Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 10, Number 1 (2014), pp. 29–34

http://www.ripublication.com/gjpam.htm

Neighborhood Distinguishing Coloring number of Graphs

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Abstract

Starting from proper coloring of vertices, various types of vertex colorings have been studied. Recognizable colorings [1], detectable colorings [2], irregular colorings [3], are coloring of vertices which distinguish the vertices in some manner. A slight variation of irregular colorings is the Neighborhood distinguishing colorings. In this case, the code of the vertex does not involve the color assigned to the vertex. Whereas irregular coloring exists in any graph, Neighborhood Distinguishing Coloring exists in a graph if and only if any two non-adjacent vertices do not have the same neighbor. The definition and some of the properties of this coloring are given in [4] . In this paper, graph G with $\chi_{NDC}(G) = 2$ are characterized. Necessary and sufficient conditions for existence of ND-coloring in trees are derived. A lower and upper bound for χ_{NDC} of union of two graphs have been found.

AMS subject classification: 05C15.
Keywords: ND-coloring, NDC-partition, NDC-code, NDC-number.

1. Introduction

A (proper) coloring of a graph G is a function $c : V(G) \to N$ having the property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G, where N is the set of positive integers. A k-coloring of G uses k colors. The chromatic number $\chi(G)$ of G is the minimum positive integer k for which there is a k-coloring of G. For a positive integer k and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of a graph G, the color code of a vertex v of G (with respect to c) is the ordered (k+1)-tuple $code_c(v) = (a_0, a_1, \ldots, a_k)$, where a_0 is the color assigned to v $(that\ is,\ a_0 = c(v))$ and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i . If the coloring c is clear, we write $code_c(v)$ as code(v) or simply, $code(v) = a_0a_1a_2 \ldots a_k$. Therefore, if $a_0 = i$, then $a_i = 0$, for $1 \leq i \leq k$ and $\sum_{i=1}^{k} a_i = deg_G v$. The coloring c is called irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring [3]. For any vertex $u \in V(G)$, u is assigned a k-tuple $(a_1, a_2, \ldots, a_k)$ where a_i is the number of vertices adjacent to u that are colored i, and if distinct vertices have distinct codes then the coloring is called ND-coloring. The minimum cardinality of a NDC-partition is called the ND-coloring number of G and is denoted by $\chi_{NDC}(G)$ [4]. Characterization of trees which admit NDC and charaterization of bipartite graphs for which $\chi_{NDC}(G) = 2$ are done. The χ_{NDC} number of the union of two graphs has been found to lie between the maximum of χ_{NDC}'s of the two graphs and the sum of the χ_{NDC}'s of the two graphs.

2. Main Results

Theorem 2.1. $\chi_{NDC}(G) = 2$ if and only if G is bipartite graph satisfying the following: Let V_1, V_2 be the partite sets of G. Then

(i) $||V_1| - |V_2|| \leq 1$.

(ii) G has one isolated vertex if $||V_1| - |V_2|| = 1$ and no isolated vertex if $|V_1| = |V_2|$.

(iii) Suppose $V_1 = \{u_1, u_2, \ldots, u_k\}$ and $V_2 = \{v_1, v_2, \ldots, v_{k+1}\}$. Then u_i has $k - i + 1$ neighbors in V_2, $1 \leq i \leq k$ and V_2 has an isolated vertex. If $|V_1| = |V_2|$ then u_i has $k - i + 1$ neighbors in V_2, $1 \leq i \leq k$ and v_i has $k - i + 1$ neighbors in V_1, $1 \leq i \leq k$.

Proof. Let G be a bipartite graph satisfying the hypothesis. Then, clearly $\chi_{NDC}(G) = 2$. Conversely, Suppose $\chi_{NDC}(G) = 2$. Then, $\chi(G) \leq \chi_{NDC}(G) = 2$. If $\chi(G) = 1$, then $G = \overline{K}_r$. In this case, G admits NDC if and only if $r = 1$. That is, $G = \overline{K}_1$. Hence $\chi_{NDC}(G) = 1$, a contradiction. Therefore, $\chi(G) = 2$. Then G is bipartite. Let V_1, V_2 be the partite sets of G. Suppose $|V_1| \geq |V_2| + 2$. Let $V_1 = \{u_1, u_2, \ldots, u_k\}$ and $V_2 = \{v_1, v_2, \ldots, v_l\}$ where $k \geq l + 2$. Since $\pi = \{V_1, V_2\}$ is χ_{NDC} - partition of G, the code of u_i is $(0, |N(u_i)|)$. Therefore, $|N(u_1)|, |N(u_2)|, \ldots, |N(u_k)|$ are all distinct. Therefore $|V_2| \geq k - 1$. i.e., $l \geq k - 1$, a contradiction. Since $l \leq k - 2$. Therefore $k \leq l + 1$. i.e., $||V_1| - |V_2|| \leq 1$. Suppose, $||V_1| - |V_2|| = 1$. Let without loss of generality $|V_1| = |V_2| + 1$. Let $|V_1| = k + 1$. Then $|V_2| = k$. Also, u_1 as k neighbors in V_2, u_2 has $k-1$ neighbors in V_2, ... ,u_{k+1} has no neighbor in V_2. i.e., G has one isolated vertex. If $|V_1| = |V_2| = k$, then G has no isolated vertex and u_i has $k-i+1$ neighbors in V_2, v_j has $k-i+1$ neighbors in V_1. Hence G satisfies the hypothesis. ■

