Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 21, Number 1 (2025), pp. 1-14

http://www.ripublication.com/gjpam.htm

Certain VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$-Cordial Thorn Graphs

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Abstract

Let G be a graph. Let V be an inner product space with basis S. We denote the inner product of the vectors ω_1 and ω_2 by $< \omega_1, \omega_2 >$. Let $\chi : V(G) \to S$ be a function. For edge $\omega_1\omega_2$ assign the label $< \omega_1, \omega_2 >$. Then χ is called a vector basis S-cordial labeling of G (VB S-cordial labeling) if $|\chi_{\omega_1} - \chi_{\omega_2}| \leq 1$ and $|\delta_i - \delta_j| \leq 1$ where χ_{ω_i} denotes the number of vertices labeled with the vector ω_i and δ_i denotes the number of edges labeled with the scalar i. A graph which admits a VB S-cordial labeling is called a vector basis S-cordial graph (VB S-cordial graph). In this paper, we investigate the VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling behavior of some thorn graphs like the thorn rod, thorn path, thorn ring, thorn star, thorn multi star, bintang graph, banana tree and coconut tree.

Keywords. Thorn rod, Thorn path, Thorn ring, Thorn star.
2020 Mathematics Subject Classification. 05C38, 05C78.

1. INTRODUCTION

In this paper, a graph $G = (V, E)$ is finite, simple, connected and undirected. The concept of graph labeling was introduced by Rosa [15] in 1967. Congruence labeling of path, cycle, friendship graph and star graph were discussed in [11]. Hemalatha and Gokilamani [7] have carried out a study on the balanced rank distribution labeling of crown and wheel graphs. Uma Maheswari and Purnalakshmi [17] have proved that the banana tree, olive tree, shrub, jelly fish and tadpole graphs are arithmetic number graphs. Oblong sum labeling of some special graphs was discussed in [14].

Results on some topological indices were brought out by Pawar and Soner [12]. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $P = \{p_1, p_2, \ldots, p_n\}$ be n-tuples of nonnegative integers. The concept of thorn graphs was introduced by Gutman [6]. The thorn graph G_p [6] is the graph obtained by attaching p_i pendent vertices (terminal vertices or vertices of degree one) to the vertex v_i of $G, 1 \leq i \leq n$. The p_i pendent vertices attached to the vertex v_i by $V_i, 1 \leq i \leq n$. Vijaya Lakshmi and Parvathi [18] have focused attention on thorn graphs from the perspective of topological indices. The idea of cordial labeling was first introduced by I. Cahit [3]. The forcing semi-H cordial numbers of certain graphs was considered in [9]. Bosmia [2] has dealt with cordial labelling arising out of graph operations on bistars. Aljouiee [1] has worked on a prime cordial labeling of the closed helm and P_n^2. Sarah Surya et al. [16] have proved that the banana tree, olive tree, jewel graph, Jahangir graph and crown graph are integer cordial. Mean square cordial labeling on star related graphs was examined in [4]. For the terminologies and different notations of graph theory, we refer the book of Harary [10] and of algebra, we refer the book of Herstein [8]. Throughout this paper, we consider the inner product space R^n and the standard inner product $< \omega_1, \omega_2 >= a_1b_1 + a_2b_2 + \cdots + a_nb_n$ where $\omega_1 = (a_1, a_2, \ldots, a_n), \omega_2 = (b_1, b_2, \ldots, b_n)$, $a_i, b_i \in R$. For a survey on graph labeling, we refer the book of Gallian [5]. We have introduced the new labeling technique for graphs called VB S-cordial labeling in [13] and same labeling technique verified for various graphs like path, cycle, comb, star, complete graph, etc., for the bases $S = \{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$ and $S = \{(1,1,1), (1,1,0), (1,0,0)\}$ in [13]. In the present paper, we investigate the VB $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial labeling behavior of some thorn graphs like the thorn rod, thorn path, thorn ring, thorn star, thorn multi star, bintang graph, banana tree and coconut tree.

