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Numerical Solution of Stiff System by Second Order Backward Difference Formula

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Abstract

In this paper, we suggest a second order Backward Differentiation Formula (BDF-2) for solving stiff systems of Ordinary Differential Equations (ODEs) using piecewise uniform mesh instead of uniform mesh. The proposed method reduces the error and average error and improve the accuracy over BDF-2 with uniform mesh. Numerical examples validate our claim.

Key words: System of stiff differential equations, Initial value problem, BDF-2, Piecewise uniform mesh.

Mathematics Subject classification:[1]

1 Introduction

Numerical solutions for ODEs are very important in scientific computation, as they are widely used to model real world problems. In this paper, we are concerned with the numerical solution of Initial Value Problems (IVPs) for first-order ODEs of the form

$$\begin{cases} u'(t)+a_{11}(t)u(t)+a_{12}(t)v(t) = f_1(t), \\ v'(t)+a_{21}(t)u(t)+a_{22}(t)v(t) = f_2(t) \end{cases} \tag{1}$$

with given initial values

$$u(0) = A\ ,\ v(0) = B \tag{2}$$

The function $a_{ij}(t)$ *for* $i, j = 1,2$ of (1) satisfy the following inequality

1 65L04, 65L05, 65L15, 65L50, 34A12, 34A09, 34A30

$For\ all\ |a_{ij}(t)| >> 1, for\ i, j = 1,2. \quad where\ t \in (0,1]$

The linear system (1) - (2) is said to be stiff if

(i) $Re(\lambda_i) < 0, \quad i = 1,...,n$ and

(ii) $max_i\ |Re(\lambda_i)| >> min_i\ |Re(\lambda_i)|$ where λ_i are the eigenvalues of stiff ODEs, n is the number of equations in the system and the ratio

$$SR = \frac{max_i\ |real\ \ part\ of\ \lambda_i\ |}{min_i\ |real\ \ part\ of\ \ \lambda_i\ |}, i = 1,2,....,n \qquad (3)$$

is called the stiffness ratio or stiffness index.

Problems involving rapidly decaying transient solutions occur naturally in a wide variety of applications, including the study of spring and damping systems, the analysis of control systems, and problems in chemical kinetics, fluid dynamics, quantum mechanics, electrical networks, etc. These are all examples of a class of problems called stiff (mathematical stiffness) systems of differential equations.

Stiff systems are considered to be difficult because explicit numerical methods designed for non-stiff problems need to be used with very small step sizes. In the quest for better methods for solving these systems, Curtiss and Hirschfelder [5] discovered the BDF. Since then, a great effort has been made in order to obtain new numerical integration methods with strong stability properties desirable for solving stiff systems.

The first use of BDF methods appears to date back to Curtiss and Hirschfelder (1952), although they were not given that name at the time. Later, Henrici [9], in Section 5.1-4 of his book, discussed "methods based on differentiation". The methods were dismissed by Henrici because they are "less accurate than the corresponding Adams-Moulton formula". For non-stiff equations this is a valid point. For stiff systems, the value of BDF lies in their superior stability properties which allow them to take much larger stepsizes than would be possible with explicit methods.

We are concerned with the BDF-2 which is a linear multistep method. The focus of this paper is to extend the BDF-2 method with piecewise uniform mesh.

For a detailed discussion on stiff nature, application, implicit methods and BDF-2 with uniform mesh of stiff system of ODEs, one may refer to [1, 2, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 20, 21, 22] and the thesis [3], to name a few.

The idea of Shishkin mesh have been done by G. I. Shishkin. A Shishkin mesh is a piecewise uniform mesh. What distinguishes a Shishkin mesh from any other piecewise uniform mesh is the choice of the so-called transition parameter(s), which are the point(s) at which the mesh size changes abruptly. This type of piecewise uniform mesh allows fast updating of the iteration matrix after a stepsize or order change. Other studies on piecewise uniform mesh are discussed by several researchers such as C. Clavero, J. L. Gracia and F. Lisbona [4], Kailash C. Patidar [12], J. J. H. Miller, E. O' Riordan and G. I. Shishkin [15], Natalia Kopteva and Eugene O' Riordan [18]. As in [23] and [24], the focus of this paper is to improve the performance of the BDF-2 method by applying it in a piecewise uniform mesh (Shishkin mesh).