Theorem 2.2. Let G be a graph of order n with $\chi_{NDC}(G) = k \geq 2$. Then G is a k-partite graph satisfying the following: Let $V_1, V_2, \cdots, V_k$ be the elements of a χ_{NDC}-partition of G.

(i) $||V_i| - |V_j|| \leq 1$, for all $i, j, 1 \leq i, j \leq k$.

(ii) If $|V_i| - |V_j| = 1$ and if $|V_i| > |V_j|$ then there exists exactly one vertex in V_i which is not adjacent with any vertex in V_j.

(iii) If $|V_i| = |V_j|$, every vertex of V_i is adjacent with some vertex of V_j and vice versa.

(iv) If $V_r = \{u_1, u_2, \cdots, u_k\}$ and $V_s = \{v_1, v_2, \cdots, v_{k+1}\}$, then u_i has $k-i+1$ neighbors in V_s, $1 \leq i \leq k$ and V_s has exactly one vertex which is not adjacent with any vertex of V_r. Also, if $|V_r| = |V_s|$, then u_i has $k-i+1$ neighbors in V_s, $1 \leq i \leq k$ and v_i has $k-i+1$ neighbors in V_r, $1 \leq i \leq k$.

Proof. The Proof follows on the same lines as in theorem 2.1. ■

Remark 2.3. The maximum number of edges that a graph G with n vertices and $\chi_{NDC}(G) = 2$ can have is $\dfrac{\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)}{2}$.

Proof.
Case(i) : n is even. Say $n = 2k$. Then the partite sets of G have equal cardinality, namely k. By condition (iii) the number of edges is $\dfrac{k(k+1)}{2} = \dfrac{\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)}{2}$.
Case(ii) : n is odd. Say $n = 2k+1$. Then the partite sets of G have cardinalities $k, k+1$. Also, G has an isolated vertex. Therefore, number of edges of G is $\dfrac{k(k+1)}{2} = \dfrac{\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)}{2}$. ■

Theorem 2.4. Let T be a tree. If there exist a vertex in T which supports two pendant vertices, then T does not admit NDC.

Proof. Obvious. ■

Theorem 2.5. Let T be a tree. Then T admits NDC if and only if any support vertex supports exactly one pendant vertex.

Proof. Suppose T is a tree in which any support vertex supports exactly one pendant vertex. Let u and v be two non-adjacent vertices of T.

Case (i) u and v are pendant vertices. Clearly $N(u) \neq N(v)$.

Case (ii) One of u or v is a pendant vertex. Then also $N(u) \neq N(v)$.

Case (iii) Both u and v are not pendant vertices. Suppose u or v is a support vertex. Then $N(u) \neq N(v)$.

Case (iv) u and v are neither pendant vertices nor support vertices. If u and v have more than one vertex as a common neighbor, then there exists a cycle, a contradiction. Therefore, u and v have atmost one vertex as a common neighbor. Therefore, $N(u) \neq N(v)$ for any two non adjacent vertices u, v of T. Therefore, T admits NDC. Conversely, suppose T admits NDC. Then, any support vertex supports exactly one pendant vertex. ■

Theorem 2.6. Let G be a uni-cyclic graph with $g(G) \geq 5$. Then, G admits NDC if and only if any support vertex supports exactly one pendant vertex.

Proof. Let G be a uni-cyclic graph with $g(G) \geq 5$ and any support vertex supports exactly one pendant vertex. Let u and v be two non-adjacent vertices.

Case (i) Let $deg(u) \geq 2$ and $deg(v) \geq 2$. If $N(u) = N(v)$, then $g(G) \leq 4$. Therefore, $N(u) \neq N(v)$.

