) - g(x)\right]t \end{cases} \tag{20}$$

The Laplace transform

Let us consider the following partial differential equation:

$$\begin{cases} Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \\ u(x,0) = f(x) \end{cases} \tag{21}$$

Where L the first order differential operator is $L = \frac{\partial}{\partial t}$, R the remaining linear operator, Nu is a nonlinear differential operator and $h(x,t)$ is source term. Applying Laplace transform to (22), we obtain:

$$£\left(Lu(x,t)\right) + £\left(Ru(x,t)\right) + £\left(Nu(x,t)\right) = £\left(h(x,t)\right) \tag{22}$$

Using the differentiation property of Laplace transform we get:

$$p£\left[u(x,t)\right] - f(x) + £\left[Ru(x,t)\right] + £\left[Nu(x,t)\right] = £\left[h(x,t)\right]$$

$$£\left[u(x,t)\right] = \frac{f(x)}{p} - \frac{£\left[h(x,t)\right]}{p} - \frac{£\left[Ru(x,t)\right]}{p} - \frac{£\left[Nu(x,t)\right]}{p} \tag{23}$$

Applying the inverse $£^{-1}$ of £ to (23), we obtain:

$$u(x,t) = f(x)£^{-1}\left(\frac{1}{p}\right) + £^{-1}\left(\frac{£\left[h(x,t)\right]}{p}\right) - £^{-1}\left(\frac{£\left[Ru(x,t)\right]}{p}\right) - £^{-1}\left(\frac{£\left[Nu(x,t)\right]}{p}\right) \tag{24}$$

From (1), we have:

$$\begin{cases} £\left(\frac{\partial u}{\partial t}\right) = \lambda £(u) - \beta £(v) \\ £\left(\frac{\partial v}{\partial t}\right) = \beta £(u) - \lambda £(v) \end{cases} \tag{25}$$

(25) $\Leftrightarrow$

$$\begin{cases} \pounds(u) = \dfrac{p+\lambda}{p^2-\lambda^2+\beta^2} f(x) - \dfrac{\beta}{p^2-\lambda^2+\beta^2} g(x) \\ \pounds(v) = \dfrac{\beta}{p^2-\lambda^2+\beta^2} f(x) - \dfrac{p-\lambda}{p^2-\lambda^2+\beta^2} g(x) \end{cases} \tag{26}$$

First case : $0<\beta<\lambda<<1$

$$\begin{cases} \pounds(u) = \left(\dfrac{a}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{b}{p+\sqrt{\lambda^2-\beta^2}}\right) f(x) + \left(\dfrac{c}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{c}{p+\sqrt{\lambda^2-\beta^2}}\right) g(x) \\ \pounds(v) = \left(\dfrac{e}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{f}{p+\sqrt{\lambda^2-\beta^2}}\right) f(x) + \left(\dfrac{g}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{h}{p+\sqrt{\lambda^2-\beta^2}}\right) g(x) \end{cases} \tag{27}$$

$(27) \Leftrightarrow (28)$:

$$\begin{cases} \pounds(u) = \dfrac{1}{2\sqrt{\lambda^2-\beta^2}}\left(\left(\dfrac{\lambda+\sqrt{\lambda^2-\beta^2}}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{-\lambda+\sqrt{\lambda^2-\beta^2}}{p+\sqrt{\lambda^2-\beta^2}}\right) f(x) + \left(-\dfrac{\beta}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{\beta}{p+\sqrt{\lambda^2-\beta^2}}\right) g(x)\right) \\ \pounds(v) = \dfrac{1}{2\sqrt{\lambda^2-\beta^2}}\left(\left(\dfrac{\beta}{p-\sqrt{\lambda^2-\beta^2}} - \dfrac{\beta}{p+\sqrt{\lambda^2-\beta^2}}\right) f(x) + \left(\dfrac{-\lambda+\sqrt{\lambda^2-\beta^2}}{p-\sqrt{\lambda^2-\beta^2}} + \dfrac{\lambda+\sqrt{\lambda^2-\beta^2}}{p+\sqrt{\lambda^2-\beta^2}}\right) g(x)\right) \end{cases}$$