The rest of the paper is organized as follows: In section 2, we briefly summarize the BDF-2 scheme. In section 3, we present the description of piecewise uniform mesh. In section 4, we briefly discuss the local truncation error of the method. In section 5, we show the accuracy of our method. Finally, in section 6 we present some concluding remarks.

2 BDF-2 piecewise uniform mesh

Approximating the equations (1) and (2) by applying the BDF-2 method we have

$$\begin{cases} u_{j+1} = \frac{4}{3}u_j - \frac{1}{3}u_{j-1} + \frac{2h}{3}[f_1(t_{j+1}, u_{j+1})] \\ v_{j+1} = \frac{4}{3}v_j - \frac{1}{3}v_{j-1} + \frac{2h}{3}[f_2(t_{j+1}, v_{j+1})] \end{cases} \quad (4)$$

where $j = 1\text{to}(N-1)$ and N is the number of mesh point.

From (4), u_{j+1} and v_{j+1} are determined implicitly. The new solution approximation needs to be computed iteratively, typically by an explict Euler method

$$\begin{cases} u_{j+1} = u_j + hf_1(t_j, u_j) \\ v_{j+1} = v_j + hf_2(t_j, v_j) \qquad \textit{where} \;\; j = 0\text{to}(N-1) \end{cases} \quad (5)$$

In the next section, the description of piecewise uniform mesh is presented.

3 Description of piecewise uniform mesh

A piecewise uniform mesh is constructed on the interval $[0,1]$ as follows:

Choose a point σ satisfying $0 < \sigma \le \frac{1}{4}$ and assume that $N = 2^m$ *with* $m \ge 12$. The point σ is called a transition point and it divides the interval $[0,1]$ into the two subintervals $[0,\sigma]$ and $[\sigma,1]$.

In stiff problems the solution may have an brief initial transient as the fast modes settle, followed by a longer period where it is the behaviour of the slow modes that dominate. In this latter phase, the fast modes are still present in the system even if they are not visible in the solution. Since the solution has fast varying component in the neighbourhood of $t = 0$, it is natural to have more number of mesh points in the neighbourhood of $t = 0$. This will give a better information about the solution near $t = 0$.

Therefore the piecewise uniform mesh is constructed by dividing $[0,\sigma]$ into $\frac{N}{4}$ equal mesh elements and $[\sigma,1]$ into $\frac{3N}{4}$ equal mesh elements.

The piecewise uniform mesh is used with the following location of the transition point

$$\sigma = \min\{\frac{1}{4}, \varepsilon \, lnN\}. \quad (6)$$

Assume that the parameter ε as

$$\varepsilon < \frac{1}{M}, \quad where\ M = min\{a_{11}(t), a_{12}(t), a_{21}(t), a_{22}(t)\}, \quad for\ all \quad t \in (0,1], \qquad (7)$$

and

$$\begin{cases} t_j = jh_1 \quad where \quad h_1 = \dfrac{4\sigma}{N}, \quad j = 0(1)\dfrac{N}{4}, \\ t_j = \sigma + (j - \dfrac{N}{4})h_2 \quad where \quad h_2 = \dfrac{4(1-\sigma)}{3N}, \quad j = (\dfrac{N}{4}+1)(1)N. \end{cases} \qquad (8)$$

If $\sigma = \frac{1}{4}$ then $h_1 = N^{-1}$ and $h_2 = N^{-1}$.

In such a case the method can be analysed using the standard techniques. We therefore assume that

$$\sigma = \varepsilon lnN \qquad (9)$$

The above scheme will give less error and less average error of the solution if the stiff ratio lies between 400 to 1000.

