Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 10, Number 1 (2014), pp. 113-128

http://www.ripublication.com

Composition Series and Decompositions of Partial Groups

M. El-Ghali M. Abdallah and Marwa. A. El-lithy

Department of Mathematics, Faculty of Science, Menofia University,
Shebin Elkom 32511, Egypt.
E-mail address: Mohamed_elghaly@yahoo.com
E-mail address: marwamath@Gmail.com

Abstract

The classical theory of subnormal series, refinements and composition series in groups is extended to the class of partial groups which is known to be precisely the classes of Clifford semigroups, or equivalently semilattices of groups. The Jordan-Holder theorem concerning composition series which is known to hold in every abelian category is now extended to partial groups which are certainly a non-abelian category. Other results including decomposability and the Remak decomposition theorem of groups are extended as well to partial groups.

1-Introduction

In [1] a partial group is defined in terms of a set of (new) axioms and proved to satisfy a structure theorem of strong sort and a representation theorem. Formally; a partial group is precisely a strong semilattice of groups (also called in literature a Clifford semigroup), and every partial group is isomorphic to a certain partial group of partial mappings. Other kinds of "partial algebras" have been introduced and existed in literature, (e.g. partial rings in [4], and partial monoids in [5]). The principal aim of introducing a particular kind of a partial algebra has been show that it is the most convenient viewing as a "generalized algebra" from both the categorical and algebraic points of views [2], [3], [4]. Some interesting categorical and algebraic results concerning semigroup congruences on partial groups (semilattices of groups) whose arrows are epimorphisms have been developed in [6] and [7]. In the present work we establish for partial groups (new) generalization of two remarkable theorems in groups. The first of those theorems is the Jordan Holder Theorem of composition series which is known to hold in every abelian category (see, e.g. [8]).

We devote the second section of this paper for developing the theorem for the category of partial groups, which is certainly non abelian category. This has required generalizing all the needed definitions and results, such as Dedekind modular law, the

Schreir refinement and the second Neother isomorphism theorems. In the third section we develop the generalized Remak decomposition theorem for partial groups.

2- Preliminaries

In this section, we prepare the ground by the needed definitions and results concerning partial groups cited from [1], [2]. Our reference in groups is [10], and for semigroups in general we refere to [9]. Also, unless stated otherwise, S will denote an arbitrary semigroup and $E(S)$ will be the set of all idempotents in S, that is all $x \epsilon S$ such that $x^2 = xx = x$.

Definition 2.1: Let $x \epsilon S$. An element $e \epsilon S$ is called a *partial identity* for x if $ex = xe = x$, and if $e'x = xe' = x$ for some $e' \epsilon S$, then $ee' = e'e = e$. A partial identity of x, when exist is unique and idempotent. It will be denoted by e_x.

Definition 2.2: Let $x \epsilon S$ and suppose that the partial identity e_x of x exists. An element $y \epsilon S$ is called a *partial inverse* of x if $x\,y = y\,x = e_x$ and $y\,e_x = e_x\,y = y$. A partial inverse of $x \epsilon S$, when exists, is unique and will be denoted by x^{-1}.

Proposition 2.1: *S* is a completely regular semigroup if and only if e_x and x^{-1} exist for every $x \epsilon S$.

Definition 2.3: *S* is called a *partial group* if the following axioms hold. For all, $y \epsilon S$;

- (PG1) e_x and x^{-1}exist,
- (PG2) $e_{xy} = e_x e_y$, that is, the mapping, $e_S: S \to S, x \mapsto e_x$, is a homomorphism (of semigroups),
- (PG3) $(xy)^{-1} = y^{-1}x^{-1}$, that is, the mapping, $i_S: S \to S, x \mapsto x^{-1}$, is an anti homomorphism.

Proposition 2.2: Let *S* be a partial group. Then

- Every idempotent in *S* is its own partial identity and partial inverse,
- $e_x^{-1} = e_x = (e_x)^{-1}$ for all $x \epsilon S$,
- $(x^{-1})^{-1} = x$ for all $x \epsilon S$.

Definition 2.4: Let *S* be a partial group. A subset B of *S* is a *subpartial group* of *S* written $B \leq S$ if B is a subsemigroup of S and e_x, x^{-1} are in B for all $x \epsilon B$. In particular, *S* and *E*(*S*) are subpartial groups of *S* for every partial group *S*.

Definition 2.5: Let $x \epsilon S$ and T be partial groups. A mapping *φ: S→T* is a *homomorphism* (of partial groups) if it is a homomorphism as a mapping of semigroup. That is, if $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \epsilon S$. *Monomorphism, epimorphism, isomorphism*, etc of partial groups are defined in the usual manner. In particular, if *φ*: *S →T* is a monomorphism of partial group, then it is called an *empedding*. If *φ*: *S→T* is an isomorphism then *S* and *T* are called *isomorphic*, written

$S \simeq T$. If φ: $S \to T$ is a homomorphism of partial groups, then image of φ, denoted by Imφ and defined by Imφ = $\{\varphi(x)$: $x \in S\}$ is clearly a subpartial group of T and φ is epimorphism if and only if Imφ= T.

Proposition 2.3: Let φ: $S \to T$ be a homomorphism of partial groups. Then, for all, $x \epsilon S$, we have

- $\varphi(e_x) = e_{\varphi(x)}$,
- $\varphi(x^{-1}) = (\varphi(x))^{-1}$.

By the well known Clifford theorem (see [9]) a Clifford semigroup (i.e. a regular semigroup with central idempotents)is precisely a strong semilattice of groups. Some other characterizations of a Clifford semigroup exist in literatures (e.g. a completely regular inverse semigroup, a semilattice of groups,...etc).In [1] the axioms PG1, PG2 and PG3 allowed to characterize a partial group as a Clifford semigroup, or equivalently a strong semilattice of groups.If S is an arbitrary partial group, then S is a (disjoint) union of its (maximal) subgroups $S = \bigcup_{x \in S} S_x$ is the maximal subgroup of S with identity e_x. In essence we have $S_x = \{y \epsilon S \,;\, e_y = e_x\}$. It follows that $S_x = S_y$ iff $e_x = e_y$. In particular $S_x = S_{e_x}$ for all $x \epsilon S$.

Theorem 2.1: The following statements about a semigroup S are equivalent:

- S is a partial group,
- S is a completely regular inverse semigroup,
- S is a Clifford semigroup,
- S is a semilattice of groups,
- S is a strong semilattice of groups.

According to the above structure theorem (of strong sort), a partial group S viewed as a strong semilattice of groups may be written in the form $S = \wp[E(S); S_f, \varphi_{f,g}]$ where $E(S)$ is the semilattice of idempotents (partial identities) in S with the usual partial ordering, $e \leq f$ iff $ef = fe = e$, S_f is the maximal subgroup of S with identity f, and for $f \geq g$ in $E(S)$, $\varphi_{f,g}$ is the homomorphism of groups $\varphi_{f,g}: S_f \to S_g,\ x \mapsto gx$. The operation in S may be given by the structure mappings as follows. If, $y \epsilon S$, say $x \epsilon S_e$ and $y = S_f$. Then $xy = \varphi_{e,ef}x \cdot \varphi_{f,ef}y$. Throughout the rest of this section S denotes an arbitrary partial group and $E(S)$ denotes the set of all partial identities (idempotentes) in S.

