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On a Mixed Boundary Value Problems of Linear Stochastic Partial Differential Equations of Parabolic Type

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Abstract

In this paper, linear stochastic partial differential equations (denoted by SPDEs hereafter) in the 3 dimensional domain with derivative boundary value conditions are considered. We consider linear SPDEs in the domain with various differential boundary conditions.

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1. Introduction

The problem discussed in this paper is in the following form:

$$\begin{cases} \mathrm{d}u = \left[a^{ij}u_{x_i x_j} + b_i u_{x_i} - cu + f\right]\mathrm{d}t + [\sigma^{il}u_{x_i} + \nu_l u + g^l]\,\mathrm{d}w_t^l \\ \text{in } \Omega \times [0,T] \times G \equiv \Omega \times G_T, \\ B(t,x,\partial_x)u|_{\partial G_T} \equiv \boldsymbol{l} \cdot \nabla u + ku|_{\partial G_T} = h \\ \text{on } \Omega \times [0,T] \times \partial G \equiv \Omega \times \partial G_T, \\ u(0,x) = u_0(x) \quad \text{on } \Omega \times G, \end{cases} \tag{1.1}$$

where G is a domain in $\boldsymbol{R}^n$ under some conditions stated later, and all the coefficients and f, $g = (g^l)_{l=1}^n$, h are functions depending on $(\omega,t,x) \in \Omega \times G_T$. in (1.1) the Einstein rule is applied for the representation of the summasion. Note that we use both of the notations $\partial_{x_i}u$ and u_{x_i} for the derivative of u with respect to x_i. There exist a large amount of contibutions to linear SPDEs in the literature. Among them, Krylov and Rozovskii [9] is one of the most attractive results. They investigated the Cauchy problem of linear SPDEs in a weak sense, whose solution takes the value in Banach and Hilbert spaces. They have also shown the existence and uniqueness of the solution to the first boundary value problem [4]. Krylov [6] investigated the spacial regularity of the solution to the first boundary value problem in the weighted Sobolev space, based on the preceding result [9] in the half space. After the result concerning the Dirichlet and differential boundary value problems of partial differential equations with measurable coefficients [5], Krylov investigated the first boundaryvalue problem of parabolic SPDEs on a half-plane in a modified weighted Sobolev spaces [7]. They also investigated the first boundary value problem in the Sobolev space with fractional derivatives [8]. In it, the solution satisfies the SPDE in a distribution sense, whose derivative is defined using Bessel potentials. However, as far as we know, there are few results concerning the boundary conditions with both tangential and normal derivatives. Shimizu [13] studied the following second boundary value problem based on the potential theory:

$$\begin{cases} \mathrm{d}u = \left[a^{ij}u_{x_i x_j} - cu + f\right]\mathrm{d}t + [\nu_l u + g^l]\,\mathrm{d}w_t^l \quad \text{in } \Omega \times G_T, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Omega \times \partial G_T, \\ u(0,x) = u_0(x) \quad \text{on } \Omega \times G, \end{cases}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)^{\mathrm{T}}$ is the outer unit normal to G, and c, f, ν and g depend only on t, which is a restrictive assumption. In the present paper, we consider more general problem (1.1). One of the difficulties exists in setting the appropriate function space. Since the order of the derivative is necessarily higher than that of weight, the approach of Krylov [6] is not applicable. Another difficulty lies in the settings of functionspace that includes the lower derivative of the solution. Weighted Sobolev space is applied, which resolved these difficulties in this paper. The case of $p \neq 2$ and the nonlinear problems will be considered in the following papers.

2. Function Spaces

Let G stands for a domain in $\boldsymbol{R}^n$. By $W_2^l(G)$ we mean a space of functions $u(x), x \in G$, equipped with the norm $\| u \|^2_{W_2^l(G)} = \sum_{|\alpha|<l} \| D^\alpha u \|^2_{L_2(G)} + \| u \|^2_{\dot{W}_2^l(G)}$,

$$\begin{cases} \| u \|^2_{\dot{W}_2^l(G)} = \displaystyle\sum_{|\alpha|=l} \| D^\alpha u \|^2_{L_2(G)} \text{ if } l \text{ is an integer}, \\ \| u \|^2_{\dot{W}_2^l(G)} = \displaystyle\sum_{|\alpha|=[l]} \int_G \int_G \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} \mathrm{d}x\mathrm{d}y \\ \text{if } l \text{ is a non}-\text{integer},\ l = [l] + \{l\},\ 0 < \{l\} < 1. \end{cases}$$

Weighted Sobolev space with a positive integer r and $\gamma \in \boldsymbol{R}$ with $0 < r + \gamma - 1/2 < r$ is defined as follows:

$$W^r_{2,\psi,\gamma}(G) \equiv \{u \in L_2(G) | \psi^\gamma D^\alpha u \in L_2(G),\ |\alpha| = r\};$$
$$W^r_{2,\psi,\gamma}(G_T) \equiv \{u \in L_2(G_T) | \psi^\gamma D^\alpha u \in L_2(G_T),\ |\alpha| = r\};$$

where $\psi = \psi(x)$ satisfies following conditions with positive constants χ and ρ_0:

$$\begin{cases} \psi \in C^1(\overline{G}), \\ \psi \leq \chi^{-1} dist(x, \partial G) \quad \text{if } dist(x, \partial G) \leq \rho_0, \\ \psi \geq \chi \quad \text{if } dist(x, \partial G) > \rho_0 \quad \text{on } G, \end{cases} \tag{2.1}$$

where $\overline{G}$ is the closure set of G, and $dist(x, \partial G)$ is the distance in the normal sense from a point $x \in \boldsymbol{R}^3$ and ∂G. When $G = \boldsymbol{R}_+^3 = \{x \in \boldsymbol{R}^3 | x' \in \boldsymbol{R}^2, x_3 > 0\}$, we take $\psi = \tilde{\psi}(x) \equiv x_3$. The norms of the functions spaces defined above are defined as follows, respectively:

$$|||u|||_{\psi,\gamma,r,G}^2 \equiv E[\| u \|_{\psi,\gamma,r,G}^2], \quad |||u|||_{\psi,\gamma,r,G_T}^2 \equiv E[\| u \|_{\psi,\gamma,r,G_T}^2],$$

where

$$\| u \|_{\psi,\gamma,r,G}^2 \equiv \| u \|_{L_2(G)}^2 + \sum_{|\alpha|=r} \| \psi^\gamma D^\alpha u \|_{L_2(G)}^2,$$

$$\| u \|_{\psi,\gamma,r,G_T}^2 \equiv \| u \|_{L_2(G_T)}^2 + \sum_{|\alpha|=r} \| \psi^\gamma D^\alpha u \|_{L_2(G_T)}^2,$$