2. MAIN RESULTS

Theorem 2.1. *The thorn rod $P_{n,m}$ is a VB $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial graph for all n, m.*

Proof. A thorn rod $P_{n,m}$ is a graph that is obtained by adding $m-1$ pendent vertices to each terminal vertex of the path P_n. Let $P_n : u_1u_2\ldots u_n$ be a path and $v_1v_2\ldots v_{m-1}$ be the adjacent vertices of u_1, $w_1w_2\ldots w_{m-1}$ be the adjacent vertices of u_n. Let $V(P_{n,m}) = \{u_i, v_j, w_j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m-1\}$ and $E(P_{n,m}) = \{u_iu_{i+1}, u_1v_j, u_nw_j \mid 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m-1\}$. Note that $p = |V(P_{n,m})| = 2m+n-2$ and $q = |E(P_{n,m})| = 2m+n-3$. Assign the vectors in the following order $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{m-1}, w_1, w_2, \ldots, w_{m-1}$.

Case (i): $p \equiv 0 \pmod 4$
Then $p = 4t$. Allocate the vector $(1,1,1,1)$ to the first t vertices. We allocate the vector $(1,1,1,0)$ to the next t vertices. Next, allocate the vector $(1,1,0,0)$ to the next t vertices. Moreover, allocate the vector $(1,0,0,0)$ to the last t vertices.
Case (ii): $p \equiv 1 \pmod 4$
Note that $p = 4t+1$. Then, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We now allocate the vector $(1,1,1,0)$ to the next t vertices. Also, allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the last t vertices.
Case (iii): $p \equiv 2 \pmod 4$
We have $p = 4t+2$. Now, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. Then, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.
Case (iv): $p \equiv 3 \pmod 4$
We see that $p = 4t+3$. Next, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. So allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Finally, allocate the vector $(1,0,0,0)$ to the next t vertices.
Clearly the above labeling technique is a VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$-cordial labeling of the thorn rod $P_{n,m}$. □

Example 2.1. *A VB* $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$*-cordial labeling of the thorn rod* $P_{7,5}$ *is given in figure 1.*

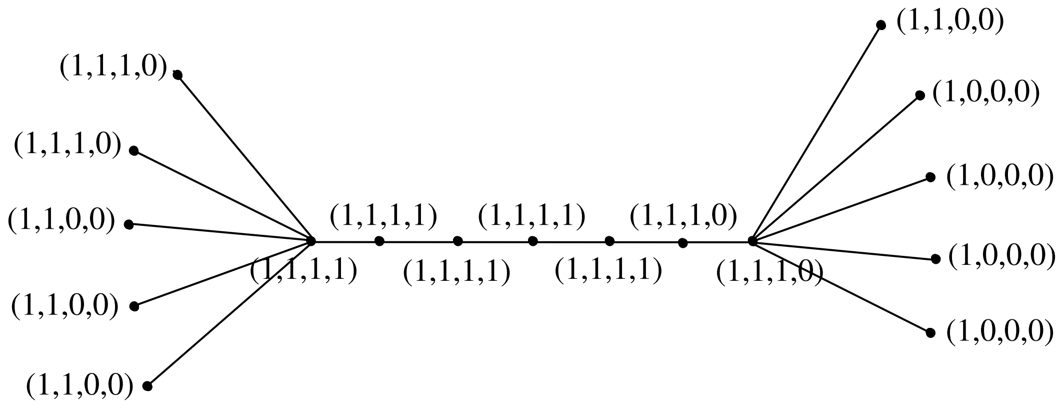

Figure 1: VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of $P_{7,5}$.

Theorem 2.2. *The thorn path* $P_{m,n,n}$ *is a VB* $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$*-cordial graph for all* n, m.

Proof. A thorn path $P_{m,n,n}$ is a graph formed from a path P_m by attaching n pendent vertices to its terminal and non-terminal vertices. Let $P_m : u_1u_2\dots u_m$ be a path and u_{1i} be the vertex the adjacent to $u_1(1 \leq i \leq n)$, u_{2i} be the vertex the adjacent to $u_2(1 \leq i \leq n)$ and so on. Let $V(P_{m,n,n}) = \{u_i, u_{ij} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(P_{m,n,n}) = \{u_iu_{i+1}, u_iu_{ij} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{u_iu_{i+1} \mid 1 \leq i \leq m-1\}$ Then $p = |V(P_{m,n,n})| = m(n+1)$ and $q = |E(P_{m,n,n})| = mn + m - 1$. We assign the vectors to $P_{m,n,n}$ in the following order $u_1, u_2, \dots, u_m, u_{11}, u_{12}, \dots, u_{1n}, u_{21}, u_{22}, \dots, u_{2n}, u_{31}, u_{32}, \dots, u_{3n}, \dots, u_{m1}, u_{m2}, \dots, u_{mn}$.

