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An Elementary Proof on the Limit Superior and Limit Inferior of $\cos(n)$ and $\sin(n)$

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Abstract

We present an elementary proof of $\limsup_{n\to\infty}\cos(n) = \limsup_{n\to\infty}\sin(n) = 1$, and $\liminf_{n\to\infty}\cos(n) = \liminf_{n\to\infty}\sin(n) = -1$. Our approach avoids the use of the Kronecker density theorem, equidistribution theorem, or any concepts from Diophantine approximations, providing a more direct and accessible argument.

Given the oscillatory nature of $\cos(x)$ and $\sin(x)$, it is natural to expect that the sequences $\{\cos(n)\}_{n=1}^{\infty}$ and $\{\sin(n)\}_{n=1}^{\infty}$ do not converge to a single limit. However, since they are bounded within $[-1, 1]$, their $\limsup$ and $\liminf$ values exist and are finite.

The inequalities $\liminf_{n\to\infty}\cos(n) < \limsup_{n\to\infty}\cos(n)$, and similarly for $\sin(n)$, highlights the presence of distinct accumulation points due to their oscillatory nature. Finding the exact values of these bounds is therefore a natural problem of interest.

Traditionally, these values are obtained using results from the Kronecker density theorem, or Diophantine approximation. For an introduction to the Kronecker density theorem, see [1]. For discussions on its applications to this problem, refer to the online discussion threads [3], which includes insights from contributors like Prahlad Vaidyanathan and André Nicholas. These approaches analyze the distribution of sets like $\{a + b\pi \mid a, b \in \mathbb{Z}\}$ and its implications for trigonometric sequences.

In this paper, we offer an alternative elementary proof that does not rely on advanced number-theoretic methods. Since this problem belong to real analysis, we believe it deserves a proof grounded purely in analytical techniques. Our approach remains accessible to readers with an advanced calculus background, where $\limsup$ and $\liminf$ are typically introduced.

Elementary proofs are valuable for making mathematical results more accessible to students and researchers unfamiliar with advanced techniques. A new or simplified perspective can enhance understanding, particularly in educational contexts. Furthermore, even for a well-known result, a novel method can offer fresh insights or inspire applications in related areas. In particular, the behavior of $\cos(n)$ and $\sin(n)$ is closely tied to the distribution of $n \pmod{2\pi}$ (a consequence of the irrationality of π). Our approach sheds light on this connection in a direct and intuitive way.

To establish our main results, we first recall the following fundamental facts about the limit superior and limit inferior of a bounded sequence.

Theorem 1. *If $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

Similarly, there exists a (possibly different) subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n.$$

Proof. See Theorem 2.3.4., page 75 of [2]. □

Proposition 1. *Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence and $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence. Then*

$$\liminf_{n\to\infty} x_n \le \liminf_{k\to\infty} x_{n_k} \le \limsup_{k\to\infty} x_{n_k} \le \limsup_{n\to\infty} x_n.$$

Proof. See Proposition 2.3.6., page 77 of [2]. □

Now let us prove the limit superior of $\cos(n)$ first.

Theorem 2. $\limsup\limits_{n\to\infty} \cos(n) = 1.$

Proof. The sequence $\{\cos(n)\}_{n=1}^{\infty}$ is bounded; hence $\limsup\limits_{n\to\infty} \cos(n) = a \in [-1, 1]$ exists. Our goal is to show $a \ge 1$, which will force $a = 1$.

First, we prove that $a \geq \frac{1}{2}$. For all $k \in \mathbb{N}$, let $u_k = [2k\pi] \in \mathbb{N}$, where $[\ \cdot\ ]$ is the nearest integer function. By the Mean Value Theorem, there exists c between u_k and $2k\pi$ such that $|\cos(u_k) - 1| = |\cos(u_k) - \cos(2k\pi)| = |\cos'(c)(u_k - 2k\pi)| = |-\sin(c)| \cdot |u_k - 2k\pi| \leq 1 \cdot \frac{1}{2} = \frac{1}{2}$. Thus, by the triangle inequality, we obtain $\cos(u_k) \geq 1 - |\cos(u_k) - 1| \geq 1 - \frac{1}{2} = \frac{1}{2}$. Applying Proposition 1, we obtain $a = \limsup_{n\to\infty} \cos(n) \geq \limsup_{k\to\infty} \cos(u_k) \geq \frac{1}{2}$.

Next, by Theorem 1, there exists a subsequence $\{\cos(n_k)\}$ such that $\lim_{k\to\infty} \cos(n_k) = a$.