We also define the following spaces:

$$\widetilde{W}_{2,\psi,\gamma,r}^r(\mathcal{G}) \equiv \{f | |||f|||_{\psi,\gamma,r,\mathcal{G}}^{*2} \equiv E \| \psi f \|_{L_2(\mathcal{G})}^2 + \sum_{|\alpha|=r} E \| \psi^\gamma D^\alpha f \|_{L_2(\mathcal{G})}^2 < \infty\}.$$

We also use the notation $L_2(\Omega \times [0, T], \mathcal{P}, \mathcal{W}) \equiv L_2(\Omega_T, \mathcal{P}, \mathcal{W}) \equiv \{u | E \| u \|_{L_2([0,T];\mathcal{W})}^2 < \infty\}$ with a function space $\mathcal{W}$ in general.

3. Main Results

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with a 3-dimensional Wiener process $(w_t, \mathcal{F}_t)$ defined on it for $t \geq 0$. Denote by $\mathcal{P}$ the σ-algebra of predictable sets on $\Omega \times (0, \infty)$ associated with $\mathcal{F}_t$. Let $\mathcal{G}$ be a domain in $\boldsymbol{R}^3$ with boundary $\partial\mathcal{G}$ of $C^{1+\alpha}$ $(0 < \alpha < 1)$, and $m \geq 0$ be a non-negative number. We impose following assumptions named conditions (A):

1. $\bar{A} \equiv \{a^{ij}\}$, $b = (b_i)$, c, f, $\overline{\sum} \equiv \{\sigma^{ij}\}$, ν and $g = (g^l)$ are $\mathcal{P}$ measurable in the set of real 3×3 matrices, in $\boldsymbol{R}^3$, in $\boldsymbol{R}$, in $\boldsymbol{R}$, in $\boldsymbol{R}$, in the set of real 3×3 matrices, in $\boldsymbol{R}^3$ and in $\boldsymbol{R}^3$, respectively;
2. $\bar{A}$ and $\bar{\Sigma}$ are defined on $\bar{\mathcal{G}}$, and are bounded and continuous both with respect to t and x in $\bar{\mathcal{G}}_T$;
3. There exist ψ, ρ_0 and $\chi > 0$ satisfying (2.1);
4. u_0 is $\mathcal{F}_0$ measurable, $u_0 \in L_2(\Omega, \mathcal{P}, W_{2,\psi,m}^{m+2}(\mathcal{G}))$;
5. $b, \nu \in L_p(\Omega \times [0, T], \mathcal{P}, L_\infty(\mathcal{G}))$ with $p > 2$;
6. Either the following conditions holds:

$$\begin{cases} \psi^{m-i} D^\alpha b \in L_\infty(\Omega; L_p([0, T]; L_q(\mathcal{G}))) \\ \text{for } |\alpha| = m + 1 - i,\ i = 1, \ldots, mp > 2,\ q \geq 6/5; \\ \psi^{m-i+1} D^\alpha b \in L_\infty(\Omega; L_p([0, T]; L_\infty(\mathcal{G}))) \\ \text{for } |\alpha| = m + 1 - i,\ i = 1, \ldots, mp > 2; \end{cases}$$

7. The following conditions hold:

$$\begin{cases} \psi^{m-i}D^\alpha c,\ \psi^m D^\beta c \in L_\infty(\Omega; L_p([0,T]; L_q(\mathcal{G}))) \\ \text{for } |\alpha| = m-i,\ |\beta| = m+1,\ i = 0,\dots,m,\ p>2,\ q \geq 6/5; \\ \psi^{m-i+1}D^\alpha v,\ \psi^m D^\beta v \in L_\infty(\Omega; L_p([0,T]; L_q(\mathcal{G}))) \\ \text{for } |\alpha| = m+2-i, |\beta| = m+2,\ i = 1,\dots,m+1,\ p>2,\ q \geq 6/5; \end{cases}$$

10. $k,\ l_i \in L_\infty(\Omega, \mathcal{P}, \partial\mathcal{G}_T)$ $(i = 1,2,3)$;
11. $\overline{\Sigma}\ \overline{\Sigma}^{*} + \chi I \leq 2a \leq \chi^{-1}I,$; where I is the unit matrix and $\overline{\Sigma}^{*}$ means the transpose of the matrix $\overline{\Sigma}$;
12. $|\boldsymbol{l}\cdot\boldsymbol{n}| \geq \delta > 0$, where $\boldsymbol{n} = (n_1, n_2, n_3)^{\mathrm{T}}$ is an outer unit normal to $\partial\mathcal{G}$;
13. $f \in L_2(\Omega_T, \mathcal{P}, \widetilde{W}^{m+1}_{2,\psi,m}(\boldsymbol{R}^3_+))$, $g \in L_2(\Omega_T, \mathcal{P}, W^{m+2}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+))$,

$h \in L_2(\Omega_T, \mathcal{P}, W_2^{\frac{3}{2}}(\boldsymbol{R}^2)) \cap C\,([0,T]; W_2^{\frac{1}{2}}(\boldsymbol{R}^2))$ (a.s.).

The next theorem is the main result of this paper.

Theorem 1. For arbitrary $T > 0$, under the conditions (A), problem (1.1) has a unique solution

$$u \in L_2(\Omega_T, \mathcal{P}, W^{m+3}_{2,\psi,m}(\mathcal{G})) \bigcap C\,([0,T], W^{m+2}_{2,\psi,m}(\mathcal{G}))\ \text{(a.s.)}$$

which satisfies

$$\begin{aligned} &|||u|||^2_{\psi,m,m+3,\mathcal{G}_T} + E \sup_{t\in[0,T]} \| u(t) \|^2_{\psi,m,m+2,\mathcal{G}} \\ &\leq C[|||f|||^{*2}_{\psi,m,m+1,\mathcal{G}_T} + |||g|||^2_{\psi,m,m+2,\mathcal{G}_T} + E \| h \|^2_{L_2([0,T],\mathcal{P},W_2^{\frac{3}{2}}(\partial\mathcal{G}))} \\ &+E \sup_{t\in[0,T]} \| h(t) \|^2_{W_2^{\frac{1}{2}}(\partial\mathcal{G})} + |||u_0|||^2_{\psi,m,m+2,\mathcal{G}}]. \end{aligned}$$

Hereafter C denotes a positive constant which does not depend on u.