Case (i): Consider the thorn path $P_{m,2,2}, P_{m,4,4}, P_{m,6,6}, \dots$

Subcase (i): $p \equiv 0 \pmod 4$

Note that $p = 4t$. Then, allocate the vector $(1,1,1,1)$ to the first t vertices. We now allocate the vector $(1,1,1,0)$ to the next t vertices. Also, allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (ii): $p \equiv 1 \pmod 4$

We have $p = 4t + 1$. Now, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next t vertices. Then, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (iii): $p \equiv 2 \pmod 4$

We see that $p = 4t + 2$. Next, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. So allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the next t vertices.

Subcase (iv): $p \equiv 3 \pmod 4$

Then $p = 4t + 3$. allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. Next, allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Moreover, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (ii): Consider the thorn path $P_{m,3,3}, P_{m,7,7}, P_{m,11,11}, \dots$

Then $p \equiv 0 \pmod 4$. Note that $p = 4t$. Also, assign the vector to the vertices as in subcase (i) of case (i).

Case (iii): Consider the thorn path $P_{m,1,1}, P_{m,5,5}, P_{m,9,9}, \dots$

Subcase (i): $p \equiv 0 \pmod 4$

Then, assign the vector to the vertices as in subcase (i) of case (i).

Subcase (ii): $p \equiv 2 \pmod 4$

Also, assign the vector to the vertices as in subcase (iii) of case (i).

Hence the above labeling method is a VB $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial labeling of the thorn path graph $P_{m,n,n}$. $\square$

Example 2.2. *A VB* $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$*-cordial labeling of* *the thorn path graph* $P_{4,4,4}$ *is given in figure 2.*

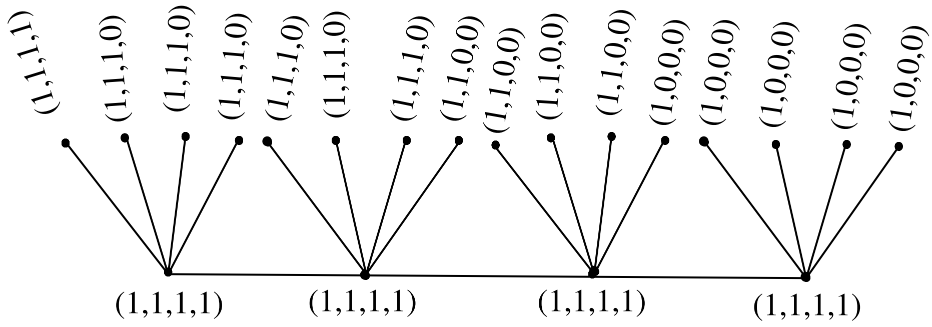

Figure 2: VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of $P_{4,4,4}$.

Theorem 2.3. *The thorn ring* $C_{m,n+2}$ *is a VB* $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$*-cordial graph for all* $n,m \geq 3$ *and for* $n = 1,2$*, except* $m(n+1) \equiv 0 \pmod 4$.

Proof. If each vertex of a cycle graph C_m and a thorn of length n is attached then it is called thorn ring $C_{m,n+2}$. Let $V(C_{m,n+2}) = \{u_i, u_{ij} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(C_{m,n+2}) = \{u_j u_{ij} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{u_i u_{i+1}, u_1 u_m \mid 1 \leq i \leq m-1\}$ Then $p = |V(C_{m,n+2})| = m(n+1)$ and $q = |E(C_{m,n+2})| = m(n+1)$. Assign the vectors to $C_{m,n+2}$ in the following order $u_1, u_2, \ldots, u_m, u_{11}, u_{12}, \ldots, u_{1n}, u_{21}, u_{22}, \ldots, u_{2n}, u_{31}, u_{32}, \ldots, u_{3n}, \ldots, u_{m1}, u_{m2}, \ldots, u_{mn}$.

