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Functional Laplace Transformations: Theory and Applications

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Abstract

This research article introduces a novel extension of the classical Laplace transform within the framework of functional quantum calculus. We define two new classes of integral transforms, namely the (u) – Laplace transform and the $h(u)$ – Laplace transform, collectively referred to as Functional Laplace Transforms. These transforms may be interpreted as variable-order Laplace-type transformations in which the transformation parameters evolve according to prescribed functional dependencies. Fundamental properties, convolution theorem, and operational rules are established. Illustrative examples demonstrate the applicability of the developed theory to generalized differential and quantum models.
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1 Introduction:

The "$q-$calculus, or quantum calculus, provides an alternative framework to infinitesimal calculus that avoids the classical limit concept. Instead, it uses a $q-$derivative defined through finite differences, leading to its characterization as 'calculus without limits'. In this work, we have further extended the results and usefulness of [11] by giving the applications in various fields.

The Laplace transform has been a cornerstone of mathematical analysis for over two centuries, providing elegant solutions to differential equations and serving as a fundamental tool in engineering, physics, and applied mathematics. However, classical transforms assume fixed parameters and uniform structures and limiting their applicability to systems with time-varying dynamics along with adaptive discretization, or quantum deformations [10].

The paper explores with extension of q – Laplace and h – Laplace Transformations to $q(u)$ – Laplace and $h(u)$ – Laplace Transformations, its properties and applications to find out some standard results.

In section 1, we have mainly deals with basic definitions and properties. Section 2, gives extension to newly defined Functional Laplace Transformations. In section 3 and 4, we have presented some theorems based on the new definitions along with some properties. While in section 5, we have given some applications. The Section 6 ends with conclusion.

2. Preliminary and Proposed Definitions:

Definition 2.1: q – derivative:

Given an arbitrary function $f(x)$ then its q – derivative is denoted by $D_q f(x)$ and is given by [7]

$$D_q f(x) = \frac{f(qx)-f(x)}{qx-x} = \frac{f(qx)-f(x)}{(q-1)x} \tag{1}$$

Definition 2.2: Functional analogue of $q(u)$ – derivative:

Given an arbitrary function $f(x)$ then its $q(u)$ – derivative is defined as $D_{q(u)} f(x)$ and is given by [11]

$$D_{q(u)} f(x) = \frac{f(q(u)x)-f(x)}{q(u)x-x} = \frac{f(q(u)x)-f(x)}{(q(u)-1)x} \tag{2}$$

Where, $Q = \{ q(u) \mid q(u) \to 1 \ as\ u \to 0 \}$

The above set is non – empty i. e. $Q \neq \emptyset$ and are well defined. e.g., If $q(u) = 1 + u\ , u \in \mathbb{R}$ then $q(u) \in Q$.

Definition 2.3: $q(u)$ –Convolution:

Given an arbitrary function $f(t) and\ g(t)$ then the $q(u)$ –Convolution of them is given by,

$$(f *_{q(u)} g)(t) = \int_0^t f(\tau) g(t - q(u)\tau) d\tau \tag{3}$$

Where, $q(u) \in Q$.

Definition 2.4: h – derivative:

Given an arbitrary function $f(x)$ then its h – derivative is denoted by $D_h f(x)$ and is given by [7]

$$D_h f(x) = \frac{f(x+h)-f(x)}{h} \tag{4}$$

Definition 2.5: $h(u)$ – derivative:

Given an arbitrary function $f(x)$ then its $h(u)$ – derivative is defined as $D_{h(u)} f(x)$ and is given by [11]

$$D_{h(u)} f(x) = \frac{f(x+h(u))-f(x)}{h(u)} \tag{5}$$

Where, $H = \{ h(u) \mid h(u) \to 0\ as\ u \to 0 \}$

The above set is non – empty i. e. $H \neq \emptyset$ and are well defined. e.g., If $h(u) = u\ , u \in \mathbb{R}$ then $h(u) \in H$.

Definition 2.6: $h(u)$ −Convolution:

Given an arbitrary function $f(t) and\ g(t)$ then the $h(u)$ −Convolution of them is given by,

$$(f *_{h(u)} g)(t) = \int_0^t f(\tau) g(t - \tau + h(u))\, d\tau \quad (6)$$

Definition 2.7: Laplace Transformation

If function $f(t)$ is continuous piecewise and is of exponential order then its Laplace –transformation [10] is given by:

$$\mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt \quad (7)$$

3. $q(u)$– Functional Extension of Laplace Transformation:

In this section, we extend the classical Laplace Transformation to $q(u)$–Laplace Transformation. We proved some theorems and properties with $q(u)$–Laplace Transformation of some standard functions.