4. Model Problem in a half space

In this section, we consider the problem which contains only the principal terms, constant coefficients in the half space $\boldsymbol{R}^3_+ = \{y = (y', y_3) | y' \in \boldsymbol{R}^2,\ y_3 > 0\}$. Coordinate system in (4.1) below is denoted by y to distinguish it from the one in the original problem (1.1).

$$\begin{cases} \mathrm{d}u = [a_0^{ij} u_{y_i y_j} + f]\mathrm{d}t + [\sigma_0^{il} u_{y_i} + g^l]\,\mathrm{d}w_t^l \\ \text{in } \Omega \times [0,T] \times \boldsymbol{R}^3_+, \\ \boldsymbol{l}_0 \cdot \nabla u|_{y_3=0} = h(t, y') \\ \text{on } \Omega \times \boldsymbol{R}^2 \times [0,T] \equiv \Omega \times \boldsymbol{R}^2_T, \\ u(0,y) = u_0 \quad \text{on } \Omega \times \boldsymbol{R}^3_+, \end{cases} \tag{4.1}$$

where $\{a_0^{ij}\}^3_{i,j=1}$, $\{\sigma_0^{il}\}^3_{i,j=1}$ and $\boldsymbol{l} = (l_{10}, l_{20}, l_{30})^{\mathrm{T}}$ are constants with $l_{30} \neq 0$.

Theorem 1. Assume $f \in L_2(\Omega_T, \mathcal{P}, \widetilde{W}^{m+1}_{2,\psi,m}(\boldsymbol{R}^3_+))$, $g \in L_2(\Omega_T, \mathcal{P}, W^{m+2}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+)), h \in L_2(\Omega_T, \mathcal{P}, W_2^{\frac{3}{2}}(\boldsymbol{R}^2)) \cap C\,([0,T]; W_2^{\frac{1}{2}}(\boldsymbol{R}^2))$ (a.s.), and $u_0 \in L_2(\Omega, \mathcal{P}, W^{m+2}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+))$ with an interger $m \geq 0$.

Then, there exists a unique solution to (4.1)

$$u \in L_2(\Omega_T, \mathcal{P}, W^{m+3}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+)) \bigcap L_2\,(\Omega_T, \mathcal{P}, W_2^1(\boldsymbol{R}^3_+)) \bigcap C\,([0,T]; W^{m+2}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+))\ (\text{a.s.})$$

satisfying the following estimates:

$$\begin{aligned}
&|||u|||^2_{\widetilde{\psi},m,m+3,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]} |||u(t)|||^2_{\widetilde{\psi},m,m+2,[0,T]\times\boldsymbol{R}^3_+} \\
&\leq C[|||f|||^{*2}_{\widetilde{\psi},m,m+1,[0,T]\times\boldsymbol{R}^3_+} + |||g|||^2_{\widetilde{\psi},m,m+2,[0,T]\times\boldsymbol{R}^3_+} \\
&+|||h|||^2_{L_2(0,T;W_2^{\frac{3}{2}}(\boldsymbol{R}^2))} + E\sup_{t\in[0,T]} \| h(t) \|^2_{W_2^{\frac{1}{2}}(\boldsymbol{R}^2)} + |||u_0|||^2_{\widetilde{\psi},m,m+2,\boldsymbol{R}^3_+}], \\
&|||u|||^2_{\widetilde{\psi},0,1,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]} |||u(t)|||^2_{L_2(\boldsymbol{R}^3_+)} \\
&\leq C[|||f|||^{*2}_{\widetilde{\psi},m,m+1,[0,T]\times\boldsymbol{R}^3_+} + |||g|||^2_{\widetilde{\psi},m,m+2,[0,T]\times\boldsymbol{R}^3_+} \\
&+|||u_0|||^2_{\psi,m,m+2,\boldsymbol{R}^3_+} + |||h|||^2_{L_2(0,T;W_2^{\frac{1}{2}}(\boldsymbol{R}^2))} + E\sup_{t\in[0,T]} \| h(t) \|^2_{W_2^{\frac{1}{2}}(\boldsymbol{R}^2)}].
\end{aligned}$$

To prove Theorem 4.1, we consider the following problems:

$$\begin{cases}
\mathrm{d}u_1 = \left[a_0^{ij} u_{1y_iy_j} + f\right]\mathrm{d}t + [\sigma_0^{il} u_{1y_i} + g^l]\mathrm{d}w_t^l \\
\text{in } \Omega \times [0,T] \times \boldsymbol{R}^3_+, \\
u_1|_{y_3=0} = 0 \quad \text{on } \Omega \times \boldsymbol{R}^2, \\
u_1(0,y) = u_0 \quad \text{on } \Omega \times \boldsymbol{R}^3_+,
\end{cases} \tag{4.2}$$

$$\begin{cases}
\mathrm{d}u_2 = a_0^{ij} u_{2y_iy_j}\mathrm{d}t \quad \text{in } \Omega \times [0,T] \times \boldsymbol{R}^3_+, \\
\boldsymbol{l}_0 \cdot \nabla u_2|_{y_3=0} = h - \boldsymbol{l}_0 \cdot \nabla u_1|_{y_3=0} \quad \text{on } \Omega \times \boldsymbol{R}^2_T, \\
u_2(0,y) = 0 \quad \text{on } \Omega \times \boldsymbol{R}^3_+,
\end{cases} \tag{4.3}$$

where $\{a_0^{ij}\}^3_{i,j=1}$, $\{\sigma_0^{il}\}^3_{i,j=1}$ and $\boldsymbol{l} = (l_{10}, l_{20}, l_{30})^{\mathrm{T}}$ are constants with $l_{30} \neq 0$. Theorem .1 results from Lemmas 4.1 and 4.2 below.

