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Distance r-colouring and Distance r-chromatic Free Fixed and Totally Free Vertices in a Graphs

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Abstract

Given a positive integer r, two vertices u, v ϵ V (G) are r-independent if d (u, v) >r. A partition of V (G) into r-independent sets is called a distance r-coloring. A study of distance r-coloring and distance r-chromatically fixed, free and totally free vertices are studied in this paper.

Key–Words: distance r-chromatic number, distance r-independent color partition, distance r-independent .

1 Introduction

Consider a network and a coloring scheme for the nodes. Two nodes are compatible if they receive the same color. The usual coloring scheme stipulates that adjacent nodes should not receive the same color. Such a scheme is helpful in storage problem of chemicals where two non compatible chemicals (two chemicals which when placed nearby will cause danger) cannot be stored in the same room.

The chromatic number of such a scheme will give the minimum number of storage spaces required for keeping all the chemicals without any problem.

In a Institution, groups may exist. People in the same group may have some common interest. People in different groups may not be well disposed to each other. The affinity of people within a group may be determined by the closeness between the people and the number of people in the group. In the graphical model of this situation, all the nodes of the same group may be given the same color and different groups may be given different colors.

Two important aspects of graphs are partitions of vertex set into sets with a prescribed property and partition of edge set with a prescribed properties. The first gives rise to different types of colorings and the second leads to decomposition in graphs. We introduce a new coloring based on the distance. Given a positive integer r, two vertices u, v∊ V (G) are r-adjacent if u, v are adjacent in G^r and are r-independent if they are independent in G^r. A partition of V (G) into r-independent sets is called a distance r-coloring. These are same as proper coloring in G^r. The chromatic number of G^r will concide with the distance r-chromatic number of G.

1.1 Distance r-Coloring in Graphs

Definition 1 For any integer $r \geq 1$, a graph G = (V, E) is said to be r-complete if every vertex in V (G) is r-adjacent to every other vertex in V (G) . The maximum cardinality of a subset S of V (G) such that is r-complete is called r-clique number of G and is denoted by ω_r (G) .

Definition 2 Let $r \geq 1$. Let u, v ∊ V (G) . A vertex u distance r-dominates a vertex v if $d (u, v) \leq r$.

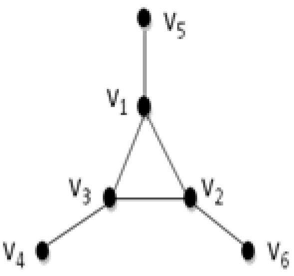
Definition 3 A subset I of V (G) is distance r-independent set if for any u, v ∊ I, $d (u, v) > r$. The maximum cardinality of a distance r-independent set of G is called the distance r-independence number of G and is denoted by β_r (G) .

Definition 4 [6] A partition of V (G) is called distance r-independent color partition of G if each element of the partition is distance r-independent. The minimum cardinality of a distance r-independent color partition of G is called distance r-chromatic number and is denoted by χ_r (G) .

Remark 5 Let G be a simple graph. Let V (G) = $\{v_1, v_2 \ldots\ldots v_n\}$. Let $\pi = \{\{v_1\}, \{v_2\}, \ldots\ldots, \{v_n\}\}$ be a distance r-color partition of a graph G. Therefore existence of distance r-color partition of any graph is guaranteed.

Remark 6 Any distance r-color partition of G is a proper color partition of G but not the other way. For example.

Example 7



$\{v_1, v_6\}, \{v_3, v_5\}, \{v_2, v_4\}$ is a proper color partition of G, but it is not a distance 2-color partition of G.

Observation 8 For any graph G, $\chi (G) \leq \chi_r (G)$.