4 The local truncation error

In general, $u(t_{j+1})$ is the exact value and u_{j+1} is the approximate numerical value and the local truncation error at the point $t_{(j+1)}$ in the BDF-2 with uniform mesh is

$$T_{j+1} = u(t_{j+1}) - u_{j+1} \qquad where j = 0,1 \ldots N-1$$

$$= u(t_{j+1}) - \frac{4}{3}u(t_j) + \frac{1}{3}u(t_{j-1}) - \frac{2h}{3} f(t_{j+1}, u(t_{j+1}))$$

$$= u(t_{j+1}) - \frac{4}{3}[u(t_{j+1}) - hu'(t_{j+1}) + \frac{h^2}{2}u''(t_{j+1}) - \frac{h^3}{3}u'''(t_{j+1})]$$

$$+ \frac{1}{3}[u(t_{j+1}) - 2hu'(t_{j+1}) + 2h^2u''(t_{j+1}) - \frac{8h^3}{3}u'''(t_{j+1})] - \frac{2h}{3}u'(t_{j+1})$$

$$\Rightarrow T_{j+1} = Ch^3u'''(t_{j+1})$$

Applying the BDF-2 with piecewise uniform mesh,

The truncation error for $0 \le j \le \frac{N}{4} - 1$ is

$$T_{j+1} = u(t_{j+1}) - u_{j+1}$$

$$= u(t_{j+1}) - \frac{4}{3}u(t_j) + \frac{1}{3}u(t_{j-1}) - \frac{2h_1}{3} f(t_{j+1}, u(t_{j+1}))$$

$$= u(t_{j+1}) - \frac{4}{3}[u(t_{j+1}) - h_1u'(t_{j+1}) + \frac{h_1^2}{2}u''(t_{j+1}) - \frac{h_1^3}{3}u'''(t_{j+1})]$$

$$+ \frac{1}{3}[u(t_{j+1}) - 2h_1u'(t_{j+1}) + 2h_1^2u''(t_{j+1}) - \frac{8h_1^3}{3}u'''(t_{j+1})] - \frac{2h_1}{3}u'(t_{j+1})$$

$$\Rightarrow T_{j+1} = Ch_1^3u'''(t_{j+1})$$

The truncation error for $\frac{N}{4} \le j \le N-1$ is

$$T_{j+1} = u(t_{j+1}) - u_{j+1}$$

$$= u(t_{j+1}) - \frac{4}{3}u(t_j) + \frac{1}{3}u(t_{j-1}) - \frac{2h_2}{3} f(t_{j+1}, u(t_{j+1}))$$

$$= u(t_{j+1}) - \frac{4}{3}[u(t_{j+1}) - h_2 u'(t_{j+1}) + \frac{h_2^2}{2} u''(t_{j+1}) - \frac{h_2^3}{3} u'''(t_{j+1})]$$

$$+ \frac{1}{3}[u(t_{j+1}) - 2h_2 u'(t_{j+1}) + 2h_2^2 u''(t_{j+1}) - \frac{8h_2^3}{3} u'''(t_{j+1})] - \frac{2h_2}{3} u'(t_{j+1})$$

$$\Rightarrow T_{j+1} = Ch_2^3 u'''(t_{j+1})$$

Therefore, the truncation error for BDF-2 with piecewise uniform mesh is

$$T_{j+1} = \begin{cases} C\,h_1^3\,u'''(t_{j+1}) & \text{for} \quad 0 \le j \le \frac{N}{4} - 1 \\ C\,h_2^3\,u'''(t_{j+1}) & \text{for} \quad \frac{N}{4} \le j \le N-1 \end{cases} \quad (10)$$

Similarly, the truncation error for the second component v can be easily derived.

$we\ define\ \|Y\|_s = sup\{|u^{(s)}(t)|, |v^{(s)}(t)|\} \quad for\ all \quad t \in (0,1].$

Let

$h^3 = max(h_1^3, h_2^3)$

Then,

$T_{j+1}(h) \le C\,h^3 \|Y\|_3$

$where\ \|Y\|_3 = sup\{|u'''|, |v'''|\} \quad for\ all \quad t \in (0,1].$

Hence, by the definition given as in [11], the order of convergence of BDF-2 with piecewise uniform mesh is two.