Definition 2.6: A subpartial group B of S is called *wide* if $E(S) \subseteq B$ and *normal*, written $B \lhd S$, if it is wide and $xBx^{-1} \subseteq B$ for all $x \epsilon S$. Evidently, $E(S) \lhd S$, and we call $E(S)$ the trivial normal subpartial group of S.

Proposition 2.4: If K is a normal subpartial group of S, then K_{e_x} is a normal subgroup of S_{e_x} for all $x \epsilon S$.

Definition 2.7: Let $\varphi: S \to T$ be a homomorphism of partial groups. Then *k-kernel* of φ, or simply k-kerφ, is the subset of S, k-kerφ= $\{x \epsilon S$: $\varphi(x) = e$ for some $e \epsilon E(T)\}$.

Proposition 2.5: If φ: $S\rightarrow T$ is a homomorphism of partial groups, we have

- k-kerφ = $\{x\in S: \varphi(x) = \varphi(e_x) = e_{\varphi(x)}$ =ker(kerφ)$\}$,
- k-kerφ is a normal subpartial group of S,
- If φ is a monomorphism then k-kerφ = $E(S)$,
- If φ is a homomorphism of groups, then k-kerφ=kerφ is the usual group-theoretic kernel of φ.

Here kerφ is the usual kernel of φ, that is kerφ = $\{(x,y)\in S\times S: \varphi(x) = \varphi(y)\}$, which is a congruence on S.(cf[9]). On the other hand, the kernel of any congruence on (a semigroup) S is the subset of S; ker ρ= $\{x\in S: x\,\rho\, e$ for some $e\in E(S)\}$ (cf.[9]).

Proposition 2.6: If is a congruence on S, then $S/\rho=\{x\rho: x\in S\}$ with the usual operation $x\rho\cdot y\rho = xy\rho$, is a partial group, called the quotient partial group induced by ρ. Moreover, $e_{x\rho} = e_x\rho$ and $(x\rho)^{-1} = x^{-1}\rho$’ for all $x\in S$.

Theorem 2.2: Let K be a normal subpartial group of S. Define $\rho_k = \{(x,y)\in S\times S: e_x = e_y$ and $x^{-1}y \in K\}$.Then

- ρ_k is an idempotent separating congruence on S and K =kerρ_k = k-ker$(\rho_k^{\#})$, where $\rho_k^{\#}: S\rightarrow S/\rho$ is the natural homomorphism,
- $x\rho_k = xK_{e_x}$, for all $x\in S$,
- $K = (E(S))\rho_k = \cup\{e_x\rho_k: e_x\in E(S)\}$.

Theorem 2.3: For every idempotent separating congruence ρ on S there exists a normal subpartial group K of S with K =kerρ = $E(S)$ρ and ρ= ρ_k.

If K is a normal subpartial group of S, we denote the quotient partial group S/ρ_k by S/K, where ρ_k is the unique idempotent separating congruence on S associated with K (Theorems 2.2, 2.3 above). We refer to S/K as the quotient of S by K.

Lemma 2.1: Let A and K be respectively wide and normal subpartial groups of S. Then

- $AK = KA$ and this is a wide subpartial group of S,
- A and K are respectively wide and normal subpartial groups of AK,
- If A is also normal, then $AK = KA$ is normal subpartial group of S.

The following is a second Noether isomorphism theorem for partial groups.

Theorem 2.4: Let A and K be respectively wide and normal subpartial groups of S. Then

- The mapping γ: $A\rightarrow AK/K$, $a\mapsto aK_{e_a}$ is an epimorphism of partial groups with k-kerγ=$A\cap K$,
- There exists a unique isomorphism α: $A/A\cap K\rightarrow AK/K$ of partial groups such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\gamma"] \arrow[d, "\rho^{\#}_{A\cap K}"'] & AK/K \\
A/A\cap K \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

3-Composition series and The Jordan-Holder theorem for partial group

In this section we extend the definitions and the well-known results concerning refinements and composition series in Ω- groups to Ω-partial groups.

Definition 3.1: A right operator partial group is a triple (S, Ω, α) consists of a partial group S, a set Ω the operator domain and a function α: $S\times\Omega\rightarrow S$, $(x, \omega))\mapsto x^{\omega}$ which is an endomorphism of S for each $\omega \in \Omega$. We then refer to S as an Ω-partial group.

An operator partial group is a generalization of a partial group, since any partial group can be regarded as an operator partial group with empty operator domain.

Concepts such as Ω-subgroups and homomorphisms of Ω-groups can be extended analogously to Ω-partial groups. In particular, if K is a normal Ω-subpartial group of an Ω-partial group S, then by Theorem 2.2 (ii), the quotient partial group S/K becomes an Ω-quotient partial group with $(x \quad)^{\omega} = x^{\omega}$. Some examples of Ω-groups can be extended to partial groups. For instance, if S is a partial group we may take the operator domain Ω the set of all endomorphisms of S acting on S in the obvious way. Similarly, we may take Ω the set of all automorphisms of S or the set of all inner automorphisms of S, where an inner automorphism of S induced by an element $s \in S$ is the mapping s^{τ}: $S\rightarrow S$, defined by $s^{\tau}(\ ) = \quad s$ (this is the conjugate of by s). In this last example, a subpartial group K of the Ω -partial group S is an Ω-subpartial group of S if and only if for all $\in K$ and $s\in S$, $s^{\tau}(\ ) \in K$, that is if and only if $\quad s\in K$, if and only if K is a normal subpartial group of S.

In the following definitions and results, unless stated otherwise, S denotes an operator partial group with operator domain. When no confusion exists, we will denote $E(S)$ by E.

Definition 3.2(Ω-series): A chain of Ω- subpartial groups $E = S_0 \lhd S_1 \lhd \ldots\lhd S_l = S$ is called an *Ω-series* in S. The S_i are the terms of the series and the quotient partial groups S_{i+1} /S_i are the factors of the series. If all the S_i are distinct, the integer l is called the *length* of the series. A subpartial group which is a term of at least one Ω-series is said to be Ω-subnormal in S. When Ω is empty; we shall simply speak of a series and a subnormal subpartial group. If, for instance, Ω =Inn$S$ (the set of all inner automorphisms of S), the terms of an Ω-series are normal in S and we shall speak of a normal series, etc.

Definition 3.3 (Refinements): In S there always exists at least one Ω-series, namely $E \lhd S$. If R and T are Ω-series of S, we call R a *refinement* of T if every term of T is also a term of R. If there is at least one term of R which is not a term of T, then R is a proper refinement of T.

We now extend the Dedekind modular law in groups (cf [10]) to partial groups. For Ω-partial groups, the proof is essentially the same.

Lemma 3.1: Let B, C and L be wide subpartial groups of a partial group with $C\subseteq L$. Then $(B\cap L)C = (BC)L$.

***Proof*:** That $(B\cap L)C\subseteq (BC)\cap L$ is clear. Let $x\in (BC)\cap L$, say $x = bc$ for some $b\in B$ and $c\in C$ with $bc\in L$ Then $bc = l$, for some $l\in L$ and so we have $be_c = bcc^{-1} = lc^{-1}\in LC\subseteq L$. Since B is wide, $e_c\in B$, whence $be_c\in B\cap L$: Thus $x = bc = b(e_c c) = (be_c)c\in (B\cap L)C$. Thus $(BC)\cap L\subseteq (B\cap L)C$. The result follows.