1.2 χ_r (G) for standard graph

1. χ_r (K_n) = 1, for all r.
2. χ_r (K_n) = n, for all r.
3. $\chi_r (K_{1,n}) = \begin{cases} 2, & if\ r=1. \\ n+1, & if\ r \geq 2. \end{cases}$
4. $\chi_r (W_n) = \begin{cases} 3, & if\ r=1\ and\ n\ is\ odd \\ 2, & if\ r=1\ and\ n\ is\ even \\ n, & if\ r \geq 2 \end{cases}$
5. $\chi_r (K_{m,n}) = \begin{cases} 2, & if\ r=1 \\ m,+n & if\ r \geq 2 \end{cases}$
6. $\chi_r (P_n) = \begin{cases} r+1, & if\ 1 \leq r < n-1 \\ n, & if\ r \geq n-1 \end{cases}$

Observation 9 If G has diameter ≤ 2, then $\chi_r (G) = \begin{cases} n, & if r \geq 2 \\ \chi(G), & if\ r=1 \end{cases}$

Observation 10 For any graph G, $1 \leq \chi_r (G) \leq n$.

Theorem 11 χ_r (G) = 1 if and only if G = (K_n) .

Theorem 12 For any graph G, χ_r (G) = n if and only if r ≥ diam (G) .

Proof: If r ≥ diam (G) , then χ_r (G) = n. Suppose χ_r (G) = n and r < diam (G) . Let u, v be two vertices of G such that d (u, v) = diam (G) > r. Then {u, v} is a distance r-independent set and so $\chi_r (G) \leq n-1$, a contradiction. Therefore r ≥ diam (G) .

Theorem 13 Let G be a graph of order n. Then $\frac{n}{\beta r(G)} \leq \chi_r (G) \leq n - \beta_r (G) + 1$.

Proof: Let χ_r (G) = s. Let $\pi = \{V_1, V_2, \ldots V_s\}$ be a χ_r – partition of V (G) . Therefore $\beta_r (G) \geq |V_i|$ for all i, ($1 \leq i \leq s$) . Now $n = |V_1| + |V_2| + \ldots\ldots + |V_s| \leq s\beta_r (G)$ .Hence $\beta_r (G) \chi_r (G) \geq n$. Let D be a β_r-set of G. Let D = $\{u_1, u_2, \ldots u_{\beta r (G)}\}$. Let π = {D, $\{u_{\beta r+1}\}, \ldots \{u_n\}$}. Then π is a distance r- color partition of G.Therefore $|\pi| \geq \chi_r (G)$. That is $n- \beta_r (G) + 1 \geq \chi_r (G)$.

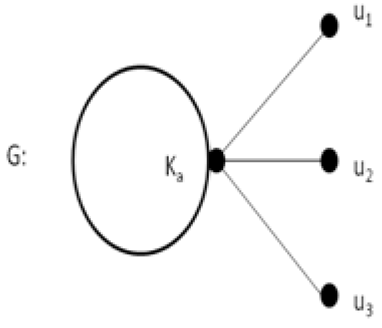
Corollary 14 Let G be a graph of order n. Then $2\sqrt{n} \leq \beta_r (G) + \chi_r (G) \leq n+1$.

Remark 15 For any graph G, $w_r (G) \leq \chi_r (G) \leq 1 + \Delta_r (G)$.

Remark 16 For any graph G, if r = diam (G) , then $\chi_r (G) = 1 + \Delta_r (G)$.

Proposition 17 Given positive integers a, b, and r with a ≤ b, there exist a connected graph G such that χ (G) =a, χ_r (G) =b.

Proof:
Case 1: a = b. For any $r \geq 1$. Then $\chi(K_a) = \chi_r(K_a) = a = b$.



Case 2: a < b. Then $r \geq 2$. Attach (b – a) pendent vertices at a vertex of K_a. Let G be the resulting graph. Then $\chi(G) = a$ and $\chi_r(G) = b$.

2 Distance r – chromaic free, fixed and totally free vertices in graphs

Definition 18 Let $P_1, P_2, \ldots\ldots P_k$ be the set of all $\chi_r(G)$ – partitions of V (G) . Let $v \in V(G)$. Then

1. V is χ_r - fixed if $\{v\} \in P_i$, for all i.
2. V is χ_r - free if for some i, j and $\{v\} \in P_i$ and $\{v\} \notin P_j$.
3. V is χ_r - totally free if $\{v\} \notin P_i$, for all i.

Remark 19 Let G be a graph of order p. Every $v \in V(G)$ is χ_r - fixed if and if only if r = diam (G) .

