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Common fixed point theorems for four self-maps satisfying E.A and (CLR) property in Fuzzy Metric Spaces

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Abstract

This paper aims to introduce the new concept of rational type four fuzzy contraction mappings in fuzzy metric spaces. We prove some fixed point theorems under rational type fuzzy contraction conditions for two pairs of maps in fuzzy metric spaces using compatibility and E.A property.

Keywords: Fixed point, fuzzy metric space, weakly compatible mapping.

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1. Introduction and Preliminaries

In 1965, the concept of fuzzy set was introduced by Zadeh [15] to represent vagueness in everyday life. In 1975, Kramosil and Michalek [7] invented new concept of fuzzy metric space which is generalization of Probabilistic metric spaces. After that George and Veeramani [4] modified the concept which is used in quantum particle Physics and Topology. George and Veeramani [4] introduced the concept of complete fuzzy metric space.

In particular the fuzzy distance between two equal points is 1 while in cases when one point is far from other one the fuzzy distance between them is closed to 0. The degree of closeness of two points in a fuzzy metric space corresponds to the probability of coincidence of these points.

In 1986, Jungek [6] introduced weakly compatible mappings. In 2002, Aamri et. al. [1] generalized the concept of non-compatibility and gave the new contraction by defining E.A property.

Definition 1.1. [15] Let X be any set. A fuzzy set A of X is a function from domain X and values in $[0,1]$.

Example 1.2. Consider $U = \{a, b, c, d\}$ and $A: U \rightarrow [0,1]$ define as $A(a) = 0$, $A(b) = 0.5$, $A(c) = 0.2$ *and* $A(d) = 1$. Then A is a fuzzy set on U. This fuzzy set also can be written as follows:

$$A = \{(a, 0)\ (b, 0.5)\ (c, 0.2)\ (d, 1)\}.$$

George, A. and Veeramani [4] gave the concept of fuzzy metric space.

Definition 1.3. [4] A 3- tuple $(X, M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following properties:

(FMS1) $M(p, q, t) > 0$,

(FMS 2) $M(p, q, t) = 1$ if and only if $p = q$;

(FMS 3) $M(p, q, t) = M(q, p, t)$;

(FMS 4) $M(p, q, t) * M(q, r, s) \leq M(p, r, t + s)$;

(FMS 5) $M(p, q, .): (0, \infty) \rightarrow (0,1]$ is continuous,

for all $p, q, r \in X$ and $s, t > 0$.

Then M is called a fuzzy metric on P. The function $M(p, q, t)$ denote the degree of nearness between p and q with respect to t.

Definition 1.4. [3] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is said a continuous t-norm if it satisfies the following conditions:

(i) is associative and commutative,

(ii) is continuous,

(iii) $a * 1 = a$ for all $a \in [0,1]$ and

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Example 1.5. $a * b = ab$ for $a, b \in [0,1]$ is a continuous t-norm.

Definition 1.6. [7] A binary operation $\Delta : [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t co-norm if it satisfies the following conditions:

(i) Δ is commutative and associative;

(ii) Δ is continuous;

(iii) $a\Delta\, 0 = a$ for all $a \in [0,1]$;

(iv) $a\Delta\, b = c\Delta\, d$ whenever $a \geq c$ and $b \geq d$, for each $a, b, c, d \in [0, 1]$.

Example 1.7. A binary operation $\Delta : [0,1] \times [0,1] \to [0,1]$ such that $a\Delta\, b = min(a + b, 1)$ is a continuous t co-norm.

Definition 1.8. [8] Let $(X, M,*)$ be a fuzzy metric space, $x \in X$ and $\phi \neq A \subseteq X$. We define $D(x, A, t) = \sup\{M\,(x, y, t): y \in A\}\,(t > 0)$ then $D(x, A, t)$ is called a degree of closeness of x to A at t.

Definition 1.9. [8] A topological space is called a (topologically complete) fuzzy metrizable space if there exists a (topologically complete) fuzzy metric inducing the given topology on it.

Definition 1.10. [4] Let $(X, M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) if for $\lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$.

Definition 1.11. [4] Let M be a fuzzy metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p,}\ x_n\,,\ t) = 1$ for all $p > 0$ and $t > 0$.

Definition 1.12. [10] Two mappings S and T of a fuzzy metric space $(X, M,*)$ into itself are said to be compatible maps if $\lim_{n\to\infty} M(STx_n\,, TSx_n, \varepsilon) = 1$ for all $\varepsilon > 0$ where $\{x_n\}$

$\in X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \omega \,\epsilon\, X$.

Definition 1.13. The self-mappings S and T of a fuzzy metric space $(X, M,*)$ are said to be commuting if $M(STx, TSx, t) = 1$ for all $x \in X$.

Definition 1.14. [13] The self-mappings S and T of a fuzzy metric space$(X, M,*)$ are said to be weakly commuting if $M(STx, TSx, t) \geq M(Sx, Tx, t)$ for each $x \in X$ and $t > 0$.