- **$q(u)$–Laplace Transformation**

For a function $f: [0, \infty) \to \mathbb{C}$ and a functional parameter $q(u)$ satisfying $q(u) \to 1$ as $u \to 0$, we define:

$$\mathcal{L}_{q(u)}\{f(t)\}(s) = \int_0^\infty f(t) E_{q(u)}(-st)\, d_{q(u)} t$$

Where, $E_{q(u)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q(u)}!}$ is the $q(u)$ −exponential function, with $[n]_{q(u)} = \frac{(q(u))^n - 1}{q(u) - 1}$ being the $q(u)$ −analogue of n [11].

[3.1] Properties:

A. Linearity:

$$\mathcal{L}_{q(u)}\{af + bg\}(s) = a\mathcal{L}_{q(u)}\{f\}(s) + b\mathcal{L}_{q(u)}\{g\}(s)$$

Proof:

$\mathcal{L}_{q(u)}\{af + bg\}(s) = \int_0^\infty (af(t) + bg(t))\, E_{q(u)}(-st)\, dt$

Using the linearity property of integration

$\Rightarrow \int_0^\infty (af(t) + bg(t))\, E_{q(u)}(-st) d_{q(u)} t$

$= a \int_0^\infty f(t) E_{q(u)}(-st)\, dt + b \int_0^\infty g(t) E_{q(u)}(-st)\, d_{q(u)} t$

$= a\mathcal{L}_{q(u)}\{f\}(s) + b\mathcal{L}_{q(u)}\{g\}(s)$

B. $q(u)$ −Scaling:

For $\alpha > 0$,

$$\mathcal{L}_{q(u)}\{f(\alpha t)\}(s) = \frac{1}{\alpha} \mathcal{L}_{q(u)}\{f(t)\}\left(\frac{s}{\alpha}\right)$$

Proof: For $\alpha > 0$:

$$\mathcal{L}_{q(u)}\{f(\alpha t)\}(s) = \int_0^\infty f\,(\alpha t)E_{q(u)}(-st)\,d_{q(u)}t$$

Let, $\tau = \alpha t$, so $dt = d\tau/\alpha$:

$$= \int_0^\infty f\,(\tau)E_{q(u)}\left(-\frac{s}{\alpha}\tau\right)\frac{d_{q(u)}\tau}{\alpha}$$

For the transform to maintain its functional form under scaling, we require the parameter to scale as $q(u) \to q(\alpha u)$. This holds for specific $q(u)$ such as $q(u) = 1 + u$ or $q(u) = e^{-u}$, where:

$$E_{q(u)}\left(-\frac{s}{\alpha}\tau\right) = E_{q(\alpha u)}\left(-\frac{s}{\alpha}\tau\right)$$

Thus:

$$\mathcal{L}_{q(u)}\{f(\alpha t)\}(s) = \frac{1}{\alpha}\int_0^\infty f\,(\tau)E_{q(u)}\left(-\frac{s}{\alpha}\tau\right)d\tau = \frac{1}{\alpha}\mathcal{L}_{q(u)}\{f(\tau)\}\left(\frac{s}{\alpha}\right)$$

C. $q(u)$ −Shift:

$$\mathcal{L}_{q(u)}\{e_{q(u)}(at)f(t)\}(s) = \mathcal{L}_{q(u)}\{f(t)\}(s-a)$$

Where, $e_{q(u)}(at)$ is the $q(u)$ −exponential defined above

Proof:

$$\mathcal{L}_{q(u)}\{e_{q(u)}(at)f(t)\}(s) = \int_0^\infty e_{q(u)}\,(at)f(t)E_{q(u)}(-st)\,d_{q(u)}t$$

Using the q −exponential addition formula [7] valid for specific $q(u)$ functions:

$$e_{q(u)}(at)E_{q(u)}(-st) = E_{q(u)}(-(s-a)t)$$

This formula holds when the q −exponential satisfies a q-analogue of the exponential addition property, which is true for the defined $q(u)$-exponential series when terms are rearranged under convergence.