Using the identity $\sin^2(n_k)+\cos^2(n_k) = 1$, we conclude that $\lim_{k\to\infty} |\sin(n_k)| = \sqrt{1-a^2}$. Thus, we have two cases

- There exists a further subsequence n_{k_l} such that $\lim_{l\to\infty} \sin(n_{k_l}) = \sqrt{1-a^2}$,
- Alternatively, $\lim_{k\to\infty} \sin(n_k) = -\sqrt{1-a^2}$.

For simplicity, we assume (without loss of generality) that $\lim_{k\to\infty} \sin(n_k) = \pm\sqrt{1-a^2}$.

Using the sum-to-product formula, for all $m \in \mathbb{N}$, we have $\cos(n_k - m) = \cos(n_k)\cos(m)+\sin(n_k)\sin(m)$. Taking limits, we get $\lim_{k\to\infty} \cos(n_k - m) = a\cos(m) \pm \sqrt{1-a^2}\sin(m)$.

For all $m \in \mathbb{N}$, since $n_k \geq k$, we know $n_k - m \geq k - m \geq 1$ when $k \geq m+1$, meaning that $\{n_k - m\}_{k=m+1}^{\infty}$ is a subsequence of $\{n\}$. Therefore, by Proposition 1, the convergent subsequence $\{\cos(n_k - m)\}_{k=m+1}^{\infty}$ must have its limit no greater than $a = \limsup_{n\to\infty} \cos(n)$. That is,

$$a\cos(m) \pm \sqrt{1-a^2}\sin(m) \leq a, \forall m \in \mathbb{N} \tag{1}$$

In particular, for $m = n_1, n_2, \ldots$, repeatedly applying (1) yields

$$a\cos(n_k) \pm \sqrt{1-a^2}\sin(n_k) \leq a, \forall k \in \mathbb{N}. \tag{2}$$

Letting $k \to \infty$ in (2), we have $a \cdot a + (\pm\sqrt{1-a^2}) \cdot (\pm\sqrt{1-a^2}) = a^2 + 1 - a^2 = 1 \leq a$, as desired. □

Next, we prove the limit superior of $\sin(n)$.

Theorem 3. $\limsup_{n\to\infty} \sin(n) = 1$.

Proof. The sequence $\{\sin(n)\}_{n=1}^{\infty}$ is bounded; hence, $\limsup_{n\to\infty} \sin(n) = b \in [-1,1]$ exists. Our goal is to show $b \geq a$, which implies $b = 1$ since we have already established $a = 1$.

Using a similar argument as before, we set $v_k = [2k\pi + \frac{\pi}{2}] \in \mathbb{N}$, which allows us to conclude that $b \geq \frac{1}{2}$.

By Theorem 1, we can find a subsequence $\{\sin(m_k)\}$ such that $\lim_{k\to\infty} \sin(m_k) = b$. Similarly, we can assume $\lim_{k\to\infty} \cos(m_k) = \pm\sqrt{1-b^2}$.

Using the sum-to-product formula, for all $m \in \mathbb{N}$, we have the identities $\sin(m_k \pm m) = \sin(m_k)\cos(m) \pm \cos(m_k)\sin(m)$. Taking limits, we get $\lim_{k\to\infty} \sin(m_k \pm m) = b\cos(m) \pm \lim_{k\to\infty} \cos(m_k)\sin(m)$. Again, applying Proposition 1,

$$b\cos(m) \pm \lim_{k\to\infty} \cos(m_k)\sin(m) \leq b. \tag{3}$$

Since $\sin(m) \neq 0$, there are four possible cases:

- $\lim_{k\to\infty} \cos(m_k) = \sqrt{1-b^2}$, $\sin(m) > 0$;
- $\lim_{k\to\infty} \cos(m_k) = \sqrt{1-b^2}$, $\sin(m) < 0$;
- $\lim_{k\to\infty} \cos(m_k) = -\sqrt{1-b^2}$, $\sin(m) > 0$;
- $\lim_{k\to\infty} \cos(m_k) = -\sqrt{1-b^2}$, $\sin(m) < 0$.

Selecting the $+$ sign in (3) for the first and the last cases, and the $-$ sign in (3) for the two remaining cases, we obtain

$$b\cos(m) + \sqrt{1-b^2}\,|\sin(m)| \leq b, \forall m \in \mathbb{N}. \tag{4}$$

This implies $\sqrt{1-b^2}\,|\sin(m)| \leq b(1-\cos(m))$. Dividing both sides by the positive number $|\sin(m)|$, we have

$$\sqrt{1-b^2} \leq b\frac{1-\cos(m)}{|\sin(m)|} = b|\frac{1-\cos(m)}{\sin(m)}| = b|\tan(\frac{m}{2})| \tag{5}$$

by the double-angle formulae $\cos(x) = 1 - 2\sin^2(\frac{x}{2})$ and $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$.