Lemma 1. For arbitrary $T > 0$ with the same assumptions as in Theorem .1, there exists a unique solution u_1 to problem (4.2) such that

$$u_1 \in L_2(\Omega_T, \mathcal{P}, W^{m+3}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+)) \bigcap L_2\,(\Omega_T, \mathcal{P}, W_2^1(\boldsymbol{R}^3_+)) \bigcap C\,([0,T], W^{m+2}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+))\ (\text{a.s})$$

satisfying the following estimates for an integer $m \geq 0$:

$$|||u_1|||^2_{\tilde{\psi},m,m+3,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]}|||u_1(t)|||^2_{\tilde{\psi},m,m+2,\boldsymbol{R}^3_+}$$
$$\le C[|||f|||^{*2}_{\tilde{\psi},m,m+1,[0,T]\times\boldsymbol{R}^3_+} + |||g|||^2_{\tilde{\psi},m,m+2,[0,T]\times\boldsymbol{R}^3_+} + |||u_0|||^2_{\tilde{\psi},m,m+2,\boldsymbol{R}^3_+}],$$
$$|||u_1|||^2_{\tilde{\psi},0,1,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]}|||u_1(t)|||^2_{L_2(\boldsymbol{R}^3_+)}$$
$$\le C[E\,\|\,\psi f\,\|^2_{L_2([0,T]\times\boldsymbol{R}^3_+)} + |||g|||^2_{L_2(\Omega_T,\mathcal{P},L_2(\boldsymbol{R}^3_+))} + |||u_0|||^2_{\tilde{\psi},m,m+2,\boldsymbol{R}^3_+}].$$

In additon, problem (4.3) has a unique solution

$$u_2 \in L_2(\Omega_T,\mathcal{P},W^{m+3}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+)) \bigcap L_2(\Omega_T,\mathcal{P},W^1_2(\boldsymbol{R}^3_+)) \bigcap C(\Omega_T,W^{m+2}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+))\ (\text{a.s.})$$

satisfying the following estimates:

$$|||u_2|||^2_{\tilde{\psi},m,m+3,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]}|||u_2(t)|||^2_{\tilde{\psi},m,m+2,\boldsymbol{R}^3_+}$$
$$\le C[|||h|||^2_{L_2(\Omega_T,\mathcal{P},W_2^{\frac{3}{2}}(\boldsymbol{R}^2))} + E\sup_{t\in[0,T]}\|\,h(t)\,\|^2_{W_2^{\frac{1}{2}}(\boldsymbol{R}^2)}$$
$$+|||u_1|||^2_{\tilde{\psi},m,m+3,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]}|||u_1(t)|||^2_{\tilde{\psi},m,m+2,\boldsymbol{R}^3_+}],$$
$$|||u_2|||^2_{\tilde{\psi},0,1,[0,T]\times\boldsymbol{R}^3_+} + E\sup_{t\in[0,T]}|||u_2(t)|||^2_{L_2(\boldsymbol{R}^3_+)}$$
$$\le C[|||h|||^2_{L_2(\Omega_T,\mathcal{P},W_2^{\frac{1}{2}}(\boldsymbol{R}^2))} + E\sup_{t\in[0,T]}\|\,h(t)\,\|^2_{W_2^{\frac{1}{2}}(\boldsymbol{R}^2)}$$
$$+E\sup_{t\in[0,T]}\|\,u_1(t)\,\|^2_{W^{m+2}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+)}].$$

Due to the lack of the space, the detail of the proof is omitted. We merely refer to making use of the embedding theorem for weighted Sobolev spaces $W^{r+2+i}_{2,\tilde{\psi},r}(\boldsymbol{R}^3_+) \subset W_2^{\frac{3}{2}+i}(\boldsymbol{R}^2)$ $(i = 0,1)$ by Nikol'skiĭ [11] in the proof.

5. Problem in the Original Domain

Now we consider (1.1) based on the regularizer method [10], which requires some assumptions on the domain.

1. The Original Domain

First, it is assumed that there exists a number $d > 0$ such that, in a sphere of radius d with center at any point $\xi \in \partial\mathcal{G}$, $\partial\mathcal{G}$ is provided in a local coordinate system by the equation $z_3 = F(z') = F(z_1, z_2)$, where F is a function with enough regularity, satisfying $F(0) = \nabla F(0) = 0$. Under the asumption $\partial\mathcal{G} \in C^{1+\alpha}$ $(0 < \alpha < 1)$ in a neighborhood of ξ, we have the inequality $|\partial F/\partial z_k| \le c|z'|^\alpha$. Further, we assume that it is possible to construct in $\mathcal{G}$ for any $\lambda > 0$, a finite or countable number of subdomains $\omega^{(k)}$ and $D^{(k)}$ possessing the following properties:

1. $\omega^{(k)} \subset D^{(k)} \subset \mathcal{G}$, $\quad \bigcup_k \omega^{(k)} = \bigcup_k D^{(k)} = \mathcal{G}$;

2. For any point $x \in \mathcal{G}$, there exists $\omega^{(k)}$ such that $x \in \omega^{(k)}$ and the distance from x to $D^{(k)} \setminus \omega^{(k)}$ is not less than $d\lambda$;
3. There exists a positive number N_0 not depending on λ such that the intersection of any $N_0 + 1$ distinct of $D^{(k)}$ is empty;
4. $\omega^{(k)}$ and $D^{(k)}$ that are seperated from $\partial\mathcal{G}$ by a positive distance larger than ρ_0 (we denote the set of their indices k by $\mathcal{M}$) are 3-dimensional cubes with common center $\xi^{(k)} \in \mathcal{G}$ whose linear dimensions are equal to λ and 2λ ($2\lambda < \rho_0$) respectively, while those adjacent to $\partial\mathcal{G}$ (the set of their indices k by $\mathcal{N}$) are defined in local coordinates at $\xi^{(k)} \in \partial\mathcal{G}$ by the inequalities:
$$\omega^{(k)} = \{z \in \boldsymbol{R}^3 \,|\, |z_i| < \frac{\lambda}{2} \ (i = 1,2);\ 0 < z_3 - F(z') < \lambda\},$$
$$D^{(k)} = \{z \in \boldsymbol{R}^3 \,|\, |z_i| < \lambda \ (i = 1,2);\ 0 < z_3 - F(z') < 2\lambda\}.$$

