Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 10, Number 1 (2014), pp. 35–42

http://www.ripublication.com/gjpam.htm

Cartesian Product of Fuzzy β-ideals of β-algebras

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Abstract

In this paper, we introduce the notion of Cartesian Product of fuzzy β-ideals on β-algebras and investigate some of their properties. Further, we study the properties of the homomorphic image of fuzzy β-ideals of a β-algebra.

AMS subject classification: 03E72, 06F35, 03G25.
Keywords: BCK/BCI-algebras, β-algebras, β-ideals, Fuzzy β-ideals.

1. Introduction

In 1966, Y. Imai and K. Iseki ([4], [5], [6]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI algebras. In 2002, J. Neggers and H.S. Kim [10] introduced the notion of β-algebras. In 2012, Y.H. Kim [7] investigated some properties of β-algebras.

Lofti A. Zadeh [13] introduced the theory of fuzzy sets. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1977, by Rosenfeld [11]. O.G. Xi [12] applied the concept of fuzzy sets to BCK algebras and got some results in 1991. W.A. Dudek and Y.B. Jun [3] fuzzified the ideals in BCC-algebras in 2001. For the general study of structures in β-algebras, the ideal theory and fuzzy ideal theory plays an important role.

This motivated us to study the notion of fuzzy ideals in β-algebras. In our paper([1], [2]) we have introduced the notion of fuzzy β-subalgebras and normal fuzzyβ-subalgebras on β-algebras and investigated some of their properties. In this paper, we introduce the notion of cartesian product of fuzzyβ-ideals of a β-algbera and investigate some of their properties. Further, we study the properties of the homomorphic image of fuzzy β-ideals of a β-algebra.

2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

Definition 2.1. [10] A β-algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

1. $x - 0 = x$.
2. $(0 - x) + x = 0$.
3. $(x - y) - z = x - (z + y) \; \forall \, x, y, z \; \in \; X$.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ be a set with constant 0 and two binary operations + and - are defined on X with the Caylay's table

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$-$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, +, -, 0)$ is a β-algebra.

Definition 2.3. Let $(X, +, -, 0)$ be a β-algebra. A partial ordering $' \leq'$ can be defined on X as follows: For any $x, y \in X, x \leq y$ if and only if $x - y = 0$.

Henceforth by aβ-algebra X, we mean a β-algebra $(X, +, -, 0)$ derived from a group or a β-algebra $(X, +, -, 0)$derived from a $B-$algebra $(X, -, 0)$.

Definition 2.4. Let X be a set of universal discourse and a fuzzy set μ in X is a function $\mu : X \to [0, 1]$. For each element x in $X, \mu(x)$ such that $0 \leq \mu(x) \leq 1$,is called the membership value of x in X.

Definition 2.5. If μ_1 and μ_2 are two fuzzy sets of X then $\mu_1 \subseteq \mu_2$ if $\mu_1(x) \leq \mu_2(x), \forall \, x \in X$.

Definition 2.6. If μ_1 and μ_2 are two fuzzy sets of X then intersection of μ_1 and μ_2 is defined as $(\mu_1 \cap \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}, x \in X$.

Definition 2.7. If μ_1 and μ_2 are two fuzzy sets of X then union of μ_1 and μ_2 is defined as $(\mu_1 \cup \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}, x \in X$.

Definition 2.8. Let μ be a fuzzy set in a set X. For $t \in [0, 1]$, the set $\mu_t = \{x \in X/\mu(x) \geq t\}$ is called a level subset of μ.

Definition 2.9. Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β-algebras. A mapping $f : X \to Y$ is said to be a β-homomorphism if $f(x + y) = f(x) + f(y)$ and $f(x - y) = f(x) - f(y), \forall x, y \in X$. Note that for a β-homomorshism $f(0_X) = f(0_Y)$.

Definition 2.10. Let μ be a fuzzy set of X. μ is said to have the supremum property if for any subset A of X, there exist a $a_0 \in A$ such that $\mu(a_0) = \sup_{a \in A} \mu(a)$.

3. Homomorphism of Fuzzy β-ideals of β-algebras

In this section we prove some simple theorems on homomorphism of Fuzzy β-ideals of β-algebras.

Definition 3.1. A non-empty subset I of a β-algebra $(X, +, -, 0)$ is called a β-ideal of X, if

1. $0 \in I$,
2. $x + y \in I, \forall x, y \in I$, and
3. if $x - y \; and \; y \in I$ then $x \in I \; \forall \; x, y \in X$.

Example 3.2. In example 2.2 of β-algebra X, the subset $I_1 = \{0, 1\}$ is a β- ideal of X. But $I_2 = \{0, 1, 3\}$ is not a β-ideal of X(since 1+3=2$\notin I_2$.)

