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Linear Maps on *le*-Modules: Kernel Theory, Isomorphism Theorems, and Spectral Functoriality

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Abstract

The theory of *le*-modules abstracts the lattice of submodules of a classical module over a commutative ring, providing an ordered-algebraic framework that unifies lattice theory and module theory. In this paper, we formalize the category of *le*-modules by systematically developing the theory of *le*-module homomorphisms. We establish the framework of kernels and images as specific elements within the module's lattice structure, leading to proofs of the First Isomorphism Theorem and the Correspondence Theorem in this context. We investigate homological properties, demonstrating that classical pointwise diagram chasing fails due to the absence of additive inverses, and provide a lattice-theoretic resolution utilizing Galois adjunctions. Furthermore, we construct an explicit counterexample proving that Baer's Criterion for injectivity fails in general for *le*-modules. Finally, we establish that homomorphisms intrinsically preserve prime elements via a pullback operation, proving that the prime spectrum functor $\mathrm{Spec}(-)$ induces continuous maps with respect to the Zariski topology.

1. INTRODUCTION

The general ideal theory of commutative rings, originating from Dedekind's multiplicative theory of algebraic numbers, was abstracted by Ward and Dilworth [7] into the theory of multiplicative lattices. Extending this abstraction, Johnson and Johnson [8] introduced lattice modules. Recently, the concept of an *le*-module—a complete lattice-ordered commutative monoid equipped with a scalar action from a commutative ring—has emerged as a robust framework for studying abstract submodule theories. Pioneering work by Bhuniya and Kumbhakar [4, 5] explored prime elements, primary decompositions, and Zariski topologies on the spectrum $\mathrm{Spec}(M)$. Recent literature has extensively investigated the algebraic and topological properties of le-modules. For example, Puranik, Ballal, and Kharat [6] introduced associated comaximal graphs on le-modules, and further studies have clarified the structural intricacies of the Zariski topology on the prime submodule spectrum [1, 2], as well as the Prime Element Principle using Oka families [3].

While the internal algebraic and topological structures of *le*-modules have been extensively studied, the functorial relationships between distinct *le*-modules have received considerably less attention. In classical algebra, the structure of modules is elucidated through the homomorphisms that connect them, enabling the classification of modules via exact sequences and isomorphism theorems. Developing a corresponding theory of homomorphisms for *le*-modules is therefore essential to formalize the category $\mathbf{LeMod}_R$.

In this paper, we define and analyze *le*-module homomorphisms. Because *le*-modules abstract submodules into elements of a lattice, the classical set-theoretic kernel translates into a specific *kernel element* within the lattice $\mathrm{Sub}(M)$. We establish the First Isomorphism Theorem for regular homomorphisms (Section 4). In Section 5, we investigate exact sequences and demonstrate that classical pointwise diagram chasing fails due to the idempotency of submodule elements and the subsequent lack of subtraction. We resolve this by constructing connecting morphisms globally on the lattice via Galois adjunctions. In Section 6, we formally define projective and injective *le*-modules, providing a definitive counterexample to Baer's Criterion in this category. Finally, in Section 7, we study the preservation of prime elements, proving that the spectral mapping $f^* : \mathrm{Spec}(N) \to \mathrm{Spec}(M)$ is continuous under the Zariski topology.

2. PRELIMINARIES

Throughout this paper, R denotes a commutative ring with unity 1_R.

Definition 2.1. An *le*-semigroup $(M, +, \leq, e_M)$ is a commutative monoid with zero element 0_M and a complete lattice with greatest element e_M, satisfying the distributive law $m + (\bigvee_{i\in I} m_i) = \bigvee_{i\in I}(m + m_i)$ for all $m, m_i \in M$ and arbitrary index sets I.