Case (ii) $deg(u) = 1$. If $deg(v) \geq 2$, then $N(u) \neq N(v)$. If $deg(v) = 1$, then u and v are pendant vertices of different supports and hence $N(u) \neq N(v)$. Therefore, G admits NDC. Conversely, let G be a uni-cyclic graph with $g(G) \geq 5$ which admits NDC. Clearly any support vertex can support exactly one pendant vertex. ■

Theorem 2.7. Let G and H be two graphs such that atmost one of G, H contains an isolate vertex. Then $\chi_{NDC}(G \cup H) \leq \chi_{NDC}(G) + \chi_{NDC}(H)$.

Proof. Let G and H be two graphs such that atmost one of them has an isolate (Note that if both G and H have isolates then $G \cup H$ will not admit NDC-partition). Assume that both G and H admit NDC-partition. Let $\pi_1 = \{V_1, V_2, \dots, V_k\}$ be a χ_{NDC} - partition of G and $\pi_2 = \{W_1, W_2, \dots, W_l\}$ be a χ_{NDC} - partition of H. Let $\pi = \pi_1 \cup \pi_2$. If $u, v \in V(G)$, then $C_\pi(u) = (a_1, a_2, \dots, a_k, 0, 0, \dots, 0)$ and $C_\pi(v) = (b_1, b_2, \dots, b_k, 0, 0, \dots, 0)$. Since $C_{\pi_1}(u) \neq C_{\pi_1}(v)$, $C_\pi(u) \neq C_\pi(v)$. If $u, v \in V(H)$, then also $C_\pi(u) \neq C_\pi(v)$. If $u \in V(G)$ and $v \in V(H)$, then $C_\pi(u) = (a_1, a_2, \dots, a_k, 0, 0, \dots, 0)$ and $C_\pi(v) = (0, 0, \dots, 0, b_1, b_2, \dots, b_l)$. $C_\pi(u) = C_\pi(v)$ if and only if $a_i = 0, 1 \leq i \leq k$ and $b_j = 0, 1 \leq j \leq l$. That is u and v are isolates in $G \cup H$, a contradiction. Therefore, $C_\pi(u) \neq C_\pi(v)$. Therefore, π is a χ_{NDC} - partition of $G \cup H$. Therefore, $\chi_{NDC}(G \cup H) \leq \chi_{NDC}(G) + \chi_{NDC}(H)$. ■

Remark 2.8. Let $G = C_5 \cup C_{11}$. Suppose $\chi_{NDC}(G) \leq 5$. Any vertex of G has degree 2. There are 16 vertices. Since there are atmost 5 classes there should be 16 different codes with 5-coordinates each of which contains either 2 in exactly one place or 1 in exactly two placces. There are 5 different codes with 2 in exactly one place. If a code has 1 in the first place then there are four choices for the other one in the code. If 1 occurs in the second place, the first placce is 0 and there are three choices for another 1. If 1 occurs in the third place, the first two places must be 0 and there are two choices for another 1. If 1 occurs in the fourth place, then the first three places are 0's and there is only one choice for another 1. Hence, there are 10 choices for different codes with two 1's. Therefore, there are only 15 possible codes with 2 in exactly one place or 1 in exactly two places. But, Sixteen different codes are required. Therefore, $\chi_{NDC}(G) > 5$. Let $V(C_{11}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and $V(C_5) = \{1', 2', 3', 4', 5'\}$. Let $\pi = \{\{8, 3', 5'\}, \{2, 4, 2'\}, \{5, 7, 10\}, \{3, 11\}, \{6, 9\}, \{1, 1', 4'\}\}$. Then π is a NDC-partition for $G = C_5 \cup C_{11}$. $\chi_{NDC}(C_5 \cup C_{11}) = 6 < \chi_{NDC}(C_5) + \chi_{NDC}(C_{11}) = 8$.

Theorem 2.9. *Max* $\{\chi_{NDC}(G), \chi_{NDC}(H)\} \leq \chi_{NDC}(G \cup H)$.

Proof. Suppose π is a NDC-partition of $G \cup H$ such that $|\pi| < \chi_{NDC}(G), \chi_{NDC}(H)$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$. Let $\pi_1 = \{V_1', V_2', \ldots, V_k'\}$ where $V_i' = V_i \cap V(G)$. Let $u, v \in V(G)$. $C_\pi(u) \neq C_\pi(v)$. Since $|N(u) \cap v_i| = \left|N(u) \cap v_i'\right|$, $C_{\pi_1}(u) \neq C_{\pi_1}(v)$. Therefore, π_1 is a NDC - partition of G. $|\pi_1| \leq k = |\pi| < \chi_{NDC}(G)$, a contradiction. Therefore, $|\pi| \geq \chi_{NDC}(G)$. Similarly, $|\pi| \geq \chi_{NDC}(H)$. Therefore, $Max\{\chi_{NDC}(G), \chi_{NDC}(H)\} \leq |\pi| = \chi_{NDC}(G \cup H)$. ■

Corollary 2.10.