Since both sides of inequality (5) are non-negative, squaring gives $1-b^2 \leq b^2\tan^2(\frac{m}{2})$, which implies $1 \leq b^2(1+\tan^2(\frac{m}{2})) = b^2\sec^2(\frac{m}{2})$. Thus, we have $\cos^2(\frac{m}{2}) \leq b^2$. Taking the square-root, we obtain

$$\cos(\frac{m}{2}) \leq |\cos(\frac{m}{2})| \leq b, \forall m \in \mathbb{N}. \tag{6}$$

In particular, for $m = 2n_1, 2n_2, \ldots$, repeatedly applying (6) yields

$$\cos(\frac{2n_k}{2}) = \cos(n_k) \leq b, \forall k \in \mathbb{N}. \tag{7}$$

Taking limits as $k \to \infty$ in (7), we conclude that $1 = a \leq b$. Thus $b = 1$, completing the proof. □

Finally, let us prove the limit inferior results.

Theorem 4. $\liminf_{n\to\infty} \cos(n) = \liminf_{n\to\infty} \sin(n) = -1$.

Proof. The proof for these two cases are similar, so we will only show $\liminf_{n\to\infty} \cos(n) = -1$.

The sequence $\{\cos(n)\}_{n=1}^{\infty}$ is bounded; hence, $\liminf_{n\to\infty} \sin(n) = c \in [-1,1]$ exists. Our goal is to show $c < 0, |c| \geq 1$, which will imply $c = -1$.

Similar to the proof on $a \geq \frac{1}{2}$, for all $k \in \mathbb{N}$, we set $w_k = [2k\pi + \pi] \in \mathbb{N}$ to establish that $c \leq -\frac{1}{2}$.

By Theorem 1, we can find a subsequence $\{\cos(l_k)\}$ such that $\lim_{k\to\infty} \cos(l_k) = c$. Similar to the previous analysis in the proof of Theorem 2, we can assume $\lim_{k\to\infty} \sin(l_k) = \pm\sqrt{1-c^2}$.

Using the sum-to-product formula, for all $m \in \mathbb{N}$, we have the identities $\cos(l_k \pm m) = \cos(l_k)\cos(m) \mp \sin(l_k)\sin(m)$. Taking limits, we get $\lim_{k\to\infty} \cos(l_k \pm m) = c\cos(m) \mp \lim_{k\to\infty} \sin(l_k)\sin(m)$. Again, by Proposition 1, we have

$$c\cos(m) \mp \lim_{k\to\infty} \sin(l_k)\sin(m) \geq c. \tag{8}$$

Since $\sin(m) \neq 0$,there are four possible cases:

- $\lim_{k\to\infty} \sin(l_k) = \sqrt{1-c^2}$, $\sin(m) > 0$;

- $\lim_{k\to\infty} \sin(l_k) = \sqrt{1-c^2}$, $\sin(m) < 0$;
- $\lim_{k\to\infty} \sin(l_k) = -\sqrt{1-c^2}$, $\sin(m) > 0$;
- $\lim_{k\to\infty} \sin(l_k) = -\sqrt{1-c^2}$, $\sin(m) < 0$.

Selecting the $-$ sign in (8) for the first and the last cases, and the $+$ sign in (8) for the two remaining cases, we have

$$c\cos(m) - \sqrt{1-c^2}\,|\sin(m)| \geq c, \forall m \in \mathbb{N}. \tag{9}$$

This implies $-\sqrt{1-c^2}\,|\sin(m)| \geq c(1-\cos(m))$. Dividing both sides by the positive number $|\sin(m)|$, we have

$$-\sqrt{1-c^2} \geq c\frac{1-\cos(m)}{|\sin(m)|} = c\left|\frac{1-\cos(m)}{\sin(m)}\right| = c|\tan(\frac{m}{2})|. \tag{10}$$

Since both sides of inequality (10) are non-positive, squaring gives $1-c^2 \leq c^2\tan^2(\frac{m}{2})$, which implies $1 \leq c^2(1+\tan^2(\frac{m}{2})) = c^2\sec^2(\frac{m}{2})$. Thus, we have $\cos^2(\frac{m}{2}) \leq c^2$. Taking the square-root, we obtain

$$\cos(\frac{m}{2}) \leq |\cos(\frac{m}{2})| \leq |c|, \forall m \in \mathbb{N}. \tag{11}$$

In particular, for $m = 2n_1, 2n_2, \ldots$, repeatedly applying (11) yields

$$\cos(\frac{2n_k}{2}) = \cos(n_k) \leq |c|, \forall k \in \mathbb{N}. \tag{12}$$

Taking limits as $k \to \infty$ in (12), we conclude that $1 = a \leq |c|$. Thus $c = -1$, completing the proof. □

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