5 Numerical example

In this section, we present some numerical results to illustrate the performance of our method. The numerical results of BDF-2 with piecewise uniform mesh will be compared with uniform mesh.

The comparison is based in terms of maximum error and average error. The numerical results are recorded interms of the following quantities and tabulated

As the formula given in [19] for uniform mesh we have,

$h = \frac{(b-a)}{N}$, where b is the end value of t and a is the initial value of t.

The calculation of error (for piecewise uniform mesh and uniform mesh) is given as,

$error_j = |u(t_j)_{(exact\ solution)} - u_{j(approximat)}|.$

For maximum error (MAXE) (for piecewise uniform mesh and uniform mesh), we use the formula,

$$MAXE^N = \max(error_j)$$

The average error for BDF-2 with uniform mesh is defined as,

$$AVE = \frac{\sum_{j=1}^{N} h\ error_j}{(b-a)},$$

where b is the end value of t and a is the initial value of t.

The average error(AVE) for BDF-2 with piecewise uniform mesh is defined as,

$$AVE1 = \frac{\sum_{j=1}^{\frac{N}{4}} (error_j)}{\frac{\sigma}{\frac{h_1}{4}}}$$

$$AVE2 = \frac{\sum_{j=\frac{N}{4}+1}^{N} (error_j)}{\frac{(1-\sigma)}{\frac{3h_2}{4}}}$$

$$AVE = max\{AVE1, AVE2\}$$

Example 5.1

$$u'(t) = 998u(t) + 1998v(t)$$
$$v'(t) = -999u(t) - 1999v(t) \quad \forall t \in [0,1],$$
$$u(0) = 1, v(0) = 1.$$

The exact solution is given by the sum of two decaying exponentials components

$$u(t) = 4e^{-t} - 3e^{-1000t}$$
$$v(t) = -2e^{-t} + 3e^{-1000t}.$$

The stiffness ratio is $1:1000$.

After a short time the solution can be closely approximated by the dominant terms as

$$u(t) = 4e^{-t}$$
$$v(t) = -2e^{-t}$$

since the fast decaying component vanished. Therefore, instead of taking uniform step size through out the interval, we consider the piecewise uniform mesh. In piecewise uniform mesh, the first $\frac{N}{4}$ interval having very small step size h_1 (compared to h_2), the fast decaying component may vanish. The remaining $\frac{3N}{4}$ interval with step size

h_2, may dominate the slow active components. This technique surely reduced the error and average error of the solution which is shown in the Table 1.
The numerical results obtained by applying the piecewise uniform mesh method (6), (7) and (8) to the $example-5.1$ are given in Table 1 and 2.

Table 1: *Value of* $MAXE(u), AVE(u)$ *for the solution component* u *for the Example 5.1*

N	MESH	MAXE(u)	AVE(u)
4096	*piecewise uniform mesh*	0.13418e-004	0.32758e-008
	uniform mesh	0.52804e-001	0.12892e-004
8192	*piecewise uniform mesh*	0.65503e-004	0.79959e-008
	uniform mesh	0.14639e-001	0.1787e-005
16,384	*piecewise uniform mesh*	0.45725e-004	0.27908e-008
	uniform mesh	0.38700e-002	0.23621e-006
32,768	*piecewise uniform mesh*	0.25907e-004	0.79062e-009
	uniform mesh	0.99612e-003	0.30399e-007
65,536	*piecewise uniform mesh*	0.13662e-004	0.20846e-009
	uniform mesh	0.25277e-03	0.38569e-008
1,31,072	*piecewise uniform mesh*	0.69862e-005	0.5330e-010
	uniform mesh	0.6367e-004	0.48576e-009

Table 2: *Values of* $MAXE(v), AVE(v)$ *for the solution component* v *for the Example 5.1*