5. The change of variables $y' = z'$, $y_3 = z_3 - F(z')$, which we denote by $y = \Phi(z)$, takes a domain $D^{(k)}$ $(k \in \mathcal{N})$ into the cube $Z_k^{-1}(D^{(k)}) = \{y \in \boldsymbol{R}_+^3 \,|\, |y_i| < \lambda \ (i = 1,2), 0 < y_3 < 2\lambda\}$, where the inverse of the composed transform $\Phi(C^{(k)}(x - \xi^{(k)}))$ in $D^{(k)}$, with $C^{(k)}$ an orthogonal matrix to transform the coordinate system into the local one around $\xi^{(k)}$, is represented by Z_k^{-1}. Then we have functions $\zeta^{(k)}(x)$ satisfying
$$0 \le \zeta^{(k)} \le 1, \quad |D^\alpha \zeta^{(k)}| \le \frac{c_\alpha}{\lambda^{|\alpha|}}, \ (|\alpha| = 0,1,\dots)$$
$$\zeta^{(k)}(x) = 1 \text{ for } x \in \omega^{(k)},\ \zeta^{(k)} = 0 \text{ for } x \in \mathcal{G} \setminus D^{(k)},$$

and $c_\alpha > 0$ independent on k. By virtue of property (iii) of the domains $D^{(k)}$, $1 \le \sum_k \left(\zeta^{(k)}(x)\right)^2 \le N_0$, and the functions $\eta^{(k)}(x) \equiv \frac{\zeta^{(k)}(x)}{\sum_j \left(\zeta^{(j)}(x)\right)^2}$ have the following properties:
$$\begin{cases} \eta^{(k)}(x) = 0 \text{ in } \mathcal{G} \setminus D^{(k)},\ |D^\alpha \eta^{(k)}(x)| \le \dfrac{c_\alpha}{\lambda^{|\alpha|}}, \\ \displaystyle\sum_k \eta^{(k)}(x) \zeta^{(k)}(x) = 1. \end{cases}$$

Now Let us define a linear operetor $R^{(k)}$. For $k \in \mathcal{M}$, it associates with
$$(f^{(k)}, g^{(k)}) \in L_2(\Omega_\tau, \mathcal{P}, \widetilde{W}_{2,\psi,m}^{m+1}(D^{(k)})) \times L_2(\Omega_\tau, \mathcal{P}, W_{2,\psi,m}^{m+2}(D^{(k)}))$$

the solution $u^{(k)} \in L_2(\Omega_\tau, \mathcal{P}, W_{2,\psi,m}^{m+3}(D^{(k)})) \cap C\,([0,\tau]; W_{2,\psi,m}^{m+2}(D^{(k)}))$ (a.s.) of the problem
$$\begin{cases} \mathrm{d}u^{(k)} = [a^{ij}|_{(t,x)=(0,\xi^{(k)})} u_{x_i x_j}^{(k)} + f^{(k)}]\mathrm{d}t \\ +[\sigma^{il}|_{(t,x)=(0,\xi^{(k)})} u_{x_i}^{(k)} + g^{(k)l}]\mathrm{d}w_t^l \quad \text{in } D_T^{(k)}, \\ u^{(k)}|_{t=0} = 0 \quad \text{(k)}. \end{cases}$$

For $k \in \mathcal{N}$, we define a mappig $R^{(k)}$ that associates with $(f^{(k)}, g^{(k)}, h^{(k)}) \in$
$$L_2(\Omega_\tau, \mathcal{P}, \widetilde{W}_{2,\widetilde{\psi},m}^{m+1}(\boldsymbol{R}_+^3)) \times L_2(\Omega_\tau, \mathcal{P}, W_{2,\widetilde{\psi},m}^{m+2}(\boldsymbol{R}_+^3))$$

$$\times\ (L_2(\Omega_\tau,\mathcal{P},W_2^{\frac{3}{2}}(\boldsymbol{R}^2))\bigcap C\,([0,\tau];W_2^{\frac{1}{2}}(\boldsymbol{R}^2))\ (\mathrm{a.s.})$$

the solution $u^{(k)}\in L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+))\cap C\,(\Omega_\tau;W^{m+2}_{2,\widetilde{\psi},m}(\boldsymbol{R}^3_+))$ (a.s.) of the problem

$$\begin{cases} \mathrm{d}u^{(k)} = [a^{ij}|_{(t,x)=(0,\xi^{(k)})}u^{(k)}_{y_iy_j} + f^{(k)}]\mathrm{d}t \\ +[\sigma^{il}|_{(t,x)=(0,\xi^{(k)})}u^{(k)}_{y_i} + g^{(k)l}]\,\mathrm{d}w^l_t \quad \text{in } \boldsymbol{R}^3_+\times[0,T], \\ u^{(k)}_{y_3}|_{y_3=0} = h^{(k)} \quad \text{on } \boldsymbol{R}^2, \\ u^{(k)}|_{t=0} = 0 \quad \text{on } \boldsymbol{R}^3_+, \end{cases}$$

which exists uniquely due to Lemma 4.1. Then, define a bounded operator R acting from

$$\mathcal{B}(\tau)\equiv L_2(\Omega_\tau,\mathcal{P},\widetilde{W}^{m+1}_{2,\psi,m}(\mathcal{G}))\times L_2(\Omega_\tau,\mathcal{P},W^{m+2}_{2,\psi,m}(\mathcal{G}))$$
$$\times\ (L_2(\Omega_\tau,\mathcal{P},W_2^{\frac{3}{2}}(\partial\mathcal{G}))\bigcap C\,([0,\tau];W_2^{\frac{1}{2}}(\partial\mathcal{G})))\ (\mathrm{a.s.})$$

into $\mathcal{A}(\tau)\equiv L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\psi,m}(\mathcal{G}))\cap C\,([0,\tau];W^{m+2}_{2,\psi,m}(\mathcal{G}))$ (a.s.) and for any $\mathcal{H}\equiv(f,g,h)\in\mathcal{B}$ and $u\in\mathcal{A}$

$$AR\mathcal{H} = \mathcal{H} + T\mathcal{H},\quad RAu = u + Wu. \tag{5.1}$$

$A=(L,B)$ shall be defined later. T and W are bounded operators in $\mathcal{B}$ and $\mathcal{A}$ respectively, the norms of which are less than 1 if τ is small enough. Then $(I+T)^{-1}$ and $(I+R)^{-1}$ exist,

$$R(I+T)^{-1} = (1+W)^{-1}R = A^{-1}$$

holds. In addition, we have $\|A^{-1}\|\leq\max\{\|(I+T)^{-1}\|\|R\|,\|(I+W)^{-1}\|\|R\|\}$, where $\|\cdot\|$ stands for the norm of the operators.