Before establishing the Zassenhaus Lemma for Ω-partial groups, we prepare the ground by some more technical lemmas.

Lemma 3.2: Let A and B be wide Ω-subpartial groups of S such that $A\lhd B$. Then for any wide Ω-subpartial group C of S, we have $C\cap A\lhd C\cap B$.

***Proof*:** Clearly both $C\cap A$ and $C\cap B$ are wide Ω-subpartial groups of S and $C\cap A$ is a wide Ω-subpartial group of $C\cap B$. Let $y\in C\cap B$ and $x\in C\cap A$. We must show that $yxy^{-1}\in C\cap A$. Since $A\lhd B$, we have $yxy^{-1}\in A$. Since x and y are in C and $C\leq S$, we have $yxy^{-1}\in C$. It follows that $yxy^{-1}\in C\cap A$. Hence $C\cap A\lhd C\cap B$.

Here we give a complete proof of Lemma 5.7, [2] for Ω-partial groups.

Lemma 3.3: Let A and K be respectively wide and normal Ω-subpartial groups of S. Then $AK = KA$ and this is a wide Ω-subpartial group of S.

- A is a wide Ω-subpartial group of AK and K is a normal Ω-subpartial group of AK.
- If A is also normal, then $AK = KA$ is a normal subpartial group of S.

Proof: (i) Clearly $E(S)\subseteq AK\cap KA$. By normality, $sK = Ks$ for all $s\in K$. Let $x, y\in AK$ say $x = a_1k_1$, $y = a_2k_2$ for some a_1, $a_2\in A$ and k_1, $k_2\in K$. We have $xy = a_1k_1a_2k_2 = a_1a_2k_3k_2 = a_1a_2k_4\in AK$ (k_3, $k_4\in K$). Since $E(S)\subseteq A\cap K$, we have $E(S) = E(S)E(S)\subseteq AK$. For any $a\in A$, $k\in K$, we have $(ak)^{-1} = k^{-1}a^{-1} = a^{-1}k_1\in AK$ ($k_1\in K$). Thus AK is a wide subpartial group of S Since A and K are Ω-subpartial groups of S, we have for $a\in A$, $k\in K$ and $\omega\in\Omega$, $(ak)^{\omega} = a^{\omega}k^{\omega}\in AK$. Hence AK is also an Ω-subpartial group of S. Let $x = ak\in AK$. Since $K\lhd S$, we have $ak = k_1a\in KA$. Hence $AK\subseteq KA$. Similarly, $KA\subseteq AK$. Thus $AK = KA$ and (i) follows.

(ii) Since $E(S)\subseteq K$, we have $A\subseteq AK$ and so A is a wide Ω-subpartial group of AK. Similarly, K is a wide Ω-subpartial group of AK. For any $ak\in K$ ($a\in A$, $k\in K$), $k_1\in K$, and by the normality of K, we have $(ak)k_1(ak)^{-1} = akk_1(k^{-1}a^{-1}) = ak(k_1k^{-1})a^{-1} = ak(k^{-1}k_2)a^{-1} = akk^{-1}k_2a^{-1} = ak_2a^{-1} = aa^{-1}k_3 = e_ak_3\in K$ (k_2, $k_3\in K$ and K is wide). This proves that K is a normal Ω-subpartial group of AK. The last assertion of (ii) follows similarly.

Lemma 3.4: Let B and C be wide Ω-subpartial groups of S such that $B \lhd C$. Then $BK \lhd CK$ for every normal Ω-subpartial group K of S.

***Proof*:** By lemma 3.3 (i), $BK = KB$ and $CK = KC$ are wide Ω-subpartial groups of S. Whence BK is a wide Ω-subpartial group of CK. To show that BK is normal in CK, let $x = bk_1 \in BK$ and $y = ck_2 \in CK$ (for some $b \in B$, $c \in C$, k_1, $k_2 \in K$). We have $yxy^{-1} = ck_2 (bk_1) k_2^{-1}c^{-1} = ck_2 (bk_1k_2^{-1})\, c^{-1} = c(k_2bk_3)\, c^{-1} = c(k_2b)\, k_3c^{-1} = c(b_1k_4)k_3c^{-1} = c(b_1k_4k_3)\, c^{-1} = cb_1\, (k_5c^{-1}) = (cb_1c^{-1})k_6 \in Bk_6 \subseteq BK$ (since $B \lhd C$, $k_3 = k_1k_2^{-1} \in K$, k_4, k_5, $k_6 \in K$, $b_1 \in B$).This proves that BK is normal in CK.

We now give a version of the Noether second isomorphism theorem (cf. Theorem2.4) in terms of Ω-partial groups and Ω-homomorphisms which is needed for the proof of Zassenhauss lemma of Ω-partial groups.

We begin by some technicalities.

Lemma 3.5: Let S be an Ω-partial group. Then for all $\omega \in \Omega$ and $x \in S$, we have

- $(e_x)^\omega = e_{x^\omega}$,
- $(x^{-1})^\omega = (x^\omega)^{-1}$.

***Proof*:** (i) We have $x^\omega (e_x)^\omega = (xe_x)^\omega = x^\omega$, similarly$(e_x)^\omega x^\omega = x^\omega$. Suppose that$yx^\omega = x^\omega y = x^\omega$, for some $y \in S$.Then$y(e_x)^\omega = y(xx^{-1})^\omega = yx^\omega (x^{-1})^\omega = x^\omega (x^{-1})^\omega = (xx^{-1})^\omega = (e_x)^\omega$. Similarly, $(e_x)^\omega y = (e_x)^\omega$. Whence, $(e_x)^\omega = e_{x^\omega}$.
(ii) We have $x^\omega (x^{-1})^\omega = (xx^{-1})^\omega = (e_x)^\omega = e_{x^\omega}$. Similarly, $(x^{-1})^\omega x^\omega = e_{x^\omega}$. Also, $e_{x^\omega}(x^{-1})^\omega = (e_x)^\omega (x^{-1})^\omega = (e_x x^{-1})^\omega = (x^{-1})^\omega$. Similarly, $(x^{-1})^\omega e_{x^\omega} = (x^{-1})^\omega$. Hence $(x^\omega)^{-1} = (x^{-1})^\omega$.

Lemma 3.6: Let K be a normal Ω-subpartial group of an Ω-partial group S. Then the natural homomorphism(epimorphism) $\rho_k^{\#}$: $S \to S/K$, $x \mapsto xK_{e_x}$, is an Ω-homomorphism.

***Proof*:** For every $\omega \in \Omega$ and $x \in S$, we have, by using Lemma 3.5 (i), $(\rho_k^{\#} x)^\omega = (xK_{e_x})^\omega = x^\omega K_{(e_x)^\omega} = x^\omega K_{e_{x^\omega}} = \rho_k^{\#} x^\omega$. The result follows.

The proof of the following theorem follows immediately by using Lemma 3.6.