Case (i): Consider the thorn ring $C_{m,3}$

Note that $C_{m,3}$ is a crown graph.

Subcase (i): $p \equiv 0 \pmod 4$

Then $p = 4t$. To set the edge label 4, the vector $(1,1,1,1)$ should be assigned to the consecutive vertices of the cycles, the maximum number of edges with label 4 is $t-1$, we get a contradiction.

Subcase (ii): $p \equiv 2 \pmod 4$

We have $p = 4t+2$. Now, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next t vertices. Then, allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (ii): Consider the thorn ring $C_{m,4}$

Subcase (i): $p \equiv 0 \pmod 4$

The proof is same as in subcase (i) of case (i).

Subcase (ii): $p \equiv 1 \pmod 4$

We see that $p = 4t + 1$. Next, allocate the vector $(1, 1, 1, 1)$ to the first $t + 1$ vertices. Also allocate the vector $(1, 1, 1, 0)$ to the next t vertices. Allocate the vector $(1, 1, 0, 0)$ to the next t vertices. Finally, allocate the vector $(1, 0, 0, 0)$ to the next t vertices.

Subcase (iii): $p \equiv 2 \pmod 4$

Then $p = 4t + 2$. Allocate the vector $(1, 1, 1, 1)$ to the first $t + 1$ vertices. We allocate the vector $(1, 1, 1, 0)$ to the next $t + 1$ vertices. Next, allocate the vector $(1, 1, 0, 0)$ to the next t vertices. Moreover, allocate the vector $(1, 0, 0, 0)$ to the last t vertices.

Subcase (iv): $p \equiv 3 \pmod 4$

Then $p = 4t + 3$. Now, allocate the vector $(1, 1, 1, 1)$ to the first $t + 1$ vertices. We allocate the vector $(1, 1, 1, 0)$ to the next $t + 1$ vertices. Next, allocate the vector $(1, 1, 0, 0)$ to the next $t + 1$ vertices. Finally, allocate the vector $(1, 0, 0, 0)$ to the last t vertices.

Case (iii): We consider the thorn ring $C_{m,5}, C_{m,9}, C_{m,13}, \ldots$

Then $p \equiv 0 \pmod 4$. Note that $p = 4t$. Now, allocate the vector $(1, 1, 1, 1)$ to the first t vertices. We allocate the vector $(1, 1, 1, 0)$ to the next t vertices. Also, allocate the vector $(1, 1, 0, 0)$ to the next t vertices. Further, allocate the vector $(1, 0, 0, 0)$ to the last t vertices.

Case (iv): Consider the thorn ring $C_{m,6}, C_{m,8}, C_{m,10}, C_{m,12}, \ldots$

Subcase (i): $p \equiv 0 \pmod 4$

Also, assign the vector to the vertices as in subcase (i) of case (iii).

Subcase (ii): $p \equiv 1 \pmod 4$

Now, assign the vector to the vertices as in subcase (ii) of case (ii).

Subcase (iii): $p \equiv 2 \pmod 4$

Assign the vector to the vertices as in subcase (iii) of case (ii).

Subcase (iv): $p \equiv 3 \pmod 4$

Then assign the vector to the vertices as in subcase (iv) of case (ii).

Case (v): We consider the thorn ring $C_{m,7}, C_{m,11}, C_{m,15}, C_{m,19}, \ldots$

Subcase (i): $p \equiv 0 \pmod 4$

Then, assign the vector to the vertices as in subcase (i) of case (iii).

Subcase (ii): $p \equiv 2 \pmod 4$

Also, assign the vector to the vertices as in subcase (iii) of case (iv).

Thus the above labeling pattern is a VB $\{(1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 0),$
$(1, 0, 0, 0)\}$-cordial labeling of the thorn ring graph $C_{m,n+2}$. □

Example 2.3. *A VB* $\{(1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0)\}$*-cordial labeling of the thorn ring graph* $C_{5,6}$ *is given in figure 3.*