(i) Let $\pi_1 = \{V_1, V_2, \ldots, V_k\}$ and $\pi_2 = \{W_1, W_2, \ldots, W_l\}$ be χ_{NDC} - partitions of two graphs G and H respectively. Let $k < l$. Let for any $v \in V(H)$, atleast one element at a co-ordinate greater than k in the code for v is non zero. Then $\chi_{NDC}(G \cup H) = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$.

Proof. Let $\pi = \{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_k \cup W_k, W_{k+1}, \ldots, W_l\}$. Let $u, v \in V(G \cup H)$. If $u, v \in V(G)$, or $u, v \in V(H)$, then $C_\pi(u) \neq C_\pi(v)$. If $u \in V(G)$ and $v \in V(H)$, then $C_\pi(u) = (a_1, a_2, \ldots, a_k, 0, 0, \ldots, 0)$ and $C_\pi(v) = (b_1, b_2, \ldots, b_k, b_{k+1}, \ldots, b_l)$. By hypothesis $b_{k+t} \neq 0$ for some $t, 1 \leq t \leq l - k$. Therefore, $C_\pi(u) \neq C_\pi(v)$. Therefore, π is a NDC-partition of $G \cup H$. Therefore, $\chi_{NDC}(G \cup H) \leq |\pi| = l = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. But, $\chi_{NDC}(G \cup H) \geq max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. Therefore, $\chi_{NDC}(G \cup H) = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. ■

(ii) Let G and H be two graphs which admits χ_{NDC} - partition and let $G \cup H$ have atmost one isolate. Let $\pi_1 = \{V_1, V_2, \ldots, V_k\}$ and $\pi_2 = \{W_1, W_2, \ldots, W_l\}$ be χ_{NDC}-partitions of two graphs G and H respectively. Let $k < l$. Let $\pi = \{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_k \cup W_k, W_{k+1}, \ldots, W_l\}$. Suppose for any $v \in V(H)$ for which $C_\pi(v)$ has 0's in all the co-ordinates from k+1 to l, $C_\pi(u) \neq C_\pi(v)$ for any $u \in V(G)$. Then, $\chi_{NDC}(G \cup H) = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$.

Proof. If $u_1, u_2 \in V(G)$, then $C_\pi(u_1) \neq C_\pi(u_2)$. So is the case if $u_1, u_2 \in V(H)$. Suppose $u_1 \in V(G)$ and $u_1 \in V(H)$ and $N(u_2) \cap W_{k+t}$ is non empty for some $t, 1 \leq t \leq l - k$. Then $C_\pi(u_1) \neq C_\pi(u_2)$. Suppose $N(u_2) \cap W_{k+t} = \phi$ for every $t, 1 \leq t \leq l - t$. Then by hypothesis, $C_\pi(u_1) \neq C_\pi(u_2)$. Therefore, π is a NDC-partition of G. Therefore,

$\chi_{NDC}(G \cup H) \leq |\pi| = l = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. But, $\chi_{NDC}(G \cup H) \geq max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. Therefore, $\chi_{NDC}(G \cup H) = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. ■

Remark 2.11. There exist Graphs G and H for which $\chi_{NDC}(G \cup H) > max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. See remark 2.

Remark 2.12. In the following, an example is given where $\chi_{NDC}(G \cup H) = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$. Let $G = C_6$ and $H = C_5$. Let $V(G) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $V(H) = \{u'_1, u'_2, u'_3, u'_4, u'_5\}$. Let $\pi = \{\{u_1, u'_1, u'_4\}, \{u_2, u'_5\}, \{u_3, u_6\}, \{u_4, u'_1\}, \{u_5, u'_3\}\}$. The codes of the vertices are, $u'_1 : (1,1,0,0,0)$, $u'_2 : (0,0,0,1,1)$, $u'_3 : (2,0,0,0,0)$, $u'_4 : (0,1,0,0,1)$, $u'_5 : (1,0,0,1,0)$, $u_1 : (0,1,1,0,0)$, $u_2 : (1,0,1,0,0)$, $u_3 : (0,1,0,1,0)$, $u_4 : (0,0,1,0,1)$, $u_5 : (0,0,1,1,0)$ and $u_6 : (1,0,0,0,1)$. Hence π is a NDC-partition of $G \cup H$. Therefore, $\chi_{NDC}(G \cup H) = |\pi| = 5 = max\{\chi_{NDC}(G), \chi_{NDC}(H)\}$.

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