Lemma 3.3. If $\{I_n\}$ be a family of asending chain of β-ideals of X, then $\cup I_n$ is also a β-ideals of X.

Proof. Straightforward. ■

Definition 3.4. Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-ideal of X if

1. $\mu(0) \geq \mu(x) \; \forall \; x \in X$,
2. $\mu(x + y) \geq min\,\{\mu(x), \mu(y)\} \;\; \forall x, y \in X$, and
3. $\mu(x) \geq min\,\{\mu(x - y), \mu(y)\} \;\; \forall x, y \in X$.

Example 3.5. In example 2.2 of β-algebra X, Define the fuzzy set $\mu_1 : X \to [0, 1]$ such that

$$\mu_1(x) = \begin{cases} 0.8 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2, 3 \end{cases}$$

then μ_1 is a fuzzy β-ideal of X. But the fuzzy set $\mu_2 : X \to [0,1]$ such that

$$\mu_2(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.6 & \text{if } x = 2, 3 \end{cases}$$

is not a fuzzy β-ideal of X.

Theorem 3.6. Let $f : X \to Y$ be a homomorphism of a β-algebra X onto a β-algebra Y. If μ is a fuzzy β-ideal of Y then the preimage $f^{-1}(\mu)$ of μ defined by $f^{-1}(\mu)(x) = \mu(f(x))\ \forall x \in X$ is a fuzzy β-ideal of X.

Proof. Let μ be a fuzzy β-ideal of Y. For $x, y \in X$,

1. For any $x \in X$, $f^{-1}(\mu(0)) = \mu(f(0)) = \mu(0) \geq \mu(x)$.

2. For any $x, y \in X$, $f^{-1}(\mu)(x + y) = \mu(f(x + y)) = \mu(f(x) + f(y)) \geq min\{\mu(f(x), \mu(f(y)\} = min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}$

3. For any $x, y \in X$. $f^{-1}(\mu)(x) = \mu(f(x)) \geq min\{\mu(f(x) - f(y)), \mu(f(y))\} = min\{\mu(f(x - y), \mu(f(y))\} = min\{(f^{-1}(\mu)(x - y), f^{-1}(\mu)(y))\}$

Hence $f^{-1}(\mu)$ is a fuzzy β-ideal of X. ■

Lemma 3.7. Let $f : X \to Y$ be a homomorphism of a β-algebra X onto a β-algebra Y. If μ is a fuzzy β-ideal of X with $\ker f \subseteq X_\mu$, then $f^{-1}(f(\mu)) = \mu$.

Proof. Let $x \in X$ and $f(x) = y$. Hence $f^{-1}(f(\mu))(x) = f(\mu)(f(x)) = f(\mu)(y) = f(\mu)(y) = \sup_{x \in f^{-1}(y)} \{\mu(x)\}$. For any $x' \in f^{-1}(y) \Rightarrow f(x') = y \Rightarrow f(x') = f(x) \Rightarrow f(x') - f(x) = 0 \Rightarrow f(x' - x) = 0 \Rightarrow x' - x \in kerf \Rightarrow x' - x \in X_\mu \Rightarrow \mu(x' - x) = \mu(0)$. Therfore $\mu(x') \geq min\{\mu(x' - x), \mu(x)\} = min\{\mu(0), \mu(x)\} = \mu(x)$. we can also prove $\mu(x) \geq \mu(x')$. Hence $\mu(x') = \mu(x)$. Therefore $f^{-1}(f(\mu))(x) = \sup_{x' \in f^{-1}(y)} \{\mu(x')\} = \mu(x) \Rightarrow f^{-1}(f(\mu)) = \mu$. ■

Theorem 3.8. Let $f : X \to Y$ be a homomorphism of a β-algebra X onto a β-algebra Y. If μ is a fuzzy β-ideal of X with supremum property and $\ker f \subseteq X_\mu$ then the image of μ, $f(\mu)$ defined by $f(\mu)(y) = \sup_{x \in f^{-1}(y)} \{\mu(x)\}, \forall y \in Y$, is a fuzzy β-ideal of Y.

Proof.

1. Now $f(\mu)(0) = \sup_{x \in f^{-1}(0)} \{\mu(x)\} = \mu(0) \geq \mu(x) \forall\ x \in X$.
   Hence $f(\mu)(0) \geq \sup_{x \in f^{-1}(y)} \{\mu(x)\} = f(\mu)(y) \forall y \in Y$.