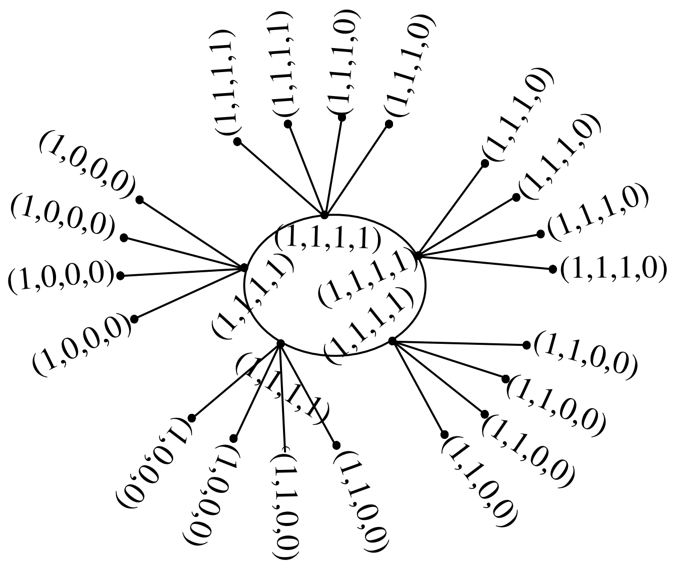

Figure 3: VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of $C_{5,6}$.

Theorem 2.4. *The thorn star* $S_{m,n,n}$ *is a VB* $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$*-cordial graph for all* m,n.

Proof. The thorn star $S_{m,n,n}$ is generalized from the star S_m joining n pendant vertices to the central vertex u and by joining n pendant vertices to its end vertices. Let $V(S_{m,n,n})=\{u,u_i,u_{ij}\mid 1\leq i\leq m \text{ and } 1\leq j\leq n\}$ and $E(S_{m,n,n})=\{uu_i,u_iu_{ij}\mid 1\leq i\leq m \text{ and } 1\leq j\leq n\}$. Note that $p=|V(S_{m,n,n})|=m(n+1)+1$ and $q=|E(S_{m,n,n})|=m(n+1)$. Assign the vectors to $S_{m,n,n}$ in the following order $u_1,u_2,\ldots,u_m,u_{11},u_{12},\ldots,u_{1n},u_{21},u_{22},\ldots,u_{2n},u_{31},u_{32},\ldots,u_{3n},\ldots,u_{m1},$ $u_{m2},\ldots,u_{mn}$.

Case (i): Consider the thorn star $S_{3,n,n},S_{7,n,n},S_{11,n,n},\ldots$

Then $p=4t$. Allocates the vector $(1,1,1,1)$ to the first t vertices. We allocate the vector $(1,1,1,0)$ to the next t vertices. Next, allocate the vector $(1,1,0,0)$ to the next t vertices. Moreover, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (ii): We consider the thorn star $S_{4,n,n},S_{6,n,n},S_{8,n,n},S_{10,n,n}\cdots$

Subcase (i): $p\equiv 0 \pmod 4$

Then, assign the vector to the vertices as in case (i).

Subcase (ii): $p\equiv 1 \pmod 4$

Note that $p=4t+1$. Then, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We allocate the vector $(1,1,1,0)$ to the next t vertices. Also, allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (iii): $p\equiv 2 \pmod 4$

We have $p = 4t + 2$. Now, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. Then, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (iv): $p \equiv 3 \pmod 4$

We see that $p = 4t + 3$. Next, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. So allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Finally, allocate the vector $(1,0,0,0)$ to the next t vertices.

Case (iii): Consider the thorn star $S_{5,n,n}, S_{9,n,n}, S_{13,n,n}, S_{17,n,n}, \dots$

Subcase (i): $p \equiv 0 \pmod 4$

Now, assign the vector to the vertices as in case (i).

Subcase (ii): $p \equiv 2 \pmod 4$

Also, assign the vector to the vertices as in subcase (iii) of case (ii).

Thus the above labeling method is a VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$-cordial labeling of the thorn star graph $S_{m,n,n}$. □

Theorem 2.5. *The thorn multi star $S_n(1,2,\dots,m)$ is a VB $\{(1,1,1,1),(1,1,1,0),$ $(1,1,0,0),(1,0,0,0)\}$-cordial graph for all m,n.*

Proof. Let $V(S_n(1,2,\dots,m)) = \{u,u_i,u_{ij} \mid 1 \le i \le n \text{ and } 1 \le j \le m\}$ and $E(S_n(1,2,\dots,m)) = \{uu_i,u_iu_{ij} \mid 1 \le i \le n \text{ and } 1 \le j \le m\}$. Note that $p = |V(S_n(1,2,\dots,m))| = m(n+1)+1$ and $q = |E(S_n(1,2,\dots,m))| = m(n+1)$. We assign the vectors to $S_n(1,2,\dots,m)$ in the following order $u_1,u_2,\dots,u_n,u_{11},u_{12},\dots,u_{1m},u_{21},u_{22},\dots,u_{2m},u_{31},u_{32},\dots,u_{3m},\dots,u_{n1},u_{n2},$ $\dots,u_{nm}$.