Taking τ small enough so that $\|T\|\leq 1/2$ holds, then we have $\|A^{-1}\|\leq\|(I+T)^{-1}\|\|R\|\leq 2\|R\|$. Hence the problem is reduced to constructing the operator R and the estimates of $\|R\|$, $\|T\|$ and $\|W\|$. In the function spaces $\mathcal{A}$ and $\mathcal{B}$, let us introduce the norms

$$\|u\|^2_{\mathcal{A}(T)}\equiv |||u|||^2_{\psi,m,m+3,\mathcal{G}_T} + E\sup_{t\in[0,T]}|||u(t)|||^2_{\psi,m,m+2,\mathcal{G}},$$
$$\|(f,g,h)\|^2_{\mathcal{B}(T)}\equiv |||f|||^{*2}_{\psi,m,m+1,\mathcal{G}_T} + |||g|||^2_{\psi,m,m+2,\mathcal{G}_T}$$
$$+E|||h|||^2_{L_2(\Omega_T,\mathcal{P},W_2^{\frac{3}{2}}(\boldsymbol{R}^2))} + E\sup_{t\in[0,T]}\|h(t)\|^2_{W_2^{\frac{1}{2}}(\boldsymbol{R}^2)}.$$

Define $R\mathcal{H}\equiv\sum_k\eta^{(k)}(x)u^{(k)}(t,x)$, where

$$u^{(k)}(t,x) = \begin{cases} R^{(k)}\zeta^{(k)}f \quad \text{for } k\in\mathcal{M}, \\ Z_kR^{(k)}Z_k^{-1}\zeta^{(k)}(f,g,h) \quad \text{for } k\in\mathcal{N}. \end{cases}$$

Now we proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1

Proof.. Let us define a map $L_0 = L_0(t,x,\partial_x)$ by

$$L_0(t,x,\partial_x): L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\psi,m}(\mathcal{G}))\bigcap C\,([0,\tau],W^{m+2}_{2,\psi,m}(\mathcal{G}))\ (\text{a.s.})$$
$$\to L_2(\Omega_\tau;\widetilde{W}^{m+1}_{2,\psi,m}(\mathcal{G}))\times L_2(\Omega_\tau;W^{m+2}_{2,\psi,m}(\mathcal{G}))$$

mapping $(f,g)\equiv((L_0u)_1,(L_0u)_2)\in L_2(\Omega_\tau;\widetilde{W}^{m+1}_{2,\psi,m}(\mathcal{G}))\times L_2(\Omega_\tau;W^{m+2}_{2,\psi,m}(\mathcal{G}))$, which satisfies

$$\mathrm{d}u = [a^{ij}(t,x)u_{x_ix_j}+f]\mathrm{d}t+[\sigma^{il}(t,x)u_{x_i}+g^l]\,\mathrm{d}w^l_t \text{in } \mathcal{G}$$

to $u\in L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\psi,m}(\mathcal{G}))\cap C\,([0,\tau],W^{m+2}_{2,\psi,m}(\mathcal{G}))$ (a.s.), and a map $L = L(t,x,\partial_x)$ by

$$L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\psi,m}(\mathcal{G}))\bigcap C\,([0,\tau],W^{m+2}_{2,\psi,m}(\mathcal{G}))\ (\text{a.s.})$$
$$\to L_2(\Omega_\tau,\mathcal{P},\widetilde{W}^{m+1}_{2,\psi,m}(\mathcal{G}))\times L_2(\Omega_\tau,\mathcal{P},W^{m+2}_{2,\psi,m}(\mathcal{G})),$$

mapping $(f,g)\in L_2(\Omega_\tau,\mathcal{P},\widetilde{W}^{m+1}_{2,\psi,m}(\mathcal{G}))\times L_2(\Omega_\tau,\mathcal{P},W^{m+2}_{2,\psi,m}(\mathcal{G}))$ which satisfies

$$\mathrm{d}u = [a^{ij}(t,x)u_{x_ix_j}+bu_{x_i}-cu+f]\,\mathrm{d}t$$
$$+[\sigma^{il}(t,x)u_{x_i}+\nu^lu+g^l]\,\mathrm{d}w^l_t \quad \text{in } \mathcal{G},$$

to $u\in L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\psi,m}(\mathcal{G}))\cap C\,([0,\tau],W^{m+2}_{2,\psi,m}(\mathcal{G}))$ (a.s.).

Let $L^{(k)}_0 = L^{(k)}_0(t,z,\partial_z)$ be the operator L_0 in the local coordinate at the point $\xi^{(k)}$, mapping

$$(f^{(k)},g^{(k)})\equiv((L^{(k)}_0u)_1,(L^{(k)}_0u)_2)$$
$$\in L_2(\Omega_\tau,\mathcal{P},\widetilde{W}^{m+1}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+))\times L_2(\Omega_\tau,\mathcal{P},W^{m+2}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+))$$

which satisfies

$$\mathrm{d}u^{(k)} = [a^{(k)ij}u^{(k)}_{z_iz_j}+f^{(k)}]\,\mathrm{d}t+[\sigma^{(k)il}u^{(k)}_{z_i}+g^{(k)l}]\,\mathrm{d}w^l_t \quad \text{in } \boldsymbol{R}^3_+,$$

to $u^{(k)}\in L_2(\Omega_\tau,\mathcal{P},W^{m+3}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+))\cap C\,([0,\tau],W^{m+2}_{2,\tilde{\psi},m}(\boldsymbol{R}^3_+))$ (a.s.), where $g^{(k)l} = \zeta^{(k)}g^l$, and $a^{(k)ij}$ $(i,j,=1{,}2{,}3)$, $\sigma^{(k)il}$ $(i,l,=1{,}2{,}3)$ are the components of the matrices $C^{(k)T}\bar{A}C^{(k)}$, $C^{(k)T}\overline{\sum}\ C^{(k)}$, respectively. We also define maps,

$$L_1\equiv L-L_0,\quad B_0(t,x,\partial_x)u = \boldsymbol{l}\cdot\nabla u|_{\partial\mathcal{G}_T},$$
$$B^{(k)}_0(t,y,\partial_y)u = \tilde{\boldsymbol{l}}(y)\cdot\nabla_y\tilde{u}(y)|_{y_3=0},\quad B_1u = k(t,x)u|_{\partial\mathcal{G}_T},$$

where $\tilde{\boldsymbol{l}}(y) = \boldsymbol{l}(Z_k(t,x))$ and is the components of $C^{(k)}\boldsymbol{l}$ with $\boldsymbol{l} = (l_1,l_2,l_3)$. It is easy to show

$$L_0R\mathcal{H} = \sum_k(\,L_0(\eta^{(k)}u^{(k)})-\eta^{(k)}L_0u^{(k)})$$
$$+\sum_k\eta^{(k)}\,[L_0(t,x,\partial_x)-L_0(0,\xi^{(k)},\partial_x)]u^{(k)}$$
$$+\sum_k\eta^{(k)}\,L_0(0,\xi^{(k)},\partial_x)u^{(k)}. \tag{5.2}$$