Theorem 3.1: [Second Noethor Isomorphism Theorem for Ω-Partial Groups]

Let A and K be respectively wide and normal Ω-subpartial groups of an Ω-partial group S. Then

- The mapping γ: $A \to AK/K$, $a \mapsto aK_{e_a}$ is an Ω-epimorphism of Ω-partial groups with k-ker $\gamma = A \cap K$,
- There exists a unique Ω-isomorphism

α: $A/A \cap K \to AK/K$ of Ω-partial groups such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "\gamma"] \arrow[d, "\rho^{\#}_{A\cap K}"'] & AK/K \\
A/A\cap K \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Theorem 3.2: [Zassenhaus's Lemma for Ω-Partial Groups]
Let A_1, A_2, B_1 and B_2 be wide Ω-subpartial groups of an Ω-partial group S such that $A_1 \lhd A_2$ and $B_1 \lhd B_2$. Let $D_{ij} = A_i \cap B_j$. Then

- $A_1D_{21} \lhd A_1D_{22}$, and $B_1D_{12} \lhd B_1D_{22}$.
- The partial groups A_1D_{22}/A_1D_{21} and B_1D_{22}/B_1D_{12} are Ω-isomorphic.

***Proof*:** (i) We have $B_1 \lhd B_2$ implies by lemma 3.2, that

$$A_2\cap B_1 \lhd A_2\cap B_2 \qquad (1)$$

Similarly

$$A_1\cap B_2 \lhd A_2\cap B_2 \qquad (2)$$

Applying lemma 3.4 with $S = A_2$ and $K = A_1$, we obtain by (1) $A_1(A_2\cap B_1) \lhd A_1(A_2\cap B_2)$, similarly $B_1(A_1\cap B_2) \lhd B_1(A_2\cap B_2)$. That is $A_1D_{21} \lhd A_1D_{22}$, and $B_1D_{12} \lhd B_1D_{22}$.

(ii) Applying the second Noether isomorphism theorem of Ω-partial groups (Theorem 3.1) with $S = A_1D_{22}$, $A = D_{22}$ and $K = A_1D_{21}$ noticing by (i) above that $K = A_1D_{21} \lhd A_1D_{22} = S$, we obtain at once the following Ω- isomorphism of Ω-partial groups $D_{22}/D_{22}\cap A_1D_{21} \simeq^{\Omega} D_{22}A_1D_{21}/A_1D_{21}$. By Lemma 3.3 (i) we have $AK = KA = A_1D_{21}D_{22} = A_1D_{22}$ and by using the modular law for Ω-partial groups (Lemma 3.1), we can easily obtain $K\cap A = D_{12}D_{21}$ whence the abave isomorphism gives,

$$D_{22}/D_{12}D_{21} \simeq^{\Omega} A_1D_{22}/A_1D_{21} \qquad (3)$$

Applying again the second isomorphism theorem (Theorem 3.1) with $S = B_1D_{22}$, $A = D_{22}$ and $K = B_1D_{21}$, we can obtain the following isomorphism

$$D_{22}/D_{12}D_{21} \simeq^{\Omega} B_1D_{22}/B_1D_{12} \qquad (4)$$

From (3) and (4) we obtain the desired isomorphism and the proof is complete.

Definition 3.4 (Isomorphic Series): Two Ω-series R and T of an Ω-partial group S are said to be *Ω-isomorphic* if there is a bijection from the set of factors of R to the set of factors of T such that corresponding factors are Ω-isomorphic.

We can now use Zassenhaus's Lemma for Ω-partial groups (Theorem 3.2) to establish the partial group analogue of the fundamental Schreier Refinement Theorem in groups ([10], 3.1.2).

Theorem 3.3: [The Schreier Refinement Theorem for Ω-Partial Groups]
Any two Ω-series of an Ω-partial group possess Ω- isomorphic refinements.

***Proof*:** Let $E(S) = H_0 \lhd H_1 \lhd .... \lhd H_l = S$ and $E(S) = K_0 \lhd K_1 \lhd ..... \lhd K_m = S$ be two Ω-series of S. Define $H_{ij} = H_i(H_{i+1} \cap K_j)$, $i = 0, 1, ..., l\text{-}1$, $j = 0, ..., m$ and $K_{ij}=K_i(H_i \cap K_{j+1})$, $i=0, 1, ..., l$, $j=0, ..., m\text{-}1$. Applying Theorem 3.2 with $A_1 = H_i$, $A_2 = H_{i+1}$, $B_1 = K_j$, and $B_2 = K_{j+1}$, we obtain, $A_1\ (A_2 \cap B_1) \lhd A_1\ (A_2 \cap B_2)$ and $B_1(A_1 \cap B_2) \lhd B_1(A_2 \cap B_2)$. That is $H_i(H_{i+1} \cap K_j) \lhd H_i(H_{i+1} \cap K_{j+1})$ and $K_j(H_i \cap K_{j+1}) \lhd K_j(H_{i+1} \cap K_{j+1})$. That is $H_{ij} \lhd H_{ij+1}$, $K_{ij} \lhd K_{i+1j}$ and $A_1(A_2 \cap B_2)/A_1(A_2 \cap B_1) \simeq^{\Omega} B_1(A_2 \cap B_2)/B_1(A_1 \cap B_2)$.

That is $H_{ij+1}/H_{ij} \simeq^{\Omega} K_{i+1j}/K_{ij}$. Hence the series $\{H_{ij}/\ i=0, 1, ..., l\text{-}1, j=0, ..., m\}$ and $\{K_{ij}/i=0, 1, ...l, j= 0, ..., m\text{-}1\}$ are Ω-isomorphic refinements of $\{H_i\ /\ i=0, 1, ..., l\}$ and $\{K_j/\ j=0, ..., m\}$ respectively.

Throughout the rest of this section unless stated otherwise S stands for an arbitrary Ω-partial group.

Definition 3.5: An Ω-series in S which has no proper refinements is called an Ω-*composition series*. If S is empty, we speak of a composition series in a partial group.

As it is known, not every group has a composition series. An example is the group of integer $\mathbb{Z}$, since every non trivial subgroup of $\mathbb{Z}$ is infinite cyclic. More formally if $0 \lhd H_1 \lhd ..... \lhd \mathbb{Z}$ is a series in $\mathbb{Z}$. Then H_1 (may be $\mathbb{Z}$ itself)is infinite cyclic with a generator, say m. Thus $H_1 = \langle m \rangle = m\,\mathbb{Z}$. Whence for any $n(\neq 0)$ in $\mathbb{Z}$, $\langle nm \rangle = nm\,\mathbb{Z} = H$ is a proper infinite cyclic subgroup of $m\,\mathbb{Z} = H_1$, and we obtain a proper refinement $0 \lhd H \lhd H_1 \lhd .... \lhd \mathbb{Z}$ of the given series.

Analogously, we may construct partial groups which are not groups with no composition series. Here is a simple example.