Case (i): Consider the thorn multi star $S_n(1), S_n(1,2,3,4,5), S_n(1,2,\dots,9),\dots$

Subcase (i): $p \equiv 1 \pmod 4$

Note that $p = 4t + 1$. Then, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We now allocate the vector $(1,1,1,0)$ to the next t vertices. Also, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (ii): $p \equiv 3 \pmod 4$

We have $p = 4t+3$. So allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. Then, allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Moreover, allocate the vector $(1,0,0,0)$ to the next t vertices.

Case (ii): Consider the thorn multi star $S_n(1,2), S_n(1,2,3,4), S_n(1,2,\dots,6),\dots$

Subcase (i): $p \equiv 0 \pmod 4$

Note that $p = 4t$. Then, allocate the vector $(1,1,1,1)$ to the first t vertices. We allocate the vector $(1,1,1,0)$ to the next t vertices. So allocate the vector $(1,1,0,0)$ to the next

t vertices. Allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (ii): $p \equiv 1 \pmod 4$

Now, assign the vector to the vertices as in subcase (i) of case (i).

Subcase (iii): $p \equiv 2 \pmod 4$

We obtain $p = 4t + 2$. Now allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. We assign the vector $(1,1,0,0)$ to the next t vertices. Then, assign the vector $(1,0,0,0)$ to the next t vertices.

Subcase (iv): $p \equiv 3 \pmod 4$

We assign the vector to the vertices as in subcase (ii) of case (i).

Case (iii): Consider the thorn multi star $S_n(1,2,3), S_n(1,2,3\ldots,7),\ldots$

Then $p \equiv 0 \pmod 4$. We allocate the vector to the vertices as in subcase (i) of case (i). Hence the above labeling method is a VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$-cordial labeling of the thorn multi star graph $S_n(1,2,\ldots,m)$. □

Example 2.4. *A VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of the thorn multi star graph $S_4(1,2,3,4)$ is given in figure 4.*

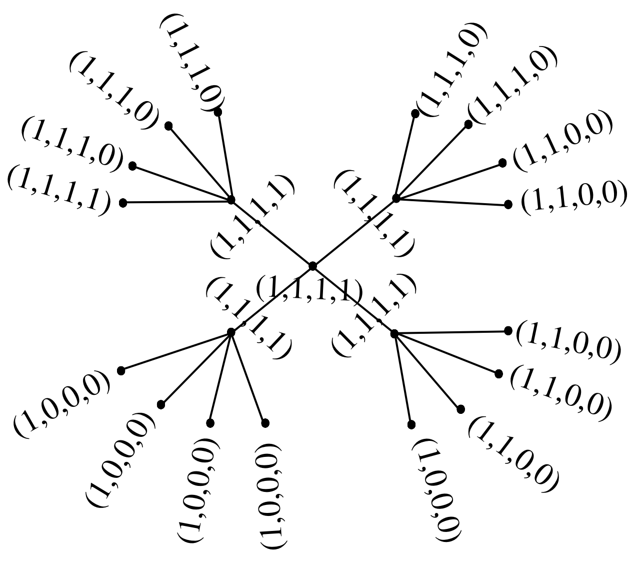

Figure 4: VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of $S_4(1,2,3,4)$.