$$\mathcal{L}_{q(u)}\{e_{q(u)}(at)f(t)\}(s) = \int_0^\infty f\,(t)E_{q(u)}(-(s-a)t)\,d_{q(u)}\mathrm{t} = \mathcal{L}_{q(u)}\{f(t)\}(s-a)$$

[3.2] Theoretical Advancement:

Theorem 3.2.1 (Convolution Theorem): $\mathcal{L}_{q(u)}\{f *_{q(u)} g\}(s) = \mathcal{L}_{q(u)}\{f\}(s) \cdot \mathcal{L}_{q(u)}\{g\}(s)$

where the $q(u)$-convolution is defined by

$$(f *_{q(u)} g)(t) = \int_0^t f(\tau) g(t-\tau) d_{q(u)}\tau$$

Proof: Using the definitions and interchanging the order of integration [10] (Using Fubini's theorem under suitable conditions), we have

$$\begin{aligned}
\mathcal{L}_{q(u)}\{f *_{q(u)} g\}(s) &= \int_0^\infty e_q^{-st} \left(\int_0^t f(\tau) g(t-\tau) d_{q(u)}\tau \right) d_{q(u)}t \\
&= \int_0^\infty \int_0^\infty e_q^{-s(\tau+\eta)} f(\tau) g(\eta) d_{q(u)}\tau d_{q(u)}\eta \\
&= \left(\int_0^\infty f(\tau) e_q^{-s\tau} d_{q(u)}\tau \right) \left(\int_0^\infty g(\eta) e_q^{-s\eta} d_{q(u)}\eta \right) \\
&= \mathcal{L}_{q(u)}\{f\}(s)\ \mathcal{L}_{q(u)}\{g\}(s).
\end{aligned}$$

Theorem 3.2.2 (Derivative Property): Let $D_{q(u)} f(t) = \frac{f(q(u)t) - f(t)}{(q(u)-1)t}$ defined in [11] Then:

$$\mathcal{L}_{q(u)}\{D_{q(u)} f(t)\}(s) = \frac{s}{1-(1-q(u))s} \mathcal{L}_{q(u)}\{f(t)\}(s) - f(0)$$

Proof: Using the Eigen function property:

$$D_{q(u)} E_{q(u)}(-st) = \frac{E_{q(u)}(-sq(u)t) - E_{q(u)}(-st)}{(q(u)-1)t} = -sE_{q(u)}(-st)$$

Then:

$$\mathcal{L}_{q(u)}\{D_{q(u)} f\}(s) = \int_0^\infty f(t) D_{q(u)} E_{q(u)}(-st) dt = -s\mathcal{L}_{q(u)}\{f\}(s) - f(0)$$

Consider, $\int_0^\infty D_{q(u)} f(t) E_{q(u)}(-st) dt$.

Using q −integration by parts (Jackson formula) [7]

$$\int_0^\infty D_{q(u)} f(t) g(t) d_{q(u)}t = [f(t)g(t)]_0^\infty - \int_0^\infty f(q(u)t) D_{q(u)} g(t) d_{q(u)}t$$

where $d_{q(u)}t = (q(u)-1)tdt$.

With $g(t) = E_{q(u)}(-st)$ and $D_{q(u)} g(t) = -sg(t)$, we get:

$$\mathcal{L}_{q(u)}\{D_{q(u)} f\}(s) = -f(0) + s \int_0^\infty f(q(u)t) E_{q(u)}(-st) d_{q(u)}t$$

But $\int_0^\infty f(q(u)t) E_{q(u)}(-st) d_{q(u)}t = \mathcal{L}_{q(u)}\{f(q(u)t)\}(s)$

and $\mathcal{L}_{q(u)}\{f(q(u)t)\}(s) = \frac{1}{q(u)} \mathcal{L}_{q(u)}\{f(t)\}(s/q(u))$.

$$\Rightarrow \mathcal{L}_{q(u)}\{D_{q(u)} f(t)\}(s) = \frac{s}{1-(1-q(u))s} \mathcal{L}_{q(u)}\{f(t)\}(s) - f(0)$$

Using series expansion, one can clearly check that as $(u) \to 1$, $\frac{s}{1-(1-q(u))s} \to s$.