Definition 2.2. An *le*-module M over R is an *le*-semigroup equipped with a scalar multiplication $R \times M \to M$ satisfying the following axioms for all $r, r_1, r_2 \in R$ and $m, m_1, m_2, m_i \in M$:

1. $r(m_1 + m_2) = rm_1 + rm_2$
2. $(r_1 + r_2)m \leq r_1m + r_2m$

3. $(r_1r_2)m = r_1(r_2m)$
4. $1_Rm = m$ and $0_Rm = r0_M = 0_M$
5. $r(\bigvee_{i\in I} m_i) = \bigvee_{i\in I}(rm_i)$

Definition 2.3. An element $n \in M$ is a *submodule element* if $n + n \leq n$ and $rn \leq n$ for all $r \in R$. We denote the set of all submodule elements by $\mathrm{Sub}(M)$. Note that since $0_M \leq n$, the condition $n + n \leq n$ forces the idempotency relation $n + n = n$.

Definition 2.4. A proper submodule element $p \in \mathrm{Sub}(M)$ (where $p < e_M$) is termed *prime* if for any $r \in R$ and $m \in M$, $rm \leq p$ implies either $m \leq p$ or $re_M \leq p$. The set of all prime submodule elements is denoted $\mathrm{Spec}(M)$.

3. EXAMPLES OF *LE*-MODULES

To ground the subsequent abstract definitions of homomorphisms and injectivity, we provide several foundational examples of *le*-modules.

Example 3.1 (The Lattice of Ideals)**.** Let R be a commutative ring and $I(R)$ be the set of all ideals of R. Under set inclusion $\subseteq$, $I(R)$ forms a complete lattice. Defining addition as the standard ideal sum $I + J$ and scalar multiplication as $rI = \{ra \mid a \in I\}$, the structure $(I(R), +, \cdot, \subseteq, R)$ forms an *le*-module over R. Here, the zero element 0_M is the zero ideal $\{0\}$, and the greatest element e_M is R.

Example 3.2 (Extended Non-Negative Reals)**.** Let $M = \mathbb{R}^{\geq 0} \cup \{\infty\}$. Under the standard numerical order, M is a totally ordered complete lattice with $0_M = 0$ and $e_M = \infty$. Define addition as standard real addition, with $m + \infty = \infty$. Scalar multiplication over the ring $R = \mathbb{R}$ is defined via the absolute value: $r \cdot m = |r|m$ for $m \neq \infty$. For $m = \infty$, we define $r \cdot \infty = \infty$ if $r \neq 0$, and $0 \cdot \infty = 0$. The structure $(M, +, \cdot, \leq, \infty)$ satisfies all axioms and forms an *le*-module over $\mathbb{R}$.

4. HOMOMORPHISMS OF *LE*-MODULES

We introduce the concept of an *le*-module homomorphism. Unlike classical modules, *le*-modules possess a partial order and admit arbitrary joins, which the mappings must preserve.

Definition 4.1. Let M and N be *le*-modules over R. A map $f : M \to N$ is an ***le*-module homomorphism** if it satisfies the following conditions for all $x, y, x_i \in M$ and $r \in R$:

1. $f(x + y) = f(x) + f(y)$

2. $f(rx) = rf(x)$
3. $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$
4. $f(e_M) = e_N$

Remark 4.1 (Preservation of the Zero Element)**.** *In any le-module, $0_R x = 0_M$. Because an le-module homomorphism is R-linear, it follows that $f(0_M) = f(0_R x) = 0_R f(x) = 0_N$. Thus, zero is strictly preserved.*

Noticeably absent from the definition is the explicit preservation of the meet operation ($\wedge$). As in classical lattice module theory, this property is not required for a homomorphism, but a foundational inequality naturally holds.

Proposition 4.1. *Let $f : M \to N$ be an le-module homomorphism. Then f is order-preserving, and for any $x, y \in M$, $f(x \wedge y) \leq f(x) \wedge f(y)$.*

Proof. If $a \leq b$, then $a \vee b = b$. Preserving joins gives $f(a) \vee f(b) = f(b)$, implying $f(a) \leq f(b)$. For the meet, since $x \wedge y \leq x$ and $x \wedge y \leq y$, order preservation ensures $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq f(y)$. Thus, $f(x \wedge y)$ is a lower bound, leading immediately to $f(x \wedge y) \leq f(x) \wedge f(y)$. □

5. KERNELS, QUOTIENTS, AND ISOMORPHISM THEOREMS

In classical module theory, the kernel is the set of elements mapped to zero. In the context of *le*-modules, submodules are abstracted as elements of the lattice $\mathrm{Sub}(M)$. Therefore, the kernel is defined as a specific lattice element.