Each term in (5.2) can be calculated explicitly. For the first term, for example,

$$L_0(\eta^{(k)}u^{(k)})-\eta^{(k)}L_0u^{(k)} = (-a^{ij}(\eta^{(k)}_{x_ix_j}u^{(k)}+2\eta^{(k)}_{x_i}u^{(k)}_{x_j}),\ -\sigma^{il}\eta^{(k)}_{x_i}u^{(k)}).$$

Then, we represent $T\mathcal{H}$ in (5.1) explicitly. Since $\sum_k\eta^{(k)}\,\zeta^{(k)} = 1$, we have $L(R\mathcal{H}) = \mathcal{H}+(T_1\mathcal{H},T_2\mathcal{H})$, where

$$(T_1\mathcal{H}, T_2\mathcal{H}) = L_1 R\mathcal{H} + \sum_k (L_0 \eta^{(k)} u^{(k)} - \eta^{(k)} L_0 u^{(k)})$$
$$+ \sum_k \eta^{(k)} [L_0(t, x, \partial_x) - L_0(0, \xi^{(k)}, \partial_x)] u^{(k)}$$
$$+ \sum_{k \in \mathcal{N}} \eta^{(k)} Z_k [L_0^{(k)}(0,0, \partial_y - \nabla F \partial_{y_3}) - L_0^{(k)}(0,0, \partial_y)]$$
$$R^{(k)} Z_k^{-1} \zeta^{(k)}(f, g, h).$$

Analogously we have $BR\mathcal{H}|_{\partial \mathcal{G}_T} = \mathcal{H} + T_3\mathcal{H}$, where

$$T_3\mathcal{H} = B_1 R\mathcal{H}|_{\partial\mathcal{G}_T} + \sum_{k \in \mathcal{N}} (B_0 \eta^{(k)} u^{(k)} - \eta^{(k)} B_0 u^{(k)})|_{\partial \mathcal{G}_T}$$
$$+ \sum_k \eta^{(k)} [B_0(t, x, \partial_x) - B_0(0, \xi^{(k)}, \partial_x)] u^{(k)}$$
$$+ \sum_{k \in \mathcal{N}} \eta^{(k)} Z_k [B_0^{(k)}(0,0, \partial_y - \nabla F \partial_{y_3}) - B_0^{(k)}(0,0, \partial_y)]$$
$$R^{(k)} Z_k^{-1} \zeta^{(k)}(f, g, h).$$

Thus $AR\mathcal{H} = \left(LR\mathcal{H}, BR\mathcal{H}|_{\partial \mathcal{G}_\tau}\right) = (f, g, h) + (T_1\mathcal{H}, T_2\mathcal{H}, T_3\mathcal{H}) = \mathcal{H} + T\mathcal{H}$. Explicit representation of Wu in (5.1) is obtained similarly. By virtue of Lemma 5.4 introduced later,

$$\| T\mathcal{H} \|_{\mathcal{B}} \leq c\eta_1(\tau, \lambda) \| \mathcal{H} \|_{\mathcal{B}}, \tag{5.3}$$
$$\| Wu \|_{\mathcal{A}} \leq c\eta_2(\tau, \lambda) \| u \|_{\mathcal{A}}, \tag{5.4}$$

hold with $\eta_i(\tau, \lambda) \to 0$ $(i = 1,2)$ as τ and λ goes to 0, and monotonically increasing with τ and λ. Taking (τ, λ) in (5.3)–(5.4) small enough makes $\| T \|$ and $\| W \|$ arbitrarily small. Next, we consider the same problem in a cylindrical domain whose lower base lies in the plane $t = t_0 < T$. The height of the cylinder in which the existence of the solution is guaranteed can be taken to be the same for any t_0. Thus we have the desired result. □

2. Auxiliary Lemmas

In the preceding subsection, we have proved Theorem 3.1 based on (5.3)–(5.4), which we finally prove.

Lemma 1. If a function u satisfies $u \in W^{r+l+j}_{2,\tilde{\psi},r+j}(\boldsymbol{R}^3_+)$ and $u(y', y_3) = 0\ on \{y \in \boldsymbol{R}^3_+ | y_3 \geq M\}$ with a positive constant M, the following estimate holds with $r > -1/2$ and $j, l \geq 0$:

$$\left\|\tilde{\psi}^r D_y^{\alpha_1} u\right\|_{L_2(\boldsymbol{R}^3_+)} \leq C \left\|\tilde{\psi}^{r+j} D_y^{\alpha_2} u\right\|_{L_2(\boldsymbol{R}^3_+)},$$

where $|\alpha_1| = r + l$, $|\alpha_2| = r + l + j$, and C is a positive constant depending on M, r, j and l.

Lemma 5.1 follows from Lemma 2.1(a) in [6] directly, and we omit the proof. Next, a similar estimate holds in the original domain with the aid of Lemma 5.1.

Lemma 2. Assume (2.1), and let a function $f \in W^{r+l+j}_{2,\widetilde{\psi},r+j}(D^{(k)})$ satisfies $D^\alpha f(x) \equiv 0$ for $|\alpha| = r + l$ on the region

$$\begin{cases} \{x \in D^{(k)} | x_3 = \xi_3^{(k)} \pm \lambda\} \text{for } k \in \mathcal{M}, \\ \{z = C^{(k)}(x - \xi^{(k)}), x \in D^{(k)} | z_3 = F(z') + 2\lambda\} \quad \text{for } k \in \mathcal{N}, \end{cases}$$

where $\xi_3^{(k)}$ is the 3rd component of $\xi^{(k)}$. Then,

$$\|\psi^r D^{\alpha_1} f\|_{L_2(D^{(k)})} \le C \left\|\psi^{r+j} D^{\alpha_2} f\right\|_{L_2(D^{(k)})}$$
$$\text{for} \quad k \in \mathcal{M} \bigcup \mathcal{N},\ r > -1/2,\ j, l \ge 0,$$

where $|\alpha_1| = r + l$, $|\alpha_2| = r + l + j$, and C is a positive constant depending on M, r, j and l.

The following lemma is substantial in proving the main theorem.