Proof: Let $V(G) = \{w_1, w_2, \ldots\ldots w_n\}$. If r = diam (G) , then every $v \in V(G)$ is χ_r - fixed. Conversly, suppose every vertex is χ_r - fixed and r < diam (G) . Without loss of generality, let $w_1, w_2 \in V(G)$ be such that $d(w_1, w_2) = \text{diam}(G)$. Then $P = \{\{w_1, w_2\}, \{w_3\}\}\ldots\ldots\{w_n\}$ is a distance r – partition of G with $|P| < \chi_r(G)$, a contradiction. Therefore r = diam (G) .

Theorem 20 Let G be Graph of order p. A vertex $v \in V(G)$ is χ_r –fixed with respect to $\chi_r(G)$, where r < diam (G) if and only if $N_r[v] = V(G)$.

Proof: Let r < diam (G) and $v \in V(G)$ be χ_r – fixed. Suppose $N_r[v] \subseteq V(G)$. Then there exist $u \in V(G)$ such that $d(u, v) > r$. Let $P = \{\{v\}, V_2, V_3, \ldots\ldots V_k\}$ be a χ_r - partition of G, where $k = \chi_r(G)$. Let $u \in V_i$.

$P_1 = \{\{u, v\}, \ldots\ldots V_i - \{u\}, \ldots\ldots V_k\}$ is a χ_r – partition of G if $|V_i| \geq 2$ and distance r-independent color partition of G of cardinality $\chi_r(G)$ - 1 if $|V_i| = 1$, which is a contradiction, since v is χ_r -fixed.

Therefore Nr[v]= V (G) . Converse is obvious.

Observation 21 In K_n, every vertex is χ_r -fixed.

Observation 22 In $K_{(1,\, n)}$, when r =1, the central vertex is χ_r -fixed and the remaining vertices are χ_r -totally free. When r =2, every vertex is χ_r -fixed.

Observation 23 For a complete bipartite graph $K_{(m,\, n)}$, with $2 \leq m \leq n$, every vertex is χ_r -free, when r =1 and every vertex is χ_r -fixed, when r =2.

Observation 24 In P_n, every vertex is χ_r -fixed, if $r \geq n - 1$. If $r < n - 1$, then some vertices are χ_r -fixed and some vertices are χ_r -totally free.

Observation 25 In C_n, every vertex is χ_r -fixed, if $r \geq \left\lfloor \frac{n}{2} \right\rfloor$. If $r < \left\lfloor \frac{n}{2} \right\rfloor$ and $n \equiv 0 \pmod{(r+1)}$, then all vertices of C_n are χ_r -totally free, otherwise χ_r -free.

Observation 26 For the Petersen graph P, every vertex is χ_r -totally free, when r =1 and every vertex is χ_r -fixed, when r =2.

Theorem 27 Let $\chi_r(G) = k$ and $v \in V(G)$.

If v is χ_r -free, then there exists a χ_r -coloring of G such that v is χ_r -saturated (a vertex u is χ_r -saturated if all the k-colors are represented in $N_r[u]$).

If v is χ_r -fixed, then v is χ_r -saturated in every χ_r -coloring of G.

Proof: Let v be χ_r -free. Then there exists a χ_r -coloring π in which $\{v\}$ is an element. Suppose $\pi=\{V_1, \ldots\ldots, V_k\}$, where $V_1 = \{v\}$ and v is not χ_r -saturated. Then there exists a colour class V_i such that $d(u, v) > r$, for all $u \in V_i$. Then $\pi_1 = \{V_2, \ldots\ldots, V_i \cup \{v\}, \ldots.. V_k\}$ is a distance r-independent color partition of G of cardinality less than $\chi_r(G)$, a contradiction. Therefore v is χ_r -saturated. The proof for (ii) is obvious.

Corollary 28 If v is either χ_r -fixed or χ_r -free, then $|N_r(v)| \geq \chi_r(G) - 1$.

Corollary 29 Let v be a χ_r -free or χ_r -fixed. Then $\chi_r(G) \leq 1 + \deg_r(v)$. Therefore $\chi_r(G) \leq 1 + \min\{\deg_r(v)$: v is χ_r -free or χ_r –fixed$\}$.

Proposition 30 Let G be a wounded spider which is not a star. If r=1, then all the vertices are χ_r – totally free. Suppose r=2.