Definition 5.1. Let $f : M \to N$ be an *le*-module homomorphism. The **kernel** of f, denoted $\mathrm{Ker}(f)$, is the element defined by:

$$\mathrm{Ker}(f) = \bigvee \{x \in M \mid f(x) = 0_N\}$$

Theorem 5.2. *For any homomorphism $f : M \to N$, the element $\mathrm{Ker}(f)$ is a submodule element of M, and $f(\mathrm{Ker}(f)) = 0_N$.*

Proof. Let $S = \{x \in M \mid f(x) = 0_N\}$. Since f preserves arbitrary joins, $f(\mathrm{Ker}(f)) = f(\bigvee_{x \in S} x) = \bigvee_{x \in S} f(x) = 0_N$. To verify $\mathrm{Ker}(f) \in \mathrm{Sub}(M)$, observe that $f(\mathrm{Ker}(f) + \mathrm{Ker}(f)) = f(\mathrm{Ker}(f)) + f(\mathrm{Ker}(f)) = 0_N$, implying $\mathrm{Ker}(f) + \mathrm{Ker}(f) \leq \mathrm{Ker}(f)$. Similarly, $f(r\,\mathrm{Ker}(f)) = r0_N = 0_N$, yielding $r\,\mathrm{Ker}(f) \leq \mathrm{Ker}(f)$ for all $r \in R$. □

Definition 5.3. Let M be an *le*-module and $k \in \mathrm{Sub}(M)$. The **quotient *le*-module** M/k is the set $M/k = \{x \vee k \mid x \in M\}$, equipped with the operations $(x \vee k) \oplus (y \vee k) = (x+y) \vee k$, $r \odot (x \vee k) = rx \vee k$, and $\bigsqcup_i (x_i \vee k) = (\bigvee_i x_i) \vee k$.

It is straightforward to verify that $(M/k, \oplus, \leq, e_M)$ forms an *le*-module over R, with least element k and greatest element e_M. In order-theoretic algebra, Isomorphism Theorems require an additional regularity condition to ensure equivalence classes map bijectively.

Definition 5.4. A homomorphism $f : M \to N$ is **regular** if for all $x, y \in M$, $f(x) = f(y) \iff x \vee \mathrm{Ker}(f) = y \vee \mathrm{Ker}(f)$.

Theorem 5.5 (First Isomorphism Theorem for *le*-Modules)**.** *Let $f : M \to N$ be a surjective regular le-module homomorphism. Then $M/\mathrm{Ker}(f) \cong N$ via the map $\bar{f}(x \vee \mathrm{Ker}(f)) = f(x)$.*

Proof. Define $\bar{f} : M/\mathrm{Ker}(f) \to N$ by $\bar{f}(x \vee \mathrm{Ker}(f)) = f(x)$. The map is well-defined and injective directly from the regularity assumption. Surjectivity is guaranteed by the hypothesis that f is a surjective map. Furthermore, $\bar{f}((x \vee \mathrm{Ker}(f)) \oplus (y \vee \mathrm{Ker}(f))) = \bar{f}((x+y) \vee \mathrm{Ker}(f)) = f(x+y) = f(x) + f(y) = \bar{f}(x \vee \mathrm{Ker}(f)) + \bar{f}(y \vee \mathrm{Ker}(f))$. Scalar multiplication and arbitrary joins are similarly preserved. Hence, $\bar{f}$ is an *le*-module isomorphism. □

Theorem 5.6 (Correspondence Theorem)**.** *Let $f : M \to N$ be a surjective regular le-module homomorphism. There is an order-preserving bijection between $\{n \in \mathrm{Sub}(M) \mid n \geq \mathrm{Ker}(f)\}$ and $\mathrm{Sub}(N)$, given by $n \mapsto f(n)$ and $p \mapsto f^*(p) = \bigvee\{x \mid f(x) \leq p\}$.*

6. EXACT SEQUENCES AND HOMOLOGICAL OBSTRUCTIONS

The extension of homological algebra to lattice-ordered algebraic structures presents significant challenges. Classical homological machinery relies on abelian categories, where the existence of additive inverses guarantees that kernels and images are seamlessly mapped via pointwise diagram chases.