This gives the classical result for convolution theorem

4. $h(u)$– Functional Extension of Laplace Transformation:

In this section, we extend the classical Laplace Transformation to $h(u)$–Laplace Transformation. We proved some theorems and properties with $h(u)$–Laplace Transformation of some standard functions.

- **$h(u)$–Laplace Transformation**

  For a functional parameter $h(u)$ with $h(u) \to 0$ as $u \to 0$, we define:

$$\mathcal{L}_{h(u)}\{f(t)\}(s) = \int_0^\infty f(t)(1 - sh(u))^{t/h(u)}\, dt$$

Where, $|1 - sh(u)| < 1$ to ensure convergence of the integral

[4.1] Properties:

A. Linearity: $\mathcal{L}_{h(u)}\{af + bg\}(s) = a\mathcal{L}_{h(u)}\{f\}(s) + b\mathcal{L}_{h(u)}\{g\}(s)$

Proof: $\mathcal{L}_{h(u)}\{af + bg\}(s) = \int_0^\infty (af(t) + bg(t)))(1 - sh(u))^{t/h(u)}\, dt$

Using the linearity property of integration

$\Rightarrow \int_0^\infty (af(t) + bg(t)))(1 - sh(u))^{t/h(u)}\, dt$

$= a\int_0^\infty f(t))(1 - sh(u))^{t/h(u)}\, dt + b\int_0^\infty g(t))(1 - sh(u))^{t/h(u)}\, dt$

$= a\mathcal{L}_{h(u)}\{f\}(s) + b\mathcal{L}_{h(u)}\{g\}(s)$

B. $h(u)$ –Scaling:

For $\alpha > 0$,

$$\mathcal{L}_{h(u)}\{f(\alpha t)\}(s) = \frac{1}{\alpha}\mathcal{L}_{h(\alpha u)}\{f(t)\}\left(\frac{s}{\alpha}\right)$$

Proof: Let $F(t) = f(\alpha t)$. By definition,

$$\mathcal{L}_{h(u)}\{f(\alpha t)\}(s) = \int_0^\infty f(\alpha t)\,(1 - sh(u))^{t/h(u)}\, dt.$$

Now we substitute $\tau = \alpha t, \Rightarrow\ t = \tau/\alpha$ and $dt = d\tau/\alpha$. Then

$$\mathcal{L}_{h(u)}\{f(\alpha t)\}(s) = \frac{1}{\alpha}\int_0^\infty f(\tau)\,(1 - sh(u))^{\tau/(\alpha h(u))}\, d\tau.$$

We Assume the kernel scales as $(1 - sh(u))^{1/(\alpha h(u))} = (1 - (s/\alpha)h(\alpha u))^{1/h(\alpha u)}$ (which holds for suitable $h(u)$, e.g., $h(u) = u$), we obtain

$$\mathcal{L}_{h(u)}\{f(\alpha t)\}(s) = \frac{1}{\alpha}\int_0^\infty f(\tau)\,(1 - (s/\alpha)h(\alpha u))^{\tau/h(\alpha u)}\, d\tau = \frac{1}{\alpha}\,\mathcal{L}_{h(\alpha u)}\{f(\tau)\}(s/\alpha).$$

C. $h(u)$ –*Shift:* $\mathcal{L}_{h(u)}\{(1 + ah(u))^{t/h(u)}f(t)\}(s) = \mathcal{L}_{h(u)}\{f(t)\}(s - a)$ under some conditions.

Proof:

$$\text{LHS} = \mathcal{L}_{h(u)}\{(1 + ah(u))^{t/h(u)}f(t)\}(s)$$

$$= \int_0^\infty (1 + ah(u))^{t/h(u)}f(t)\,(1 - sh(u))^{t/h(u)}\, dt.$$

If we assume the identity $(1 + ah(u))(1 - sh(u)) = 1 - (s - a)h(u)$ (valid up to $O(h(u)^2)$ as $h(u) \to 0$), then

$$= \int_0^\infty f(t)\,(1-(s-a)h(u))^{t/h(u)}\,dt = \mathcal{L}_{h(u)}\{f(t)\}(s-a).$$

Thus the shift property holds exactly when the $h(u)^2$ term is neglected, i.e., in the asymptotic sense for very small $h(u)$.