Case (i): Exactly one of the arms of $k_{(1,\, n)}$ is subdivided. Let u be the central vertex and let the arm uv be subdivided at v_1. Then u and v_1 are the only vertices which are χ_r –fixed and all other vertices are χ_r- free.

Case (ii): A tleast two arms of $K_{(i,\, n)}$ are subdivided. Then only the central vertex is χ_r –fixed and all others are χ_r- free.

Suppose r=3.

Case (i): Exactly one of the arms of $k_{(1,\, n)}$ is subdivided. In this case, diameter of the graph is 3.Therefore every vertex is χ_r –fixed.

Case (ii): A tleast two arms of $K_{(1, n)}$ are subdivided. In this case, the central vertex and all the vertices which are adjacent to the central vertex are χ_r –fixed. The vertices which are at a distance 2 from the central vertex are χ_r – totally free. Suppose r=4. In this case, at least two of the arms of $k_{(1, n)}$ are subdivided and diameter of the graph is 4. Therefore every vertex of the graph is χ_r –fixed.

Definition 31 Let $P_1, P_2, \ldots P_k$ be the set of all χ_r –partitions of G. Let u, v be distinct vertices of V (G) . Then
(i) u, v are relatively χ_r – fixed if for each i, $1 \le i \le k$, u, v belong to the same set in P_i.
(ii) u, v are relatively χ_r- free if there exists i, j, $i \neq j$ such that u, v belong to the same set in P_i and they belong to different sets in P_j.
(iii) u, v are relatively totally χ_r- free if for each i belong to different set in P_i.

Remark 32 Suppose u, v are relatively χ_r –fixed. Then {u} as well as {v} do not belong to any P_i .And hence both u, v are totally χ_r- free. Suppose u, v are χ_r –fixed. Then {u} as well as {v} belong to every P_i .Therefore u, v are relatively χ_r – totally free. Let $S \subseteq V(G)$. If $|S| = 1$ and if S={v}, then S is χ_r –fixed, χ_r- free, or χ_r – totally free if v is χ_r - fixed or χ_r- free, or χ_r – totally free.Suppose $| S | \geq 2$. Then S is χ_r –fixed, χ_r- free, χ_r – totally free according as any two distinct points of S are relatively χ_r – fixed, χ_r- free or χ_r – totally free. Suppose S is χ_r – fixed, or χ_r- free, Then S is χ_r – independent.

Observation 33 Let $| S | \geq 2$. Then S is χ_r – fixed if and only if for each i, $1 \le i \le k$, $S \subset V_t$, for some V_t in some χ_r – partition P_i.

Note 34 A graph G is uniquely r-colorable if and only if every set in any χ_r – partition of V (G) is χ_r – fixed.

Theorem 35 A graph G is uniquely r-colorable if and only if every pair of distinct points u, v in V (G) are relatively χ_r – fixed or relatively χ_r – totally free.

Proof: Suppose G is uniquely r-colorable. Let u, v $\in$ V (G) . Since G is uniquely r-colorable, there exists a unique χ_r – partition $\prod = \{ V_1, V_2, \ldots V_{\chi_r} \}$. Suppose u, v $\in V_1$. Then u and v relatively χ_r – fixed. Suppose u, v are in different sets of $\prod$. Then u, v are relatively χ_r – totally free. Conversely, suppose every pair of distinct points are relatively χ_r – fixed or relatively χ_r – totally free. Define a relation τ on V (G) as follows: u τ v if and only if u and v are relatively χ_r – fixed. τ is an equivalence relation on V (G) . Let $\frac{V(G)}{\tau} = \{ V_1, V_2, \ldots V_k \}$. Since u and v are relatively χ_r – fixed, d (u, v) > r. Therefore $V_1, V_2, \ldots V_k$ are r-independent. Suppose $\prod = \{ U_1, U_2, \ldots U_t \}$ be a χ_r – partition of V (G) , where t= χ_r (G) . Let x, y $\in U_i$ $(1 \le i \le t)$. Then x and y are relatively χ_r – fixed. Therefore x and y belong to the same partition say V_j in $\frac{V(G)}{\tau}$. If z $\in U_i$, then y and z are relatively χ_r – fixed. Since y $\in V_j$, z also belongs to V_j.