Theorem 2.6. *The Bintang graph $BG_{n,m}$ is a VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$-cordial graph for all n,m.*

Proof. The Bintang graph $BG_{n,m}$ is constructed from two cycle graphs where one vertex of the cycle becomes the center of the graph while the other vertices are given

graph star S_m. Let $V(BG_{n,m}) = \{u_i \mid 1 \leq i \leq n\} \cup \{v_i, u_{ij} \mid 2 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ and $E(BG_{n,m}) = \{u_iu_{i+1}, u_nu_1 \mid 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}, v_nv_1 \mid 2 \leq i \leq n-1\} \cup \{u_{ij}, v_{ij} \mid 2 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. Then $p = |V(BG_{n,m})| = 2(n-1)(m+1)+1$ and $q = |E(BG_{n,m})| = 2n + 2(n-1)m$. Assign the vectors in the following order $u_1, u_2, \ldots, u_n, v_2, v_3, \ldots, v_n, u_{11}, u_{12}, \ldots, u_{1m}$,
$u_{21}, u_{22}, \ldots, u_{2m}, u_{31}, u_{32}, \ldots, u_{3m}, \ldots, u_{n1}, u_{n2}, \ldots, u_{nm}$.

Case (i): n is odd

Then $p \equiv 1 \pmod 4$. We see that $p = 4t+1$. We consider the two subcases.

Subcase (i): $m = 1, 2$

Allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We allocate the vector $(1,1,1,0)$ to the next t vertices. Next, allocate the vector $(1,1,0,0)$ to the next t vertices. Moreover, allocate the vector $(1,0,0,0)$ to the last t vertices.

Subcase (ii): $m \geq 3$

Note that $p = 4t+1$. Then, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We now allocate the vector $(1,1,1,0)$ to the next t vertices. Also, allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the last $t+1$ vertices.

Case (ii): n is odd

Then $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$. We see that $p = 4t+1$ or $p = 4t+3$. We consider the two subcases.

Subcase (i): $m = 1, 2$

When $p \equiv 1 \pmod 4$. Then $p = 4t+1$. Now, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next t vertices. Then, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.

When $p \equiv 3 \pmod 4$. Then $p = 4t+3$. So assign the vector $(1,1,1,1)$ to the first $t+1$ vertices. Assign the vector $(1,1,1,0)$ to the next t vertices. We allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Moreover, assign the vector $(1,0,0,0)$ to the last $t+1$ vertices.

Subcase (ii): $m \geq 3$

Also, allocate the vector $(1,1,1,1)$ to the first t vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. So allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Finally, allocate the vector $(1,0,0,0)$ to the next t vertices.

Then clearly the above labeling technique is a VB $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial labeling of the Bintang graph $BG_{n,m}$. □

Theorem 2.7. *The banana tree $BT_{n,k}$ is a VB $\{(1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0)\}$-cordial graph for all n, k.*

Proof. The banana tree $BT_{n,k}$ is a graph constructed by connecting a single leaf from n distinct copies of a k-star graph with a single vertex distinct from the star graphs. Let $V(BT_{n,k}) = \{u, u_i, v_i, u_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq k\}$ and $E(BT_{n,k}) = \{uu_i, u_iv_i, v_iu_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. Note that $p = |V(BT_{n,k})| = n(k+2)+1$ and $q = |E(BT_{n,k})| = n(k+2)$. Then we assign the vectors to $BT_{n,k}$ in the following order $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, u_{11}, u_{12},$
$\ldots, u_{1k}, u_{21}, u_{22}, \ldots, u_{2k}, \ldots, u_{n1}, u_{n2}, \ldots, u_{nk}$.

Case (i): Consider the banana tree $BT_{2,k}, BT_{6,k}, BT_{10,k}, BT_{14,k}, \ldots$

Subcase (i): $p \equiv 1 \pmod 4$

Then $p = 4t+1$. Allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We assign the vector $(1,1,1,0)$ to the next t vertices. Next, assign the vector $(1,1,0,0)$ to the next t vertices. Moreover, assign the vector $(1,0,0,0)$ to the next t vertices.

Subcase (ii): $p \equiv 3 \pmod 4$

Note that $p = 4t+3$. Then, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We now allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. Also, allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Finally, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (ii): We consider the banana tree $BT_{3,k}, BT_{5,k}, BT_{7,k}, BT_{9,k}, \ldots$

Subcase (i): $p \equiv 0 \pmod 4$

We have $p = 4t$. Now, allocate the vector $(1,1,1,1)$ to the first t vertices. Allocate the vector $(1,1,1,0)$ to the next t vertices. Then, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the next t vertices.