Example 3.1: Let S be the partial group with maximal subgroups S_e, S_f that is $S = S_e \cup S_f$, $E(S) = \{e, f\}$ is the semilattice $e>f$, where S_e and S_f are (disjoint) copies of $\mathbb{Z}$, with the homomorphism $\varphi_{e;f}: S_e \rightarrow S_f$ be the natural isomorphism. Let $E(S) \lhd S_1 \lhd .... \lhd S_m=S$ be a series in S. Clearly $\{e\} \lhd (S_1)_e \lhd (S_2)_e \lhd ..... \lhd (S_m)_e=S_e$ and $\{f\} \lhd (S_1)_f \lhd (S_2)_f \lhd .... \lhd (S_m)_f= S_f$ are series in S_e and S_f respectively. By the above discussion since S_e and S_f are isomorphic to $\mathbb{Z}$, we may find(assuming that $(S_1)_e$ and $(S_1)_f$ are nontrivial infinite cyclic) the following two proper refinements of S_e and S_f respectively $\{e\} \lhd H_e \lhd (S_1)_e \lhd (S_2)_e \lhd .... \lhd (S_m)_e=S_e$, $\{f\} \lhd H_f \lhd (S_1)_f \lhd (S_2)_f \lhd .... \lhd (S_m)_f= S_f$ such that H_e and H_f are infinite cyclic and isomorphic. Let $H = H_e \cup H_f$ be the partial group with $E(H) = \{e;f\!: e>f\}$. Thus H is clearly a subpartial group of S and $E(S) \lhd H \lhd S_1 \lhd .... \lhd S_m=S$ is a proper refinement of the given series of S. Whence S has no composition series. In general, we may conclude that any partial group S whose maximal subgroups are isomorphic copies of $\mathbb{Z}$ with the connecting maps the natural isomorphisms has no composition series.

Definition 3.6: A non trivial Ω-partial group S (that is $S \neq E(S)$) is said to be Ω-*simple* if it has no proper non trivial normal Ω-subpartial groups.

By Propositions 2.3 and 4.1 in [2], [see also, section 2, Theorem 2.2 (ii) in this paper], it follows immediately that if K is a normal subpartial group of a partial group S, then every subpartial group of the quotient partial group S/K must be a quotient partial group H/K for some subpartial group H of S with $K \lhd H$. These remaks hold as well for Ω-partial groups. We can now establish the Ω-partial group analouge of 3.13 in [10]. Again S stands for an arbitrary Ω -partial group.

Lemma 3.7: An Ω-series in S is an Ω-composition series if and only if all its factors are Ω-simple.

***Proof*:** Let $E(S) \lhd S_1 \lhd .... \lhd S_m = S$ be an Ω-series in S and suppose that not all its factors are Ω-simple. Whence there is a factor say S_{j+1}/S_j (for some j) which is not Ω-simple. Thus there is a non trivial normal subpartial group K of S_{j+1}/S_j. By the above remaks, we have $K = H/S_j$ for some proper Ω-subpartial group H of S_{j+1} with $S_j \lhd_{\neq} H \lneqq S_{j+1}$. Since $K = H/S_j$ is normal in S_{j+1}/S_j, we must have H is also normal in S_{j+1}. Then $S_j \lhd_{\neq} H \lhd_{\neq} S_{j+1}$. It follows that the given series has a proper refinement $E(S) \lhd S_1 \lhd ..... \lhd S_j \lhd H \lhd S_{j+1} \lhd ..... \lhd S_m = S$ and so the series is not an Ω-composition series. This proves the only if part. Conversely, let an Ω-series in S be not a composition series. Then it has a proper refinement. Whence for some consecutive terms $K \lhd L$ in the series there exists a proper normal Ω-subpartial group H of L with $K \lhd H$ and $K \neq H$. It follows that H/K is a proper non trivial normal Ω-subpartial group of L/K. Thus the factor L/K is not Ω-simple and the proof is complete.

Here we give our main result in this section, namely a Jordan-Holder Theorem for Ω-partial groups.

Theorem 3.4: If R is an Ω-composition series and T is any Ω-series of the Ω-partial group S, then T has a refinement which is a composition series and is Ω-isomorphic with R. In particular, if T is a composition series it is Ω-isomorphic with R.

***Proof*:** According to Theorem 3.3, there exist Ω-isomorphic refinements, say, R' and T' of R and T respectively. Since, by definition, R has no proper refinement, we must have $R = R'$, and so T' is Ω-isomorphic to R. It follows that any factor of T' is Ω-isomorphic to the corresponding factor of R. Whence, by Lemma 3.7, all factors of T' are Ω-simple and again by Lemma 3.7, T' is an Ω-composition series isomorphic to R. In particular, if T is a composition series, then its refinement T' which is isomorphic to R is itself T. The result obtains.

Recall that a partially ordered set is a pair $(P, \leq)$ (or simply P), where P is a set and $\leq$ is a binary relation in P which is: reflexive ($a \leq a$ for all $a \in P$), antisymmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$) and transitive ($a \leq b$ and $b \leq c \Rightarrow a \leq c$). If $A \subseteq P$, then an element $m \in A$ is a maximal element of A, if $a \in A$ and $m \leq a$ implies $m = a$. The partially ordered set R is said to satisfy the maximal condition if each nonempty subset A of P has a maximal element. Also we say that P satisfies the ascending chain condition if there does not exist an infinite properly ascending chain $p_1 < p_2 < ....$ in P. Evidently these two properties of P are identical. The minimal condition and the descending chain condition can be defined dually. Maximal and minimal conditions can be considered on any family of sets viewing as a partially ordered set by set theoretic inclusion ($\subseteq$). In particular we consider these conditions on the family of all subnormal Ω-subpartial groups of an Ω-partial group S. As in groups, the maximal and minimal conditions on that family may be denoted by max-Ωs and min-Ωs respectively. It is known that in an Ω-group G, max-Ωs and min-Ωs are equivalent to the existence of an Ω-composition series [cf.[10], 3.1.5]. We conclude this section by extending that result to Ω-partial groups.

Theorem 3.5: An Ω-partial group S has an Ω- composition series if and only if it satisfies max-Ωs and min-Ωs.

***Proof*:** Let S have an Ω-composition series of length l, and suppose, however that there exists an infinite properly ascending chain $K_1 < K_2 < \ldots$ of subnormal subpartial groups of S. Considering the chain $E \leq K_1 < K_2 < \ldots . < K_{l+1}$, every K_i, being Ω-subnormal in S, is also Ω-subnormal in K_{i+1} (by Lemma 3.2). Again by Lemma 3.2, the given chain can be made into an Ω-series of S by inserting terms of a suitable Ω-series between K_i and K_{i+1} and between K_{i+1} and S. Obviously, the length of the resulting series, and hence of any of its refinements is at least $l+1$, and so a refinement of this series isomorphic to our composition series cannot exist. A contradiction with Theorem 3.4. Thus S must have max-Ωs. Similarly, S has min-Ωs. Conversely, suppose that G has max-Ωs and min-Ωs but does not have an Ω-composition series. Hence $S \neq E(S)$, and we may apply max-Ωs to the set of proper subnormal Ω-subpartial groups of S and obtain maximal member, say S_1. By maximality, S_1 is normal in S and S/S_1 is Ω-simple. Since S has no Ω-composition series, the set of proper subnormal Ω-subpartial groups of S_1 is non empty, otherwise $E(S) \lhd S_1 \lhd S_2 = S$ would be a composition series, contradicting our assumption. Whence we may apply again max-Ωs to this set and obtain a maximal proper member, say S_2. Again S_2 is normal in S_1, S_1/S_2 is Ω-simple and $S_2 \neq E(S)$. As this process cannot terminate, we obtain an infinite descending chain of Ω-subnormal subpartial groups $\ldots < S_2 < S_1 < S_0 = S$, a contradiction to min-Ωs.