Using $\pounds^{-1}$, we get:

$$\begin{cases} u(x,t) = \dfrac{\lambda f(x) - \beta g(x)}{\sqrt{\lambda^2-\beta^2}} \left(\dfrac{e^{t\sqrt{\lambda^2-\beta^2}} - e^{-t\sqrt{\lambda^2-\beta^2}}}{2}\right) + f(x)\left(\dfrac{e^{t\sqrt{\lambda^2-\beta^2}} + e^{-t\sqrt{\lambda^2-\beta^2}}}{2}\right) \\ v(x,t) = \dfrac{\beta f(x) - \lambda g(x)}{\sqrt{\lambda^2-\beta^2}} \left(\dfrac{e^{t\sqrt{\lambda^2-\beta^2}} - e^{-t\sqrt{\lambda^2-\beta^2}}}{2}\right) + g(x)\left(\dfrac{e^{t\sqrt{\lambda^2-\beta^2}} + e^{-t\sqrt{\lambda^2-\beta^2}}}{2}\right) \end{cases} \tag{29}$$

Finally, we get:

$$\begin{cases} u(x,t) = \dfrac{\lambda f(x) - \beta g(x)}{\sqrt{\lambda^2-\beta^2}} sh\left(t\sqrt{\lambda^2-\beta^2}\right) + f(x) ch\left(t\sqrt{\lambda^2-\beta^2}\right) \\ v(x,t) = \dfrac{\beta f(x) - \lambda g(x)}{\sqrt{\lambda^2-\beta^2}} sh\left(t\sqrt{\lambda^2-\beta^2}\right) + g(x) ch\left(t\sqrt{\lambda^2-\beta^2}\right) \end{cases} \tag{30}$$

Second case : $0<\lambda<\beta<<1$

$$\begin{cases} \pounds(u) = \dfrac{p}{p^2+\left(\sqrt{\beta^2-\lambda^2}\right)^2} f(x) + \dfrac{\lambda f(x) - \beta g(x)}{p^2+\left(\sqrt{\beta^2-\lambda^2}\right)^2} \\ \pounds(v) = \dfrac{\beta f(x) - \lambda g(x)}{p^2+\left(\sqrt{\beta^2-\lambda^2}\right)^2} + \dfrac{p}{p^2+\left(\sqrt{\beta^2-\lambda^2}\right)^2} g(x) \end{cases} \tag{31}$$

Using $\pounds^{-1}$, we get:

$$\begin{cases} u(x,t)=\dfrac{\lambda f(x)-\beta g(x)}{\sqrt{\beta^2-\lambda^2}}\sin\left(t\sqrt{\beta^2-\lambda^2}\right)+f(x)\cos\left(t\sqrt{\beta^2-\lambda^2}\right) \\ v(x,t)=\dfrac{\beta f(x)-\lambda g(x)}{\sqrt{\beta^2-\lambda^2}}\sin\left(t\sqrt{\beta^2-\lambda^2}\right)+g(x)\cos\left(t\sqrt{\beta^2-\lambda^2}\right) \end{cases} \quad (32)$$

Third case: $\lambda=\beta$

From (2.1), we have:

$$\begin{cases} \pounds(u)=\dfrac{1}{p}f(x)+\lambda\left(f(x)-g(x)\right)\dfrac{1}{p^2} \\ \pounds(v)=\lambda\left(f(x)-g(x)\right)\dfrac{1}{p^2}+\dfrac{1}{p}g(x) \end{cases}$$

Finally using $\pounds^{-1}$, we get:

$$\begin{cases} u(x,t)=f(x)+\lambda\left(f(x)-g(x)\right)t \\ v(x,t)=g(x)+\lambda\left(f(x)-g(x)\right)t \end{cases}$$