Subcase (ii): $p \equiv 1 \pmod 4$

Then $p = 4t+1$. We assign the vector to the vertices as in subcase (i) of case (i).

Subcase (iii): $p \equiv 2 \pmod 4$

We see that $p = 4k+2$. Next, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. So allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the next t vertices.

Subcase (iv): $p \equiv 3 \pmod 4$

Then $p = 4t+1$. We assign the vector to the vertices as in subcase (ii) of case (i).

Case (iii): Consider the banana tree $BT_{4,k}, BT_{8,k}, BT_{12,k}, BT_{16,k}, \ldots$

Then clearly, $p \equiv 1 \pmod 4$. We see that $p = 4t+3$. We assign the vector to the vertices as in subcase (i) of case (i).

Hence the above labeling technique is a VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$
$(1,0,0,0)\}$-cordial labeling of the banana tree $BT_{n,k}$. □

Theorem 2.8. *The coconut tree* $CT_{m,n}$ *is a VB* $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$
$(1,0,0,0)\}$*-cordial graph for all* n, m.

Proof. A coconut tree $CT_{m,n}$ is a graph that is created by attaching new pendent edges to a path at one of its end vertices. Let $V(CT_{m,n}) = \{u_i, v_j \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(CT_{m,n}) = \{u_iu_{i+1}, u_1v_j \mid 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq n\}$. Note that $p = |V(CT_{m,n})| = m+n$ and $q = |E(CT_{m,n})| = m+n-1$. Assign the vectors in the following order $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$.

Case (i): $p \equiv 0 \pmod 4$

Then $p = 4t$. Allocate the vector $(1,1,1,1)$ to the first t vertices. We allocate the vector $(1,1,1,0)$ to the next t vertices. Next, allocate the vector $(1,1,0,0)$ to the next t vertices. Moreover, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (ii): $p \equiv 1 \pmod 4$

Note that $p = 4t+1$. Then, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. We now allocate the vector $(1,1,1,0)$ to the next t vertices. Also, allocate the vector $(1,1,0,0)$ to the next t vertices. Finally, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (iii): $p \equiv 2 \pmod 4$

We have $p = 4t+2$. Now, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. Then, allocate the vector $(1,1,0,0)$ to the next t vertices. Further, allocate the vector $(1,0,0,0)$ to the last t vertices.

Case (iv): $p \equiv 3 \pmod 4$

We see that $p = 4t+3$. Next, allocate the vector $(1,1,1,1)$ to the first $t+1$ vertices. Allocate the vector $(1,1,1,0)$ to the next $t+1$ vertices. So allocate the vector $(1,1,0,0)$ to the next $t+1$ vertices. Finally, allocate the vector $(1,0,0,0)$ to the next t vertices.

Clearly the above labeling technique is a VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$ $(1,0,0,0)\}$-cordial labeling of the coconut tree $CT_{m,n}$. □

Example 2.5. *A VB* $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$*-cordial labeling of the coconut tree* $CT_{7,4}$ *is given in figure 5.*

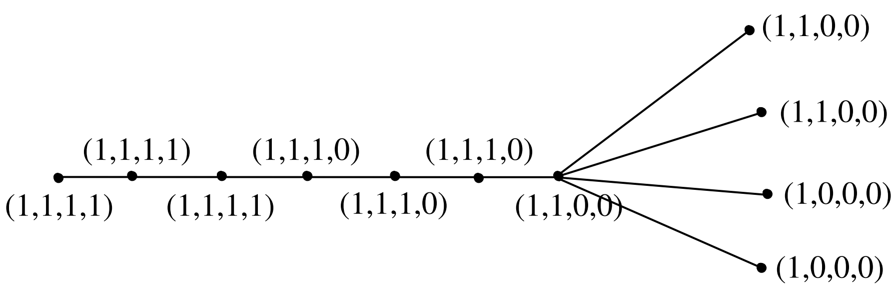

Figure 5: VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial labeling of $CT_{7,4}$.

3. CONCLUSION

In the present paper, we have investigated the VB $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),$
$(1,0,0,0)\}$-cordial labeling behavior of some thorn graphs such as the thorn rod, thorn path, thorn ring, thorn star, thorn multi star, bintang graph, banana tree and coconut tree. It would be interesting to continue this type of labelling by performing computations for more graph families.

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