[4.2] Theoretical Advancement

Theorem 4.2.1: Given an arbitrary function $f(t) and\ g(t)$ then the $h(u)$ −Convolution of them is given by,

$$\mathcal{L}_{h(u)}\{f *_{h(u)} g\}(s) = \mathcal{L}_{h(u)}\{f\}(s)\cdot\mathcal{L}_{h(u)}\{g\}(s)$$

Proof: We know that, given an arbitrary function $f(t) and\ g(t)$ then the $h(u)$ −Convolution of them is given by,

$$(f *_{h(u)} g)(t) = \int_0^t f(\tau)g(t-\tau+h(u))\,d\tau$$

Taking $h(u)-$ Laplace transform on both sides of the above equation, we get

$$\mathcal{L}_{h(u)}\{f *_{h(u)} g\}(s) = \int_0^\infty\left(\int_0^t f(\tau)g(t-\tau+h(u))\,d\tau\right)(1-sh(u))^{t/h(u)}\,dt.$$

Using the Change order of integration (Fubini's Theorem) [10]:

$$\mathcal{L}_{h(u)}\{f *_{h(u)} g\}(s) = \int_0^\infty f(\tau)\big(\int_\tau^\infty g(t-\tau+h(u))(1-sh(u))^{t/h(u)}\,dt\big)d\tau.$$

Now we substitute $u = t-\tau+h(u)$ in inner integral (so $t=\tau+u-h(u)$, $\Longrightarrow dt = du$, limits: $t=\tau \Longrightarrow u = h(u)$, $t\to\infty \Longrightarrow u\to\infty$):

$$= \int_0^\infty f(\tau)\left(\int_{h(u)}^\infty g(u)(1-sh(u))^{(\tau+u-h(u))/h(u)}\,du\right)d\tau.$$

Now split the exponent:

$$(1-sh(u))^{(\tau+u-h(u))/h(u)} = (1-sh(u))^{\tau/h(u)}\cdot(1-sh(u))^{u/h(u)}\cdot(1-sh(u))^{-1}.$$

Thus:

$$\mathcal{L}_{h(u)}\{f *_{h(u)} g\}(s) = (1-sh(u))^{-1}\int_0^\infty f(\tau)(1-sh(u))^{\tau/h(u)}\,d\tau\cdot\int_{h(u)}^\infty g(u)(1-sh(u))^{u/h(u)}\,du.$$

If g is such that $\int_{h(u)}^\infty g(u)(1-sh(u))^{u/h(u)}du = (1-sh(u))\int_0^\infty g(u)(1-sh(u))^{u/h(u)}du$ in other words, if $g(u)$ decays appropriately, then the factor $(1-sh(u))^{-1}$ cancels and we obtain:

$$\left(\int_0^\infty f(\tau)(1-sh(u))^{\tau/h(u)}d\tau\right)\left(\int_0^\infty g(u)(1-sh(u))^{u/h(u)}du\right) = \mathcal{L}_{h(u)}\{f\}(s)\cdot\mathcal{L}_{h(u)}\{g\}(s).$$

Theorem 4.2.2: Let $f:[0,\infty)\to\mathbb{C}$ be piecewise continuous and of exponential order σ, i.e.,

$$\exists M>0,\quad |f(t)|\le Me^{\sigma t}\quad \forall t\ge 0 \quad\Longrightarrow\quad \lim_{u\to 0}\mathcal{L}_{h(u)}\{f(t)\}(s) = \mathcal{L}\{f(t)\}(s)$$

and the convergence is uniform on compact subsets of $\{s: \Re(s) > \sigma\}$.

Proof:

By the Lebesgue Dominated Convergence Theorem, since:

1. $f(t)K_{h(u)}(t) \to f(t)e^{-st}$ pointwise as $u \to 0$
2. $|f(t)K_{h(u)}(t)| \le Me^{-\delta t}$ (integrable dominating function)

We can interchange limit and integral:

$$\lim_{u\to 0}\int_0^\infty f(t)K_{h(u)}(t)\,dt = \int_0^\infty f(t)\lim_{u\to 0}K_{h(u)}(t)\,dt = \int_0^\infty f(t)e^{-st}\,dt$$

$\Longrightarrow \quad \lim_{u\to 0}\mathcal{L}_{h(u)}\{f(t)\}(s) = \mathcal{L}\{f(t)\}(s).$

5. Applications of $h(u)$ and $q(u)$– Functional Laplace Transform:

In this section we, calculate the $h(u)$ and $q(u)$– Functional Laplace Transform of some standard functions and compare the results.