N	MESH	MAXE(v)	AVE(v)
4096	*piecewiseuniform mesh*	0.6545e-005	0.1598e-008
	uniform mesh	0.2640e-001	0.6446e-005
8192	*piecewiseuniform mesh*	0.3278e-004	0.4001e-008
	uniform mesh	0.7319e-002	0.8935e-006
16,384	*piecewiseuniform mesh*	0.2287e-004	0.1396e-008
	uniform mesh	0.1935e-002	0.1181e-006
	piecewiseuniform mesh	0.1296e-004	0.3954e-009

32,768	*uniform mesh*	0.4981e-003	0.5120e-007
	piecewise uniform mesh	0.6831e-005	0.1042e-009
65,536	*uniform mesh*	0.1264e-003	0.1929e-008
1,31,072	*piecewise uniform mesh*	0.3493e-005	0.2665e-010
	uniform mesh	0.3184e-004	0.2429e-009

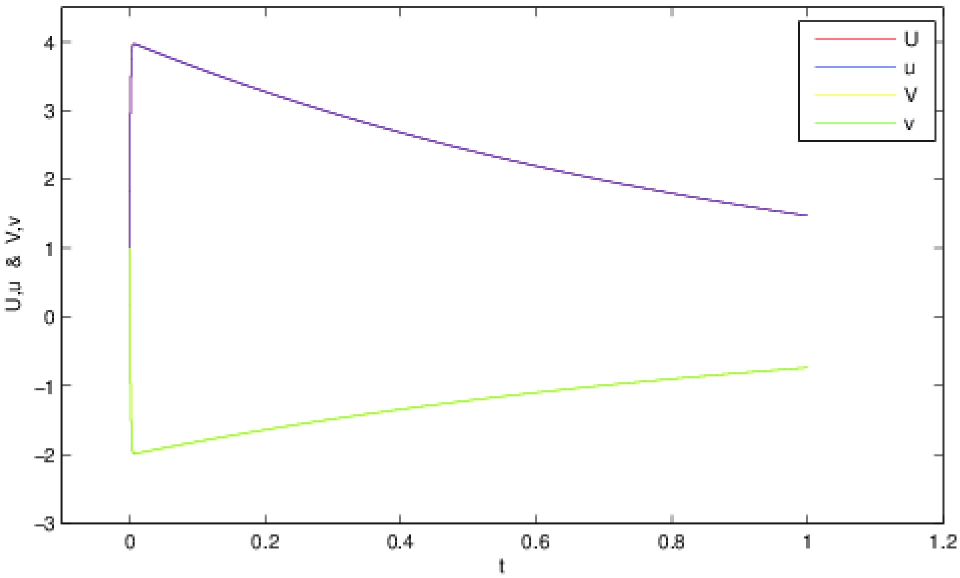

Figure 1:

For the example-5.1 with $N = 1{,}31{,}072, \varepsilon = \frac{1}{998}$ the solution obtained by the suggested numerical method is displayed in Figure-1. Here U, V and u, v represents the numerical and exact solution respectively.

Example 5.2

$u'(t) = 1195u(t) - 1995v(t)$

$v'(t) = 1197u(t) - 1997v(t) \quad \forall t \in [0,1],$

$u(0) = 1, v(0) = 1.$

Exact solution of the above problem is

$u(t) \;=\; 10e^{-2t} - 8e^{-800t}$

$v(t) = -6e^{-2t} + 8e^{-800t}$.

The stiffness ratio is $1:400$.

The numerical results obtained by applying the piecewise uniform mesh method (6), (7) and (8) to the $example-5.2$ are given in Table 3 and 4.