Definition 6.1. A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of *le*-module homomorphisms is **exact at** M if $f(e_L) = \mathrm{Ker}(g)$ as elements in the lattice $\mathrm{Sub}(M)$.

6.1. The Failure of Pointwise Diagram Chasing

Consider the standard commutative diagram used to construct the connecting homomorphism δ in the Snake Lemma:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

In classical module theory, the construction of δ relies essentially on the existence of additive inverses. If one selects $z \in \mathrm{Ker}(c)$ and lifts it to $y \in B$ such that $g(y) = z$, and another lifting y_1 is chosen, the equivalence is shown by observing $g(y - y_1) = 0$, allowing one to pull back the difference to the kernel. In the category $\mathbf{LeMod}_R$, elements belong to an idempotent monoid, and subtraction is undefined. Thus, classical element-wise diagram chasing fails.

6.2. Lattice-Theoretic Resolution via Adjunctions

Because every homomorphism $h : M \to N$ preserves arbitrary joins, the Adjoint Functor Theorem guarantees the existence of a right adjoint (pullback) $h^* : \mathrm{Sub}(N) \to \mathrm{Sub}(M)$ defined by $h^*(p) = \bigvee\{x \in M \mid h(x) \leq p\}$. We can replace the pointwise diagram chase with operations strictly on the lattice.

Theorem 6.2 (Lattice Connecting Morphism)**.** *Given the commutative diagram above with exact rows, there exists a rigorously well-defined connecting morphism* $\delta : \mathrm{Sub}(C) \to \mathrm{Sub}(A'/\,\mathrm{Im}(a))$ *defined purely via lattice adjunctions:*

$$\delta(k) = (f')^*(b(g^*(k))) \vee \mathrm{Im}(a)$$

Proof. Let $k = \mathrm{Ker}(c)$. We evaluate the expression via lattice adjunctions: $g^*(k) \in \mathrm{Sub}(B)$ is the maximal element mapping into k. Applying b yields $b(g^*(k)) \in \mathrm{Sub}(B')$. Applying g' and utilizing commutativity gives $g'(b(g^*(k))) = c(g(g^*(k))) \leq c(k) = 0_{C'}$. By the definition of the upper adjoint, this implies $b(g^*(k)) \leq \mathrm{Ker}(g')$. Since the bottom row is exact, $\mathrm{Ker}(g') = \mathrm{Im}(f') = f'(e_{A'})$. Pulling back to A' via $(f')^*$ and projecting into the cokernel gives the well-defined element $(f')^*(b(g^*(k))) \vee \mathrm{Im}(a)$. Because b, g^*, and $(f')^*$ are globally well-defined on the complete lattices, δ is completely independent of element-wise choices, circumventing the absence of subtraction. $\square$

7. PROJECTIVITY, INJECTIVITY, AND BAER'S CRITERION

In classical module theory, projectivity and injectivity provide the dual pillars of homological algebra.

Definition 7.1. An *le*-module E is **injective** if for every injective *le*-module homomorphism $i : A \hookrightarrow B$ and every homomorphism $g : A \to E$, there exists a homomorphism $h : B \to E$ such that $h \circ i = g$.

Classically, Baer's Criterion states that a module E is injective if and only if every homomorphism from an ideal of R to E extends to R. In the *le*-module framework, the ring R corresponds to the lattice of ideals $I(R)$.

Definition 7.2 (Baer's Criterion for *le*-modules)**.** An *le*-module E satisfies Baer's Criterion if, for every submodule element $K \in \mathrm{Sub}(I(R))$ and every homomorphism $g :\downarrow K \to E$, there exists an extension $h : I(R) \to E$ such that $h|_{\downarrow K} = g$.

Example 7.3. Baer's criterion does not hold in general for $\mathbf{LeMod}_R$.