Example 5.1] Find $h(u)$ and $q(u)$– Functional Laplace Transform, for $(t) = 1$ and $h(u) = u,\ q(u) = 1 + u$.

Answer:

Case I] For $q(u) = 1 + u$ and $f(t) = 1$, the $q(u)$ −Laplace transform is defined as

$$\mathcal{L}_{1+u}\{1\}(s) = \int_0^\infty E_{1+u}(-st)\,dt,$$

where $E_{1+u}(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_{1+u}!}$ is the q-exponential function with $[n]_{1+u} = \frac{(1+u)^n-1}{u}$.

By expanding the integrand in powers of u and integrating term wise (using a suitable regularization), we obtain the asymptotic expansion

$$\mathcal{L}_{1+u}\{1\}(s) = \frac{1}{s} + \frac{u}{2} + \frac{u^2 s}{12} + O(u^3).$$

This expansion shows that the functional $q(u)$ −Laplace transform reduces to the classical Laplace transform $\frac{1}{s}$ as $u \to 0$, with corrections that are linear in u at leading order.

Case II] The h(u)-Laplace transform of $f(t) = 1$ with $h(u) = u$ is defined as

$$\mathcal{L}_{h(u)}\{f(t)\}(s) = \int_0^\infty f(t)(1 - sh(u))^{t/h(u)}\,dt$$

Where, $|1 - sh(u)| < 1$ to ensure convergence of the integral

$\Longrightarrow$

$$\mathcal{L}_{h(u)}\{f(t)\}(s) = \int_0^\infty (1 - sh(u))^{t/h(u)}\,dt$$

$\Longrightarrow$

$$\mathcal{L}_u\{1\}(s) = -\frac{u}{\ln(1 - su)}$$

Which is valid for $|1 - su| < 1$ to ensure convergence of the integral

Expanding for small u yields

$$\mathcal{L}_u\{1\}(s) = \frac{1}{s} - \frac{u}{2} - \frac{su^2}{12} + O(u^3),$$

which shows that the transform reduces to the classical Laplace transform $1/s$ as $u \to 0$.

6. Conclusion:

We have extend the Classical Laplace transformation to Functional Laplace Transformation or Variable order Laplace Transformation which will be useful to offer a powerful new paradigm for mathematical modeling, combining the elegance of transform methods with the flexibility of adaptive parameters to address complex, time-varying systems across physics, engineering, and applied mathematics.

References

[1] Bailey W N, (1972), "Hypergeometric Series", Reprinted by Hafner, New York.

[2] Elzaki, T. M., (2011), "The new integral Transform Elzaki Transform". Global Journal of Pure and Applied Mathematics, 7(1), pp. 57 – 64.

[3] Ernst T, (1999), "A New Notation for $q-$ Calculus and a new $q-$ Taylor Formula", U. U. D. M. Report (1999):25, ISSN 1101-3591, Department of Mathematics, Uppsala University.

[4] Ernst T, (2000), "The History of $q-$ Calculus and a New Method", Department of Mathematics, Uppsala University.

[5] Ernst T, (2002), "A New Method for $q-$ Calculus", Uppsala Dissertations in Mathematics, (25).

[6] Ernst T., (2003), "A Method for $q-$ Calculus", Journal of Nonlinear Mathematical Physics, 10(4), pp. 487 – 525.

[7] Kac V. and Cheung P., (2002), "Quantum Calculus", Universitext, Springer-Verlag, New York.

[8] Kupershmidt B A, (2000), "$q-$Newton Binomial: from Euler to Gauss", J. Nonlinear Math. Phys. 7(2), pp. 244 – 262.

[9] Lebedev N N, (1972), "Special Functions and Their Applications", Dover.

[10] Loknath D., Dambaru B., (2015), "Integral Transforms and Their Applications", CRC Press.

[11] Manjarekar, Jafari, (2024) "A new concept of $q-$calculus with respect to another function", J. Innov. Appl. Math. Comput. Sci. 4(2), pp. 113 – 121.

[12] Manjarekar S., Nikam,(2025), "On some extended results in Functional q(t) − calculus: Theoretical and Numerical Approach", Communications in Mathematics and Applications:, Vol. 16(3).