Table 3: ***Value of*** $MAXE(u), AVE(u)$ ***for the solution component*** u ***for the Example 5.2***

N	MESH	MAXE(u)	AVE(u)
4096	*piecewiseuniform mesh*	0.71173e-003	0.17376e-006
	uniform mesh	0.70797e-001	0.17284e-004
8192	*piecewiseuniform mesh*	0.96129e-004	0.11734e-007
	uniform mesh	0.17862e-001	0.21804e-005
16,384	*piecewiseuniform mesh*	0.60188e-004	0.36736e-008
	uniform mesh	0.46062e-002	0.28114e-006
32,768	*piecewiseuniform mesh*	0.39257e-004	0.1198e-008
	uniform mesh	0.11751e-002	0.3586e-007
65,536	*piecewiseuniform mesh*	0.22714e-004	0.34659e-009
	uniform mesh	0.29707e-003	0.45329e-008
1,31,072	*piecewiseuniform mesh*	0.12231e-004	0.93315e-010
	uniform mesh	0.74703e-004	0.56994e-009

Table 4: ***Value of*** $MAXE(v), AVE(v)$ ***for the solution component*** u ***for the Example 5.2***

N	MESH	MAXE(v)	AVE(v)
4096	*piecewiseuniform mesh*	0.4279e-003	0.1045e-006
	uniform mesh	0.4248e-001	0.1037e-004
8192	*piecewiseuniform mesh*	0.5781e-004	0.7057e-008
	uniform mesh	0.1072e-001	0.1308e-005
16,384	*piecewiseuniform mesh*	0.3614e-004	0.2206e-008
	uniform mesh	0.2764e-002	0.1687e-006
32,768	*piecewiseuniform mesh*	0.2356e-004	0.7190e-009
	uniform mesh	0.7051e-003	0.2152e-007
65,536	*piecewiseuniform mesh*	0.1363e-004	0.2080e-009
	uniform mesh	0.1782e-003	0.2720e-008
1,31,072	*piecewiseuniform mesh*	0.7339e-005	0.5599e-010
	uniform mesh	0.4482e-004	0.3420e-009

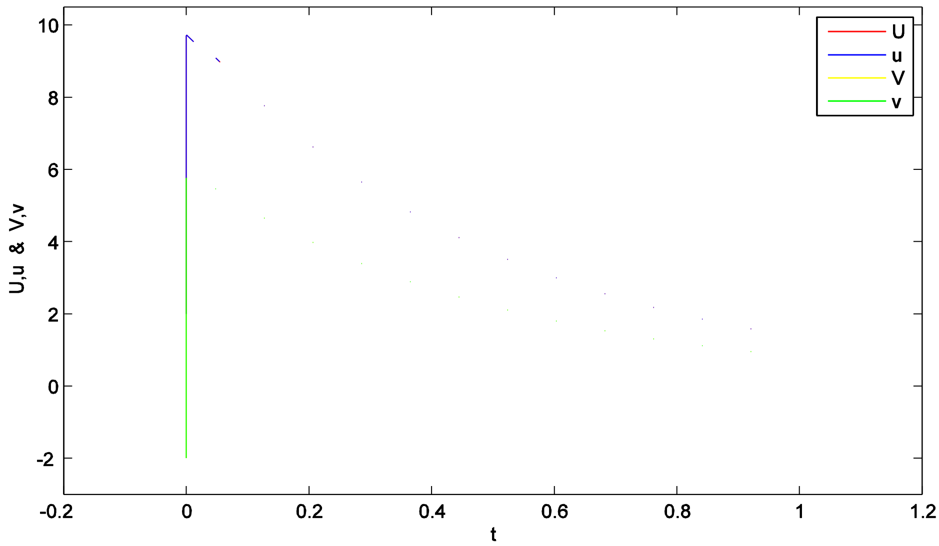

Figure 2:

For the example-5.2 with $N = 1{,}31{,}072, \varepsilon = \frac{1}{1195}$ the solution obtained by the suggested numerical method is displayed in Figure-2. Here U, V and u, v represents the numerical and exact solution respectively.

6 Conclusion

We have suggested a BDF-2 for solving stiff system of ODEs on piecewise uniform mesh. The numerical results shows that the BDF-2 with piecewise uniform mesh is more efficient than BDF-2 method with uniform mesh.

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