Lemma 3. The estimates

$$\| R\mathcal{H} \|_{\mathcal{A}} \le c \| \mathcal{H} \|_{\mathcal{B}}, \quad \| B_1 u \|_{\mathcal{B}} \le \eta_3(\tau, \lambda) \| u \|_{\mathcal{A}},$$
$$\| L_1 u \|_{\mathcal{B}} \le \eta_4(\tau) \| u \|_{\mathcal{A}}$$

hold with a non-decreasing functions $\eta_3(\tau, \lambda)$ and $\eta_4(\tau)$ which go to 0 as their arguments go to 0.

Proof.. We only show the estimate of $\| R\mathcal{H} \|_{\mathcal{A}}$. First, the following inequality is readily seen.

$$\| R\mathcal{H} \|_{\mathcal{A}} \le C \sum_k [\ \| \eta^{(k)} u^{(k)} \|_{L_2(\Omega_\tau; W^{m+3}_{2,\psi,m}(D^{(k)}))}$$
$$+ E \sup_{t \in [0,\tau]} \| \eta^{(k)} u^{(k)}(t) \|_{W^{m+2}_{2,\psi,m}(D^{(k)}))}].$$

For $k \in \mathcal{M}$, we make use of the result of Cauchy problem of the linear SPDEs [4], while for $k \in \mathcal{N}$,

$$|||\eta^{(k)} u^{(k)}|||^2_{L_2(\Omega_\tau, \mathcal{P}, W^{m+3}_{2,\psi,m}(D^{(k)}))} + E \sup_{t \in [0,\tau]} |||\eta^{(k)} u^{(k)}(t)|||^2_{W^{m+2}_{2,\psi,m}(D^{(k)})}$$
$$\le C \begin{bmatrix} \| u^{(k)} \|^2_{L_2(\Omega_\tau, \mathcal{P}, L_2(D^{(k)}))} + E \sup\limits_{t \in [0,\tau]} \| u^{(k)}(t) \|^2_{L_2(D^{(k)})} \\ + \left\| \psi^m \sum\limits_{i=0}^{m+3} \sum\limits_{|\alpha_1| = m+3-i, |\alpha_2| = i} D^{\alpha_1} \eta^{(k)} \cdot D^{\alpha_2} u^{(k)} \right\|^2_{L_2(\Omega_\tau, \mathcal{P}, L_2(D^{(k)}))} \\ + E \sup\limits_{t \in [0,\tau]} \left\| \psi^m \sum\limits_{i=0}^{m+2} \sum\limits_{|\beta_1| = m+2-i, |\beta_2| = i} D^{\beta_1} \eta^{(k)} \cdot D^{\beta_2} u^{(k)}(t) \right\|^2_{L_2(D^{(k)})} \end{bmatrix},$$

by virtue of Lemma 4.1. Now we show the estimate of the third term as an example. Due to the inequality

$$\left\| u_{x_i x_j}^{(k)} \right\|_{L_2(\Omega_\tau,\mathcal{P},L_2(D^{(k)}))}^2 \le \tau E \sup_{t\in[0,\tau]} \left\| u_{x_i x_j}^{(k)} \right\|_{L_2(D^{(k)})}^2 \quad (i,j = 1,2,\dots)$$

and Lemma 4.1, we have

$$\left\| \psi^m \sum_{i=0}^{m+3} \sum_{|\alpha_1|=m+3-i,|\alpha_2|=i} D^{\alpha_1}\eta^{(k)} \cdot D^{\alpha_2} u^{(k)} \right\|_{L_2(\Omega_\tau,\mathcal{P},L_2(D^{(k)}))}^2$$
$$\le \sum_{i=3}^{m+3} \sum_{|\alpha_1|=m+3-i,|\alpha_2|=i} \left\| \psi^{i-3} D^{\alpha_2} u^{(k)} \right\|_{L_2(\Omega_\tau,\mathcal{P},L_2(D^{(k)}))}^2 \left\| \psi^{m-i+3} D^{\alpha_1}\eta^{(k)} \right\|_{L_\infty(D^{(k)})}^2$$
$$+ \sum_{i=1}^{3} \sum_{|\beta_1|\le 3} \| D^{\beta_1}\nabla^{3-i} u^{(k)} \|_{L_2(\Omega_\tau,\mathcal{P},L_2(D^{(k)}))}^2 \| \psi^m \nabla^{m+i}\eta^{(k)} \|_{L_\infty(D^{(k)})}^2$$
$$\le C[\| \zeta^{(k)} f \|_{\psi,m,m+1,D_\tau^{(k)}}^{*2} + \| \zeta^{(k)} g \|_{\psi,m,m+2,D_\tau^{(k)}}^2 + \| \zeta^{(k)} u_0 \|_{\psi,m,m+2,D^{(k)}}^2$$
$$+ \| \zeta^{(k)} h \|_{L_2(\Omega_\tau,\mathcal{P},W_2^{\frac{3}{2}}(\boldsymbol{R}^2))}^2 + E \sup_{t\in[0,\tau]} \| \zeta^{(k)} h(t) \|_{L_2([0,\tau],\mathcal{P},W_2^{\frac{1}{2}}(\boldsymbol{R}^2))}].$$

For the estimate of $\| L_1 u \|_{\mathcal{B}}$, we use the inequality

$$\| u \|_{L_q'(D^{(k)})} \le C \| \nabla u \|_{L_2(D^{(k)})}^{\alpha} \| u \|_{L_2(D^{(k)})}^{1-\alpha}$$

with $\alpha = 3/q$ for for $u \in L_{q'}(D^{(k)}) \cap L_2(D^{(k)})$ in general, Young's and Poincaré inequalities. Due to Lemma 5.2 and the inequality

$$\sum_{k\in\mathcal{M}\cup\mathcal{N}} \| \eta^{(k)}\zeta^{(k)} u \|_{\mathcal{A}} \le c \| u \|_{\mathcal{A}},$$

we have the desired result. Other terms can be estimated similarly. □

Finally, we verify estimates (5.3) and (5.4). Together with Lemma 5.3, let us make use of the following facts:

For $(t,x) \in [0,\tau] \times D^{(k)}$, $|a^{ij}(t,x) - a^{ij}(0,\xi^{(k)})| \le \eta_5(\tau,\lambda)$ holds with η_5 having the same property with $\eta_1(\tau,\lambda)$; $|\nabla F(x)| \le C\lambda^\alpha$ on $Z_k(D^{(k)})$.

Thus, we obtain (5.3). Making use of Lemmas 5.2 and 5.3 and the facts listed above, we also have (5.4). This completes all parts of the proof of the main theorem.

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