Therefore $U_i \subseteq V_j$. Since $U_i \cup V_j$ is not r-independent, U_{i1}, U_{i2} cannot be contained in the same set in $\frac{V(G)}{\tau}$. There are at least t-sets in $\frac{V(G)}{\tau}$. Since $U_1 \cup U_2 \cup \ldots\ldots \cup U_t = V$ (G) , we get that k = t and $U_1, U_2, \ldots\ldots U_t$ are same as $V_1, V_2, \ldots\ldots V_t$ in some order. Therefore $\frac{V(G)}{\tau}$ is the unique χ_r – partition of V (G) . Therefore G is uniquely r-colorable.

Theorem 36 Let χ_r (G) = k and let S_1 and S_2 be χ_r – fixed sets of vertices of G such that S_1, S_2 belongs to P_i, for some i, $1 \le i \le k$, then $\langle S_1 \cup S_2 \rangle$ is connected.

Proof: Let $G_1 = \langle S_1 \cup S_2 \rangle$. Suppose G_1 is disconnected with components H_1 and H_2 . Let $H_1 \subseteq S_1$. Then H_1 is r-independent. Therefore H_1 is r-independent. Therefore $\langle H_1 \rangle$ is disconnected, which is a contradiction. Hence $S_2 \cap H_1$ is non-empty. Similarly we can prove that H_2 contains points of both S_1 and S_2 . Let $T_1 = H_1 \cap S_1$, $T_2 = H_2 \cap S_2$. Then $T_1 \neq \phi$ and $T_2 \neq \phi$. Interchanging the colors of points of T_1 with the colors of points of T_2, we have a χ_r –coloring in which S_1 and S_2 are not χ_r –fixed, which is a contradiction. Therefore G_1 is connected.

Remark 37 If $S_1, S_2, \ldots\ldots S_t$ are χ_r –fixed sets of G such that each S_j belong to P_i for some i, $1 \le i \le k$, then the subgraph induced by $S_1 \cup S_2 \cup \ldots\ldots \cup S_t$ is (t-1) -connected.

Remark 38 Every uniquely r-colorable graph G is χ_r (G) – 1-connected.

Theorem 39 A graph G is uniquely r-colorable if and only if every pair of distinct points u, v in V (G) are relatively χ_r –fixed or relatively χ_r – totally free.

Proof: Suppose G is uniquely r-colorable. Let u, v ∈ V (G) . Since G is uniquely r-colorable, there exists a unique χ_r – partition $\pi = \{ V_1, V_2, \ldots\ldots V_{\chi_r} \}$. Suppose u, v ∈ V_1 .Then u and v relatively χ_r –fixed. Suppose u, v are in different sets of π. Then u, v are relatively χ_r – totally free. Conversely, suppose every pair of distinct points are relatively χ_r –fixed or relatively χ_r – totally free. Define a relation τ *on* $V(G)$ as follows: u τ v if and only if u and v are relatively χ_r –fixed. τ is an equivalence relation on V (G) . Let $\frac{V(G)}{\tau} = \{ V_1, V_2, \ldots\ldots V_k \}$. Since u and v are relatively χ_r –fixed, d (u, v) > r. Therefore $V_1, V_2, \ldots\ldots V_k$ are r-independent. Suppose $\pi = \{ U_1, U_2, \ldots\ldots U_t \}$ be a χ_r –partition of V (G) , where t = χ_r (G) . Let x, y ∈ U_i, $(1 \le i \le t)$. Then x and y are relatively χ_r –fixed. Therefore x and y belong to the same partition say V_j in $\frac{V(G)}{\tau}$. If z ∈ U_i, then y and z are relatively χ_r –fixed. Since y ∈ V_j, z also belongs to V_j . Therefore $U_i \subseteq V_j$. Since $U_i \cup V_j$ is not r-independent, U_{i1}, U_{i2} cannot be contained in the same set in $\frac{V(G)}{\tau}$. There are at least t-sets in $\frac{V(G)}{\tau}$. Since $U_1 \cup U_2 \cup \ldots\ldots \cup U_t = V$ (G) , we get that k = t and $U_1, U_2, \ldots\ldots U_t$ are same as $V_1, V_2, \ldots\ldots$ Vt in some order. Therefore $\frac{V(G)}{\tau}$ is the unique χ_r – partition of V (G) . Therefore G is uniquely r-colorable.