4-Some Decompositions of Partial Groups

In this section we introduce the notion of decomposability of Ω-partial groups in a way that allows to extend the direct decomposition of Ω-groups known as Remak decomposition to Ω-partial groups. The (Remak) theorem on Ω- groups that relates the existence of a Remak decomposition to the minimal condition on Ω-direct factors is generalized to Ω-partial groups. Our reference on direct decomposition and Remak decomposition of Ω-groups is [10].

Lemma 4.1: Let S be an Ω-partial group and let H and K be wide Ω-subpartial groups of S. Let $H \odot K = \cup_{e \in E} H_e \times K_e$ where $H_e \times K_e$ is the usual product of the (maximal) groups H_e and K_e. Then $H \odot K$ is an Ω-partial group with a semilattice of idempotents $E \odot E = \{(e, e): e \in E\}$ and structure maps; for $(e, e) \geq (f, f)$ (i.e. $e \geq f$) $\varphi_{(e, e), (f, f)}: H_e \times K_e \to H_f \times K_f$, $(h, k) \mapsto (\varphi_{e, ef} h, \varphi_{e, ef} k) = (hf, kf)$. Moreover $H \odot E = \cup_{e \in E} H_e \times \{e\}$ and $E \odot K = \cup_{e \in E} \{e\} \times K_e$ are normal Ω-subpartial groups of $H \odot K$.

***Proof*:** It is easy to show that $H \odot K$, so defined, is actually a partial group with the desired semilattice of idempotents and structure mappings. That $H \odot K$ is also an Ω-

partial group follows at once uppon defining for $(h, k) \in H\odot K$, $\omega\in\Omega$, $(h, k)^{\omega} = (h^{\omega}, k^{\omega})$. The proof of the other assertions is straight forward.

Definition 4.1: The Ω-partial group $H \odot K$ constructed in Lemma 4.1 is called the (*external*) *direct product* of the Ω-partial groups H and K.

Lemma 4.2: Let H and K be normal Ω-subpartial groups of the Ω-partial group S. Suppose that there exists an Ω-isomorphism $\psi: S \to H\odot K$. Then we have:

- $H' = \psi^{-1}\ (H \odot E)$ and $K' = \psi^{-1}\ (E \odot K)$ are normal Ω-subpartial groups of S,
- $H' \cap K' = E$,
- Every element $x\in S$, can be uniquely expressed as a product $x = hk$ with $h\in H'_{e_x}$ and $k\in K'_{e_x}$,
- For each $e\in E$, S_e is the (internal) direct product of the Ω-subgroups H'_e and K'_e That is $S_e = H'_e \times K'_e$ for every $e\in E$.

Proof: Part (i) follows from the hypothesis and Lemma 4.1. Likewise, ψ is an isomorphism gives $H' \cap K' = \psi^{-1}(H\odot E)\cap\psi^{-1}(E\odot K) = \psi^{-1}((H\odot E)\cap(E\odot K)) = \psi^{-1}(E\odot E) = E$, which proves (ii). Let $x\in S$, there exist $e\in E$, $h\in H_e$, $k\in K_e$ such that $\psi(x) = (h, k)$, $e_x = \psi^{-1}(e, e)$. We have $(h, e) \in H\odot E$ and $(e, k) \in E\odot K$ and so $\psi^{-1}(h, e) \in H'_{e_x}$ and $\psi^{-1}(e, k) \in K'_{e_x}$. Setting $h' = \psi^{-1}(h, e)$ and $k' = \psi^{-1}(e, k)$, we obtain $x = \psi^{-1}(h, k) = \psi^{-1}((h, e)\odot (e, k)) = \psi^{-1}(h, e)\ \psi^{-1}(e, k) = h'k'$. On the other hand, suppose that $h_1k_1 = h_2k_2$ for some $h_1, h_2 \in H'_e$ and $k_1, k_2 \in K'_e$, for some $e\in E$. Thus $h_1{}^{1}h_1k_1 = h_1{}^{1}h_2k_2$, which gives $k_1 = hk_2$, where $h = h_1^{-1}h_2 \in H'$. Again, $k_1k_2^{-1} = hk_2k_2^{-1}$. Thus $k = h$, where $k = k_1k_2^{-1} \in K'$. It follows that $h = k \in H' \cap K' = E$ and so $h = k = e$. Thus $e = h_1^{-1}h_2$ that is $h_1 = h_2$ Similarly, $k_1 = k_2$. Thus (iii) follows. The proof of (iv) follows immediately from (i), (ii) and (iii), since for any subpartial group K of S, we have $K \lhd S$ implies $K_e \lhd S_e$ for every $e\in E$ (cf. Proposition 2.4).

Definition 4.2: If H and K are normal Ω-subpartial groups of S, satisfying (ii) and (iii) (and hence (iv) in Lemma 4.2) we say that S is the (*internal) direct product* of H and K.

The converse of the Lemma 4.2 is also true. That is if the Ω-partial group S is the internal direct product of the normal Ω-subpartial groups H and K, then S is isomorphic to the direct product $H\odot K$. Formally, we have

Theorem 4.1: Suppose that S is an Ω-partial group and H and K are normal Ω-subpartial groups of S such that the conditions (ii) and (iii) (and hence (iv)) of Lemma 4.2 hold. Then S is Ω-isomorphic to the direct product $H\odot K$.

***Proof*:** For each $e\in E$, S_e is the internal direct product of the normal Ω-subgroups H_e and K_e. Hence S_e is Ω-isomorphic to the direct product $H_e\times K_e$. Since $H_e\times K_e = (H\odot K)_e$ and the mapping $E\to E\odot E$, $e\mapsto(e, e)$ is (Ω-) isomorphism of semilattices, it follows by Lemma 4.2.[6] that S and $H\odot K$ are (Ω-) isomorphic.

Actually, we may construct a proof of Theorem 4.1 that does not depend on Lemma 4.2 [6], by defining explicitly the desired isomorphism. Formally, we may define a mapping ψ: $S \rightarrow H \odot K$ as follows, for each $x \in S$, there exist unique $h \in H_{e_x}$ and $k \in K_{e_x}$ with $x = hk$. Define $\psi(x)=(h, k)$. To show that ψ is (Ω-) homomorphism, let x, $y \in S$ have the unique products $x = h_1 k_1$ and $y = h_2 k_2$ with $h_1 \in H_{e_x}$, $k_1 \in K_{e_x}$ and $h_2 \in H_{e_y}$, $k_2 \in K_{e_y}$. Then, we have $h_1 k_1 h_2 k_2 = h_1 (h_2 k_1') k_2$ for some $k_1' \in K$, where $k_1 h_2 = h_2 k_1'$. Thus $e_{xy} = e_y e_{k_1'}$ or $e_{xy} = e_{xy} e_{k_1^{-1}}$ and $h_2 k_1' k_1^{-1} = k_1 h_2 k_1^{-1} \in H$ (by normality of H) say $h_2 k_1' k_1^{-1} = h_3$ for some $h_3 \in H$, whence $e_y k_1' k_1^{-1} = h_2^{-1} h_3 \in H \cap K = E$ (by(ii)). Then $e_y k_1' k_1^{-1} = e_{xy} k_1'$ which gives $e_{xy} k_1' = e_{xy} k_1$. We have $e_{xy} k_1' = e_{xy} k_1$ and so $e_{xy} k_1' k_2 = e_{xy} k_1 k_2 = k_1 k_2$. Therefore $\psi(xy) = \psi\big((h_1 h_2)(k_1' k_2)\big) = \psi\left(h_1 h_2 \left(e_{xy} k_1' k_2\right)\right) = \psi\big((h_1 h_2)(k_1 k_2)\big) = (h_1 h_2, k_1 k_2) = (h_1, k_1) \odot (h_2, k_2) = \psi(x)\psi(y)$. Hence ψ is a homomorphism. Clearly ψ is one-to-one and onto.