2. Let $y_1, y_2 \in Y$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$.

$$\begin{aligned} f(\mu)(y_1 + y_2) &= sup\left\{\mu(x)/x \in f^{-1}(y_1 + y_2)\right\} \\ &\geq sup\left\{\mu(x_1 + x_2)/x_1 \in f^{-1}(y_1)\ and\ x_2 \in f^{-1}(y_2)\right\} \\ &\geq sup\left\{min\left\{\mu(x_1), \mu(x_2)\right\}/x_1 \in f^{-1}(y_1)\ and\ x_2 \in f^{-1}(y_2)\right\} \\ &= min\left\{sup\left\{\mu(x_1)/x_1 \in f^{-1}(y_1)\right\}, sup\left\{\mu(x_2/x_2 \in f^{-1}(y_2)\right\}\right\} \\ &= min\left\{\sup_{x_1 \in f^{-1}(y_1)}\{\mu(x_1)\}, \sup_{x_2 \in f^{-1}(y_2)}\{\mu(x_2)\}\right\} \\ &= min\{f(\mu)(y_1), f(\mu)(y_2)\} \end{aligned}$$

3. Suppose that for some y_1 and $y_2 \in Y$, $f(\mu)(y_1) < min\{f(\mu)(y_1 - y_2), f(\mu)(y_2)\}$. Since f is onto there exist x_1 and $x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Hence $f(\mu)(f(x_1)) < min\{f(\mu)(f(x_1) - f(x_2)), f(\mu)(f(x_2))\}$
$= min\{f(\mu)(f(x_1 - x_2)), f(\mu)(f(x_2))\}$.
$\Rightarrow f^{-1}(f(\mu))(x_1) < min\{f^{-1}(f(\mu))(x_1 - x_2), f^{-1}(f(\mu))(x_2)\}$
$\Rightarrow \mu(x_1) < min\{\mu(x_1 - x_2), \mu(x_2)\}$(since by lemma 3.7 $f^{-1}(f(\mu)) = \mu$.)

Hence $f(\mu)$ is a fuzzy β-ideal of Y. ■

Theorem 3.9. Let $f : X \to X$ be an endomorphism on X. Let μ be a fuzzy β-ideal of X. Then $\mu_f : X \to [0, 1]$ defined by $\mu_f(x) = \mu(f(x)), \forall x \in X$, is a fuzzy β-ideal of X.

Proof. Let μ be a fuzzy β-ideal of Y. For $x, y \in X$,

1. For any $x \in X, \mu_f(0)) = \mu(f(0)) = \mu(0) \geq \mu(x)$.

2. For any $x, y \in X, \mu_f(x+y) = \mu(f(x+y)) = \mu(f(x)+f(y)) \geq min\{\mu(f(x), \mu(f(y)\} = min\{\mu_f(x), \mu_f(y).\}$

3. For any $x, y \in X$, $\mu_f(x) = \mu(f(x)) \geq min\{\mu(f(x) - f(y)), \mu(f(y))\} = min\{\mu(f(x-y), \mu(f(y))\} = min\{\mu_f(x-y), \mu_f(y))\}$ Hence μ_f is a fuzzy β-ideal of X. ■

4. Cartesian Product of Fuzzy β-ideals of β-algebras

Theorem 4.1. Let μ_1 and μ_2 be two fuzzy β-ideals of a β-algebra X. The Cartesian product $\mu_1 \times \mu_2$ of μ_1 and μ_2 is defined as $(\mu_1 \times \mu_2)(x_1, x_2) = min\ \{\mu_1(x_1), \mu_2(x_2)\}\ \forall\, x_1, x_2 \in X$. Then $\mu_1 \times \mu_2$ is a fuzzy β-ideal of $X \times X$.

Proof.

1. Now For any $x_1, x_2 \in X$, we have $(\mu_1 \times \mu_2)(0, 0) = min\{\mu_1(0), \mu_2(0)\} \geq min\{\mu_1(x_1), \mu_2(x_2)\} = (\mu_1 \times \mu_2)(x_1, x_2) \forall\, x_1, x_2 \in X$.

2. For $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$,

$$\begin{aligned}(\mu_1 \times \mu_2)(x + y) &= (\mu_1 \times \mu_2)((x_1, x_2) + (y_1, y_2)) \\ &= (\mu_1 \times \mu_2)(x_1 + y_1, x_2 + y_2) \\ &= min\{\mu_1(x_1 + y_1), \mu_2(x_2 + y_2)\} \\ &\geq min\{min\{\mu_1(x_1), \mu_1(y_1)\}, min\{\mu_2(x_2), \mu_2(y_2)\}\} \\ &= min\{min\{\mu_1(x_1), \mu_2(x_2)\}, min\{\mu_1(y_1), \mu_2(y_2)\}\} \\ &= min\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)\} \\ &= min\{(\mu_1 \times \mu_2)(x), (\mu_1 \times \mu_2)(y)\}.\end{aligned}$$