Theorem 40 Let $\chi_r(G) = k$ and let S_1 and S_2 be χ_r–fixed sets of vertices of G such that S_1, S_2 belongs to P_i, for some i, $1 \le i \le k$, then $\langle S_1 \cup S_2 \rangle$ is connected.

Proof: Let $G_1 = \langle S_1 \cup S_2 \rangle$. Suppose G_1 is disconnected with components H_1 and H_2. Let $H_1 \subseteq S_1$. Then H_1 is r-independent . Therefore $\langle H_1 \rangle$ is disconnected, which is a contradiction. Hence $S_2 \cap H_1$ is non-empty. Similarly we can prove that H_2 contains points of both S_1 and S_2 . Let $T_1 = H_1 \cap S_1$, $T_2 = H_2 \cap S_2$. Then $T_1 \neq \phi$ and $T_2 \neq \phi$. Interchanging the colors of points of T_1 with the colors of points of T_2, we have a χ_r–coloring in which S_1 and S_2 are not χ_r–fixed, which is a contradiction. Therefore G_1 is connected.

Remark 41 If $S_1, S_2, \ldots\ldots S_t$ are χ_r–fixed sets of G such that each S_j belong to P_i for some i, $1 \le i \le k$, then the subgraph induced by $S_1 \cup S_2 \cup \ldots\ldots \cup S_t$ is (t-1) – connected.

Remark 42 Every uniquely r-colorable graph G is $\chi_r(G) - 1$-connected.

Definition 43
(i) A set $S \subseteq V(G)$ is χ_r– critical if $\chi_r(G\text{-}S) < \chi_r(G)$.
(ii) A graph G is χ_r– critical if $\chi_r(H) < \chi_r(G)$, for every proper subgraph H of G.

Observation 44 If $S \subseteq P_i$, for some i, then S is χ_r– critical and $\chi_r(G\text{-}S) = \chi_r(G) - 1$.

Proof: Let $\pi = \{ V_1, V_2, \ldots\ldots V_{\chi_r(G)} \}$ and $S=\{V_1\}$. Let $\pi' =\{ V_2, \ldots\ldots V_{\chi_r(G)} \}$. This is a r-coloring of G-S. Therefore $\chi_r(G\text{-}S) \le \chi_r(G) - 1$. Suppose $\chi_r(G\text{-}S) < \chi_r(G) - 1$. Let $\pi = \{ U_1, U_2, \ldots\ldots U_t \}$, $t = \chi_r(G\text{-}S)$ be a χ_r– partition of G-S.Then $\pi'' =\{ U_1, U_2, \ldots\ldots U_t, S\}$ is a r-coloring of G.Therefore $\chi_r(G) \le |\pi'''| < \chi_r(G)$, which is a contradiction. Therefore $\chi_r(G\text{-}S) = \chi_r(G) - 1$. Hence the theorem.

Definition 45 Let G be a graph. A subgraph H of G is called a (k, r) - chromatic subgraph of G if $\chi_r(H) = k$.

Proposition 46 Let $\chi_r(G) = k$. Let $v \in V(G)$. Then $\chi_r(G\text{-}v) = k\text{-}1$ if and only if v is contained in each k-r-chromatic subgraph of G.

Remark 47
For any $v \in V(G)$, The following are equivalent.
(i) $\{v\}$ is χ_r-critical.
(ii) $\{v\}$ is χ_r- fixed or χ_r- free.
(iii) $\{v\}$ is contained in each $\chi_r(G) - r$ – chromatic subgraph of G.

Proof:
(i) => (ii) is obvious. Assume (ii) . $\{v\} \in P_i$, for some i. Therefore $\chi_r(G\text{-}v) = \chi_r(G) - 1$ which means v is χ_r-critical. Hence (ii) => (i) .

Claim: (i) => (iii) Since (i) => (ii) , $\chi_r(G\text{-}v) = \chi_r(G) - 1$. Therefore by the proposition

46, (iii) follows. Assume (iii) to be true.