In view of Lemma 4.2 and Theorem 4.1, an Ω-partial group S is isomorphic to the (external) direct product $H \odot K$ if and only if S is the (internal) direct product of H' and K'as defiend in Lemma 4.2. If one of those two equivalent statements holds, we shall identify H with H', K with K' and S with $H \odot K$, and refer to S simply as the direct product of the Ω-(sub)partial groups H and K. Formally, we shall write $S = H \odot K$ to indicate that S is the (internal) direct product of the normal Ω-subpartial groups H and K. This product can be generalized naturally to any family of (normal) Ω-partial groups.

Throughout, unless stated otherwise, S stands for an arbitrary Ω-partial group. Recall that a subpartial group H of S is proper if $H \neq S$, and non trivial if $H \neq E(S)$.

Definition 4.3: A wide Ω-subpartial group H of S is called an *Ω-direct factor* of S if there exists a wide Ω-subpartial group K such that $S = H \odot K$, K is then called an *Ω-direct complement* of H in S. If there are no proper non trivial Ω-direct factors of S, then S is called *Ω-indecomposable* (or just indecomposable if $\Omega = \phi$). If S is not Ω-indecomposable it is called Ω- decomposable.

Definition 4.4: We call a wide Ω-subpartial group H of S *essentially proper* (appreviated *ess. proper*) if H_e is a proper subgroup of S_e for all $e \in E(S)$, and *essentially non trivial* (abbreviated *ess. non trivial)* if H_e is a non trivial subgroup of S_e for all $e \in E(S)$. That is H is ess. proper if $He \neq Se$ for all $e \in E$ and ess. non trivial if $He \neq \{e\}$ for all $e \in E$. If there are no ess. proper ess. non trivial Ω-direct factors of S, then we say that S is *essentially Ω-indecomposable* (abbreviated *ess. Ω-indecomposable*). Again if $\Omega = \phi$ we speak of essentially indecomposable (ess. indecomposable). Clearly, S is (Ω-) indecomposable implies that S is ess.(Ω-) indecomposable. If S is not ess.(Ω-) indecomposable, that is if there is some ess. proper ess. nontrivial (Ω-) direct factor of S, then S is called ess.(Ω-) decomposable. Again, we clearly have:

If S is ess.(Ω-) decomposable, then S is (Ω-) decomposable. From, the definition, it follows clearly that, if $S = H \odot K$ for some (Ω-)subpartial groups H and K, then H is ess. proper ess.non trivial if and only if K is ess. proper ess. non trivial. Evidently an ess. proper subpartial group is necessarily proper, but the converse is not true. To show this we can (and do) construct a variety of examples.

Example 4.1: Let S be a partial group with $\varphi_{e,\,f}$ a monomorphism but not an isomorphism for some $e, f \in E$ with $e>f$. Let H be the subpartial group of S such that $H_g = S_g$ for all $g \in E$ with $g \neq f$ and $H_f = \varphi_{e,\,f} S_e = f\, S_e$. Since $\varphi_{e,\,f}$ is monomorphism we have H_f is a proper subgroup of S_f but $H_g = S_g$, $\forall g \neq f$. Thus H is a proper subpartial group of S but not ess.proper.(For instsnce if $S = S_e \cup S_f$ with $e>f$, $S_e \approx 2\mathbb{Z}$, $S_f \approx \mathbb{Z}$, and $\varphi_{e,\,f}$ is the usual emppeding $2\mathbb{Z} \hookrightarrow \mathbb{Z}$, then $H = 2\,\mathbb{Z} \cup 2\,\mathbb{Z}$ is a proper subpartial group of S but not ess.proper).

As in the case of Ω-groups, it is obviouse that every Ω-simple partial group is Ω-indecomposable, whereas the converse is not true, that is Ω-indecomposable partial groups need not be simple. Here is a simple example.

Example 4.2: For any prime p, let $S_e \simeq \mathbb{Z}_p$ and $S_f \simeq \mathbb{Z}_{p^2}$, be disjoint isomorphic copies of $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ respectivly. Let S be the partial group $S = S_e \cup S_f$ with semilattice $e>f$, and structure map $\varphi_{e,\,f}: S_e \rightarrow S_f$ the natural empedding. Clearly each of $S_e = \mathbb{Z}_p$ and $S_f = \mathbb{Z}_{p^2}$ is indecomposable (see also later results) and hence also ess.indecomposable, but clearly S is not simple. (Actually $H = \mathbb{Z}_p \cup \mathbb{Z}_p$ is a non trivial normal subpartial group of S).

The inheritance of ess.(Ω -) decomposability (ess.(Ω -) indecomposability) between S and its maximal subgroups may be formalized as follows.

Theorem 4.2: We have

- S is ess.(Ω-) decomposable if and only if S_e is (Ω-) decomposable for every $e \in E(S)$,
- S is ess.(Ω-) indecomposable if and only if S_e is (Ω-) indecomposable for some $e \in E(S)$.

Proof**:** (i) Suppose that S is ess.(Ω-)decomposable. There exists ess.non trivial ess.proper wide (Ω-) subpartial group H of S such that $S = H \odot K$, for some (Ω-) subpartial group K of S, (and so, as shown earlier, H and K are necessarily normal in S). Now let $e \in E(S)$. We have $S_e = H_e \times K_e$. By assumption $H_e \neq \{e\}$ and $H_e \lhd_{\neq} S_e$ and so S_e is (Ω-) decomposable. This establishes the only if part of (i). Conversely, suppose that S_e is (Ω-) decomposable for every $e \in E(S)$. Thus for each e, there exists a non trivial proper normal (Ω-) subgroup, say H_e of S_e such that $S_e = H_e \times K_e$, for some say (Ω-) subgroup K_e of S_e. We have $H_e \neq \{e\} \Leftrightarrow K_e \neq S_e$ and $H_e \lneqq S_e \Leftrightarrow K_e \neq \{e\}$. Whence $K_e \neq \{e\}$ and $K_e \neq S_e$. Now set $H = \cup_{e \in E} H_e$. For $e \geq f$ in $E(S)$, we have a homomorphism $\varphi_{e,\,f}: S_e \rightarrow S_f$, $x \mapsto xf$, which may be viewed naturally as a homomorphism $\varphi_{e,\,f}: H_e \times K_e \rightarrow H_f \times K_f$, $(h, k) \mapsto (hf, kf)$, where for any $x \in H_e \times K_e$, say $x = h_e \cdot k_e$ (unique product), with $h_e \in H_e$ and $k_e \in K_e$, we have $h_e k_e \in H_e \Leftrightarrow k_e = e$ by uniqueness of products. Thus $x \in H_e$ implies that $x = h_e\, e \in H_e \times K_e$ and $\varphi_{e,\,f}(x) = h_e f \in H_f$. Whence, $\varphi_{e,\,f}: S_e \rightarrow S_f$ induces a homomorphism $\varphi_{e,\,f}: H_e \rightarrow H_f$, $x \mapsto xf$ (for all $e \geq f$ in E). Thus $H = \cup_{e \in E} H_e$ is a wide ess.nontrivial ess.proper (Ω-) subpartial group of S. This holds similarly

for if we define $K=\cup_{e\in E}K_e$. The construction of H and K implies clearly, that $S = H\odot K$, and hence that S is ess.(Ω-)decomposable. This completes the proof of (i).
(ii) By taking the contrapositives of the two conditionals in (i), the bicoditional in (ii) follows at once.