Example 1

If $\lambda=\beta=\dfrac{1}{2}$, $f(x)=\cos x$ and $g(x)=\sin x$, then we get with the two methods the following solution:

$$\left(u(x,t),v(x,t)\right)=\left(\cos x+\frac{1}{2}(\cos x-\sin x)t,\cos x+\frac{1}{2}(\cos x-\sin x)t\right)$$

Example 2

If $\lambda=\dfrac{1}{4},\beta=\dfrac{1}{3}$, $f(x)=\cos x$ and $g(x)=\sin x$, then we get with the two methods the following solution:

$$\begin{cases} u(x,t)=\dfrac{3\cos x-4\sin x}{\sqrt{7}}\sin\left(\dfrac{\sqrt{7}}{12}t\right)+\cos\left(\dfrac{\sqrt{7}}{12}t\right)\cos x \\ v(x,t)=\dfrac{4\cos x-3\sin x}{\sqrt{7}}\sin\left(\dfrac{\sqrt{7}}{12}t\right)+\cos\left(\dfrac{\sqrt{7}}{12}t\right)\sin x \end{cases}$$

Example 3

If $\lambda=\dfrac{1}{2},\beta=\dfrac{1}{4}$, $f(x)=\cos x$ and $g(x)=\sin x$, then we get with the two methods the following solution:

$$\begin{cases} u(x,t) = \dfrac{2\cos x - \sin x}{\sqrt{3}} sh\left(\dfrac{\sqrt{3}}{4}t\right) + ch\left(\dfrac{\sqrt{3}}{4}t\right)\cos x \\ v(x,t) = \dfrac{\cos x - 2\sin x}{\sqrt{3}} sh\left(\dfrac{\sqrt{3}}{4}t\right) + ch\left(\dfrac{\sqrt{3}}{4}t\right)\sin x \end{cases}$$

Conclusion

In this paper, we showed that using the both method, we get the same solution.

References

[1] K. ABBAOUI and Y. CHERRUAULT, "The Decomposition method applied to the Cauchy problem"Kybernetes, (28, 1), pp68-74, 1999.

[2] K. ABBAOUI and Y. CHERRUAULT, " Convergence of Adomian method applied to differential equations" Math. Comput.Modelling (28, 5), pp 103-109, 1994.

[3] N. NGARHASTA, B. SOME, K. ABBAOUI and Y. CHERRUAULT, "New numerical study of Adomian method applied to a diffusion model" Kybernetes, Vol. 31, n^0 1, pp 61-75, 2002.

[4] S. Khelifa and Yves CHERRUAULT, "The Decomposition method for solving first order partial differential equations", Kybernetes 31(6) (2002) 844-871.

[5] Pierre BAKI-TANGOU, Gabriel BISSANGA, "Application of Adomian Decomposition method to solving the Duffing-Van der Pol equation" Communications in Mathematical Analysis. Vol. 4, n^0 pp.30-40(2008).

[6] PARE Youssouf, Francis BASSONO and Blaise SOME, "A new technique for numerical resolution of few nonlinear integral equations of Fredholm by SBA method" Vol 70, $n^0$1, 2012, pp 21-33.

[7] Francis BASSONO, PARE Youssouf, Gabriel BISSANGA and Blaise SOME, "Application of the Adomian decomposition method and the perturbation method to solving a sytem of perturbation equations" Far East Journal of Applied Mathematics, Vol 72, 2012, Issue 2, pp 91-99.

[8] M. Hussain and Majid Khan, "Modified Laplace Decomposition Method", Applied Mathematical sciences, Vol. 4.2010, n^0 36, 1769-1783.

[9] S. AKhuri, "Laplace decomposition algorithm applied to clas of nonlinear differential equations, JMath, Appl 4 (2001) 141-155.

[10] ElcinYusufoglu(Agadjanov), "Numerical solution of Duffing equation by the Laplace decomposition algorithm" Appl. Math. Comput. 177(2006) 572-580.