Claim: (iii) => (i) By the proposition 46, χ_r (G-v) = χ_r (G) – 1. Therefore v is χ_r - critical. Hence (iii) = > (i) .

Observation 48 Let S be a χ_r - fixed set and |S| ≥ 2. Then no vertex of S is either χ_r - fixed or χ_r - free. Therefore S contains no critical vertices.

Proposition 49 Let χ_r (G) = k and e = uv be an edge in G. If each of u and v is χ_r - free, then there exists a k-r-coloring of G such that
(i) u is the only vertex receiving a particular color and
(ii) no vertex r-adjacent to u other than v receives the color of v.

Proof: Let χ_r (G) = k and u be χ_r - free. Then, there exists a χ_r - partition P_i such that {u} ∈ P_i . Let P_i ={{u}, $V_2, \ldots V_k$ } and v ∈ V_2 (say) . Let w ∈ N (u) -{v}. Suppose w ∈ V_2. Since v is χ_r - free, there exist V_i, i ≥ 3 such that w is not r-adjacent to any vertex of V_i . Therefore w can be shifted to V_i . Repeating this process for every vertex in $V_2 \cap$ (N (u) -{v}) , (ii) is proved..

Theorem 50 Let x = uv and let d (u, v) > r in G-x. Then the following are equivalent.
(i) x is χ_r -critical.
(ii) Each of u and v is either χ_r - fixed or χ_r - free in G.
(iii) u and v are relatively χ_r - free in G-x.
(iv) x is contained in each χ_r -r-chromatic subgraph of G.

Proof:
Claim: (i) => (iii) Suppose x is χ_r -critical. Then χ_r (G-x) < χ_r (G) . Suppose u and v are not relatively χ_r - fixed in G-x. Then there exists a χ_r - partition P of G-x such that u and v belong to different χ_r - color classes. Then χ_r (G-x) = χ_r (G) , a contradiction, since x is χ_r -critical. Therefore u and v are relatively χ_r - fixed in G-x. Therefore (i) => (iii) .

Claim: (iii) => (i) Suppose u and v are relatively χ_r - free in G-x. Therefore there is no χ_r (G-x) -partition of G-x in which u and v are colored differently. Therefore χ_r (G) = χ_r (G-x) +1 => x is χ_r -critical. Hence (iii) => (i) .

Claim: (i) => (ii) Since x is χ_r -critical, χ_r (G-x) < χ_r (G) . Therefore u and v must receive the same color in any χ_r (G-x) -partition of (G-x) . Therefore we get a χ_r - partitions of G by giving a new color to u or v. Therefore u and v are either χ_r - fixed (or) χ_r - free in G. Hence (i) => (ii) .

Claim: (ii) => (i)
Case (i): Suppose u and v are χ_r - fixed. Therefore {u} and {v} belong to every P_i . In G-x, we can give the same color to u and v, since d (u, v) > r in G-x. Therefore χ_r (G-x) < χ_r (G) and hence x is χ_r -critical.

Case (ii): Suppose u is χ_r - fixed and v is χ_r - free. Then there exists a P_i such that {u} and {v} belong to P_i . Similar reasoning as in case (i) , shows that x is χ_r -critical.

Case (iii): Suppose each of u and v are χ_r - free. There exists a χ_r - partition say P = $\{\{u\}, V_2, \ldots, V_k\}$ let $v \in V_2$ in which $\{u\} \in P$. Any vertex $w \neq v \in V_2$ such that w is adjacent to u can be shifted to V_i, $i \geq 3$. Repeating this process, we get a χ_r - partition $P_1 = \{\{u\}, V_2', \ldots V_k'\}$ in which any vertex $w \neq v \in V_2'$ is not adjacent to v. Let $P_2 = \{ V_2' \cup \{u\}, V_3', \ldots V_k' \}$ be a χ_r - partition of G-x. Since d (u, v) > r in G-x, χ_r (G-x) < χ_r (G) . Therefore x is χ_r -critical .

3 Conclusion

We have made a study of distance r-coloring and distance r-chromatic free, fixed and totally free vertices in a graph. It is further continued in our subsequent investigations in this direction. Storage problem of chemicals and other applications are also attempted.

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