We now give the partial group analogue of the known result in Ω-groups (cf.[10], 3. 3.1) concerning minimal and maximal conditions on Ω-direct factors. As usual S denotes an Ω-partial group. In the next result the set of all Ω- direct factors of S is viewed as a partially ordered set with set theoretic inclusion $\subseteq$ as the partial ordering.

Theorem 4.3: The maximal and minimal conditions on Ω-direct factors of S are equivalent properties.

***Proof*:** Let S satisfy the minimal condition on Ω-direct factors and let $\mathcal{L}$ be a nonempty set of Ω-direct factors of S. Let $\mathcal{F}$ be the set of all Ω-subpartial groups of S which are direct complements of at least one element of $\mathcal{L}$. By assumption, $\mathcal{F}$ has a minimal element N and so $S = M\odot N$ for some $M\in\mathcal{L}$. If M is maximal in $\mathcal{L}$, the result obtains, otherwise, there exists $M_1 \in\mathcal{L}$ such that M is a proper Ω-subpartial group of M_1 (in notation, $M<M_1$) and so $S=M_1\odot N_1$ for some $N_1\in\mathcal{F}$. By the definition of the operation $\odot$, we obtain $M_1= M_1\cap S = M_1\cap(M\odot N) = (\cup_{e\in E} M_{1e}) \cap\cup_{e\in E}(M_e\times N_e) =\cup_{e\in E}\cup_{e\in E} (M_{1e}\cap(M_e\times N_e)) = \cup_{e\in E} (M_e\times (M_{1e}\cap N_e)) = M\odot(M_1\cap N)$ whence with the same procedure, $S=M_1\odot N_1=\cup_{e\in E}(M_{1e}\times N_{1e}) =\cup_{e\in E}(M_e\times N_{1e}\times(M_{1e}\cap N_e)) =M\odot N_1\odot(M_1\cap N)$. Intesrecting with N gives, $N = N_2\odot(M_1\odot N)$, where $N_2=(M\odot N_1)\cap N$. Hence, $S = M\odot N = ((M \odot(M_1\cap N)) \odot N_2 = M_1\odot N_2$. Thus $N_2 \in\mathcal{F}$ and so $N_2=N$ by the minimality of N in $\mathcal{L}$. Consequently, $N\leq M \odot N_1$and hence $N_e \leq M_e\times N_{1e}$ for all $e\in E$. Thus, $S = M\odot N =\cup_{e\in E} (M_e \times N_e)$. Now, $S_e = M_e\times N_e =M_e\times(M_e\times N_{1e}) = M_e\times N_{1e}$, for all $e\in E$. Equivalently, $S=M\odot N_1$ which gives $S =M\odot N_1 = M_1\odot N_1$. Again by the very definition of the product $\odot$and since $M\leq M_1$, it follows immediately that $M = M_1$ and hence M is not a proper subpartial group of M_1, a contradiction. Therefore, M must be maximal in $\mathcal{L}$. This proves that if S satisfies the minimal condition on Ω-direct factors, then it satisfies the maximal condition on those factors. A similar argument can be constructed to establish the opposite direction.

In (Ω-)groups the equivalent maximal and minimal conditions on direct factors lead naturally to a certain kind of decomposition the so called "Remak decomposition" (cf.[10], 3.3.2). In view of Theorem 4.3, one may excepect extension of that result to (Ω-)partial groups. First we give a definition.

Definition 4.5: An Ω-partial group S is said to have a *Remak decomposition* if it can be expressed as a direct product of finitely many nontrivialΩ-indecomposable subpartial groups.

Theorem 4.4: If the Ω-partial group S has the minimal condition on direct factors, it has a Remak decompositon.

Proof: Assume that the hypothesis holds, but S has no Remak decomposition. Thus, in particular, S cannot be Ω-indecomposable and so S must be decomposable.

Accordingly, the set $\mathcal{L}$ of all proper nontrivial Ω-direct factors of S is non empty, whence there is some minimal element S_1 of $\mathcal{L}$ which then induces an Ω-decomposition $S=S_1\odot H_1$. By minimality, S_1 is Ω-indecomposable. This with the assumption that S has no Remak decomposition implies clearly that H_1 cannot be Ω-indecomposable. Hence H_1 must be Ω-decomposable. Also, H_1 inherits the minimal condition form S and the above argument applying now to H_1 yields an Ω-decomposition $H_1=S_2\odot H_2 >S_2$, with S_2 Ω- indecomposable and $S = S_1\odot S_2\odot H_2$. Repetition of this procedure yields an infinite descending chain $H_1 >H_2 >\ldots.$ of Ω-direct factors of S, which cannot exist by the assuming minimal condition. So, S has a Remak decomposition.

References

[1] A.M. Abd-Allah and M.E.-G.M.Abdallah, On Clifford semigroups, Pure Math. Manuscript 7(1988), 1-17.

[2] A.M. Abd-Allah and M.E.-G.M.Abdallah, Congruences on Clifford semigroups, Pure Math. Manuscript 7(1988), 19-35.

[3] A.M. Abd-Allah and M.E.-G.M.Abdallah, On tensor product of Clifford semigroups, Pure Math. Manuscript 7(1988), 37-52.

[4] A.M. Abd-Allah and M.E.-G.M.Abdallah, Categories of generalized rings associated with partial mappings, Pure Math. Manuscript 7(1988), 53-65.

[5] A.M. Abd-Allah and M.El-Ghali.M.Abdallah, Semilattices of monoids, Indian J.Math. 33(1991), no.3, 325-333.

[6] M.El-Ghali M.Abdallah, L.N. Gab-Allah, and Sayed K.M. Elagan, On semilattices of groups whose arrows are epimorphisms, International Journal of Mathematics and Mathematical Sciences, Volume 2006, Artical ID 30673, no.9, Pages1-26.

[7] M.El-Ghali M.Abdallah, L.N. Gab-Allah, and Sayed K.M. Elagan, Minimal conditions on Clifford semigroup congruences, International Journal of Mathematics and Mathematical Sciences, Volume 2006, Artical ID 76951, no.8, Pages 1-9.

[8] C.Faith, Algebra II: Ring Theory, Springer Verlag (1976).

[9] M.Petrich, Inverse Semigroups, Pure and Applied Mathematics (New York), John Wiley & Sons, New York, 1984.

[10] D.J.S.Robinson, A Course in the Theory of Groups, Graduate Texts in Mathematics, Vol.80, Springer, New York, 1982.