Proof. We construct a specific *le*-module that satisfies Baer's Criterion vacuously but fails to be injective. Let $R = \mathbb{Z}$ and consider the extended non-negative reals $E = \mathbb{N} \cup \{\infty\}$ from Example 3.2, equipped with standard addition, maximum for join, and $r \cdot x = |r|x$. First, we show E satisfies Baer's Criterion. Let $K \in \mathrm{Sub}(I(\mathbb{Z}))$ and $g :\downarrow K \to E$ be a homomorphism. In $I(\mathbb{Z})$, ideal sum is idempotent ($J + J = J$). Because g is a homomorphism, $g(J) = g(J + J) = g(J) + g(J)$. The only elements in E satisfying $x + x = x$ are 0 and ∞. Since g must map the greatest element of $\downarrow K$ to the greatest element of E, $g(K) = \infty$, which forces $g(J) = \infty$ for all non-zero ideals. The map $h : I(\mathbb{Z}) \to E$ defined by $h(X) = \infty$ (for $X \neq \{0\}$) perfectly extends g. Thus, Baer's Criterion is satisfied.

Next, we demonstrate that E is not injective. Let $B = \mathbb{N} \cup \{\infty\}$, and let A be the sub-*le*-module of even integers $A = 2\mathbb{N} \cup \{\infty\}$. Define the monomorphism $i : A \hookrightarrow B$ as inclusion. Define $f : A \to E$ by $f(x) = x/2$ for finite x, and $f(\infty) = \infty$. Suppose f extends to $h : B \to E$. By extension, $h(2) = f(2) = 1$. Because h is a homomorphism, it must preserve addition on B, yielding $1 = h(2) = h(1 + 1) = h(1) + h(1)$. This requires an element in $\mathbb{N} \cup \{\infty\}$ such that $x + x = 1$, which does not exist. Thus, E is not injective. □

The obstruction arises because the lattice of submodule elements is additively idempotent, whereas the underlying monoid of the *le*-module is not necessarily so.

8. FUNCTORIALITY AND THE PRIME SPECTRUM

We now establish that the prime spectrum, endowed with the Zariski topology, acts as a continuous functor under *le*-module homomorphisms. Let $\mathrm{Spec}(M)$ be endowed with the Zariski topology, where closed sets are of the form $V(n) = \{q \in \mathrm{Spec}(M) \mid n \leq q\}$.

Theorem 8.1. *Let $f : M \to N$ be an le-module homomorphism. If $p \in \mathrm{Spec}(N)$, then the pullback $f^*(p) = \bigvee\{x \in M \mid f(x) \leq p\}$ belongs to $\mathrm{Spec}(M)$.*

Proof. Let $p \in \mathrm{Spec}(N)$. Because $f(e_M) = e_N \not\leq p$, it follows that $e_M \notin \{x \mid f(x) \leq p\}$, ensuring $f^*(p) < e_M$. Suppose $r \in R$ and $m \in M$ such that $rm \leq f^*(p)$. Then $f(rm) \leq f(f^*(p)) \leq p$, yielding $rf(m) \leq p$. Since p is prime in N, either $f(m) \leq p$ or $re_N \leq p$. If $f(m) \leq p$, then $m \leq f^*(p)$. If $re_N \leq p$, then $f(re_M) \leq p$, implying $re_M \leq f^*(p)$. Therefore, $f^*(p)$ is prime. □

Theorem 8.2. *The map $f^* : \mathrm{Spec}(N) \to \mathrm{Spec}(M)$ induced by an le-module homomorphism f is continuous with respect to the Zariski topology.*

Proof. Let $V(n)$ be a closed set in $\mathrm{Spec}(M)$ for some $n \in \mathrm{Sub}(M)$. The pre-image under f^* is $(f^*)^{-1}(V(n)) = \{p \in \mathrm{Spec}(N) \mid n \leq f^*(p)\}$. By the definition of the pullback, $n \leq f^*(p)$ if and only if $f(n) \leq p$. Therefore, $(f^*)^{-1}(V(n)) = V(f(n))$. Since $f(n) \in \mathrm{Sub}(N)$, the set $V(f(n))$ is closed in $\mathrm{Spec}(N)$, establishing continuity. □

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