3. For $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$,

$$\begin{aligned}(\mu_1 \times \mu_2)(x) &= (\mu_1 \times \mu_2)((x_1, x_2) \\ &= min\{\mu_1(x_1), \mu_2(x_2)\} \\ &\geq min\{min\{\mu_1(x_1 - y_1), \mu_1(y_1)\}, min\{\mu_2(x_2 - y_2), \mu_2(y_2)\}\} \\ &= min\{min\{\mu_1(x_1 - y_1), \mu_2(x_2 - y_2)\}, min\{\mu_1(y_1), \mu_2(y_2)\}\} \\ &= min\{(\mu_1 \times \mu_2)((x_1, x_2) - (y_1, y_2)), (\mu_1 \times \mu_2)(y_1, y_2)\} \\ &= min\{(\mu_1 \times \mu_2)(x - y), (\mu_1 \times \mu_2)(y)\}.\end{aligned}$$

Hence $\mu_1 \times \mu_2$ is fuzzy β- ideal of $X \times X$. ■

Theorem 4.2. Let μ_1 and μ_2 be two fuzzy sets in a β-algebra X. If $\mu_1 \times \mu_2$ is a fuzzy β-ideal of $X \times X$ then

1. Either $\mu_1(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x) \forall\, x \in X$.
2. If $\mu_1(0) \geq \mu_1(x) \forall\, x \in X$, then either $\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$.
3. If $\mu_2(0) \geq \mu_2(x) \forall\, x \in X$, then either $\mu_1(0) \geq \mu_2(x)$ or $\mu_1(0) \geq \mu_1(x)$.
4. Either μ_1 or μ_2 is a fuzzy β-ideal of the β-algebra X.

Proof. (1) Suppose that $\mu_1(0) < \mu_1(x)$ and $\mu_2(0) < \mu_2(x)$ $for\ some$ $x_1, x_2 \in X$, then $(\mu_1 \times \mu_2)(x_1, x_2) = min\{\mu_1(x_1), \mu(x_2)\} > min\{\mu_1(0), \mu_2(0)\} = (\mu_1 \times \mu_2)(0, 0)$. A contradiction to $\mu_1 \times \mu_2$ is a fuzzy β-ideal of X. Hence $\mu_1(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x) \forall\, x \in X$.

Proof. (2) Suppose that $\mu_2(0) < \mu_1(x_1)$ and $\mu_2(0) < \mu_2(x_2) \forall\, x_1, x_2 \in X$, then

$$\begin{aligned}(\mu_1 \times \mu_2)(x_1, x_2) &= min\{\mu_1(x_1), \mu_2(x_2)\} \\ &> min\{\mu_2(0), \mu_2(0)\} \\ &= \mu_2(0) = min\{\mu_1(0), \mu_2(0)\} \\ &= (\mu_1 \times \mu_2)(0, 0).\end{aligned}$$

Which is a contradiction to $\mu_1 \times \mu_2$ is a fuzzy β-ideal of X. Hence either$\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$.

Proof. (3) Similar to Proof (2).

Proof. (4) By (1) let $\mu_1(0) \geq \mu_1(x) \forall x \in X$ then by(2) $\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$. Now $\mu_1(x) = min\{\mu_1(x), \mu_2(0)\} = (\mu_1 \times \mu_2)(x, 0)$. Hence for $x, y \in X$ we have

$$\begin{aligned}
\mu_1(x + y) &= min\{\mu_1(x + y), \mu_2(0)\} \\
&= (\mu_1 \times \mu_2)(x + y, 0) \\
&= (\mu_1 \times \mu_2)(x + y, 0 + 0) \\
&= (\mu_1 \times \mu_2)((x, 0) + (y, 0)) \\
&= min\{(\mu_1 \times \mu_2)(x, 0), (\mu_1 \times \mu_2)(y, 0)\} \\
&= min\{min\{\mu_1(x), \mu_2(0)\}, min\{\mu_1(y), \mu_2(0)\}\} \\
&= min\{\mu_1(x), \mu_1(y)\}.
\end{aligned}$$

$$\begin{aligned}
\mu_1(x) &= min\{\mu_1(x), \mu_2(0)\} \\
&= (\mu_1 \times \mu_2)(x, 0) \\
&\geq min\{(\mu_1 \times \mu_2)((x, 0) - (y, 0), (\mu_1 \times \mu_2)(y, 0)\} \\
&= min\{(\mu_1 \times \mu_2)(x - y, 0), (\mu_1 \times \mu_2)(y, 0)\} \\
&= min\{min\{\mu_1(x - y), \mu_2(0)\}, min\{\mu_1(y), \mu_2(0)\} \\
&= min\{\mu_1(x - y), \mu_1(y)\}.
\end{aligned}$$

Hence μ_1 is a fuzzy β-ideal of X. Similarly we can prove that μ_2 is a fuzzy β-ideal of X. ■

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