Definition 1.15. [6] The self-mappings S and T of a fuzzy metric space $(X, M,*)$ are said to be compatible if and only if $\lim_{n\to\infty} M(STx_n\,, TSx_{n\,,}t) = 1$ whenever $\{x_n\}$ is a

sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some x in X and $t > 0$.

Definition 1.16. [9] The self- mappings S and T of a fuzzy metric space $(X, M,*)$ are said to be compatible of type (K) iff $\lim_{n\to\infty} \mathrm{M}(\mathrm{SSx_n}, \mathrm{Tx}, \mathrm{t}) = 1$ and $\lim_{n\to\infty} \mathrm{M}(\mathrm{TTx_n}, \mathrm{Sx}, \mathrm{t}) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$, for some x in X and $t > 0$.

Definition 1.17. [1] Let (X, d) be a metric space. Two self- mappings S , T satisfy E.A property if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 1.18. [1] Two self-mappings S and T are said to satisfy the (CLR) property if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x$ for some $x \epsilon X$.

Definition 1.19. [1] Let f and g be two self-maps of a fuzzy metric space then they are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \epsilon X$.

Similarly, the property (CLR_T) and the property (CLR_S) hold if in the above definition the mapping $g: X \to X$ has been replaced by the mapping $T: X \to X$ and $S: X \to X$.

Definition 1.20. [14] Let A, B, S, T be the four self- mappings defined on a symmetric fuzzy metric space $(X, M,*)$. Then the pairs (A, S) and (B, T) are said to satisfy common limit range property (with respect to S and T) if there exist two sequences $\{a_n\}$ and $\{b_n\}$ in X such that $\lim_{n\to\infty} Aa_n = \lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Bb_n = \lim_{n\to\infty} Tb_n = t$ with $t = Su = Tw$ for some $t, u, w \in X$.

Preposition 1.21: Let A, B, S and T are four self- mappings of a fuzzy metric space $(X, M,*)$ that satisfy the requirements

1. $S(X) \subseteq A(X)$ and $T(X) \subseteq B(X)$.
2. Pairs (S, A) and (T, B) are commuting.
3. One of S, A, T and B is continuous.
4. $M(Sx, Ty, t) \leq c\, \lambda(x, y)$ where
   $\lambda(x, y) = \max\{M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t)\}$ for all $x, y \in X$ and $0 \leq c \leq 1$.
5. X is complete.

Then A, B, S and T have a unique common fixed point.

2. Main Results

In this section, we prove theorems for four self-maps satisfying E.A and (CLR) property in fuzzy metric spaces.

Theorem 2.1. Let A, B, S and T be self-mappings of a fuzzy metric space $(X, M,*)$ satisfying

2.1.1: $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$.

2.1.2: $M(Sx, Ty, t) \leq \phi(\lambda(x, y))$,

where ϕ is an upper semi continuous, contractive modulus and $\lambda(x, y) = \max\{M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), \frac{(M(Ax,Ty,t)+M(By,Sx,t))}{2}\}$.

2.1.3: Pairs (S, A) and (T, B) are weakly compatible.

2.1.4: Pairs (S, A) and (T, B) satisfy the E.A property.

If any one of AX, BX, SX and TX is a complete subspace of X then A, B, S and T have a unique common fixed point.

Proof: Since pair (S, A) satisfies the E.A property then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = w$ for some w in X.

Since $S(X) \subseteq B(X)$, then there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$.

Hence, $\lim_{n\to\infty} By_n = w$.

We will show that $\lim_{n\to\infty} Ty_n = w$.

Let $\lim_{n\to\infty} Ty_n = r$.

From (2.1.2), we have $M(Sx_n, Ty_n, t) \leq \phi\big(\lambda(x_n, y_n)\big)$.

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(Sx_n, Ty_n, t) \leq \phi \lim_{n\to\infty}\big(\lambda(x_n, y_n)\big), \qquad \textbf{(2.1)}$$

Where

$$\lim_{n\to\infty}\big(\lambda(x_n, y_n)\big)$$
$$= \lim_{n\to\infty}\left(\max\left\{M(Ax_n, By_n, t), M(Ax_n, Bx_n, t), M(By_n, Ty_n, t), \frac{1}{2}[M(Ax_n, Ty_n, t)\right.\right.$$
$$\left.\left. + M(By_n, Sx_n, t)]\right\}\right) = \max\left\{M(w, w, t), M(w, w, t), M(w, r, t), \frac{1}{2}[M(w, r, t) + M(w, w, t)]\right\}$$
$$= M(w, r, t).$$

Thus from (2.1.2), we have $M(w, r, t) \leq \phi\big(M(w, r, t)\big) < M(w, r, t)$, a contradiction.

If $w \neq r$, then $M(w,r,t) > 0$ and hence a contractive modulus $\phi\big(M(w,r,t)\big) < M(w,r,t)$.

Therefore, $w = r$.

Hence, $Ty_n = w$.

Suppose that QX is complete subspace of X.

Then, $w = Qu$ for some u in X.

Subsequently, we have

$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = w = Bu.$

Now we will prove that $Tu = Bu$.

Let us assume that $Tu \neq Bu$.

Taking limit as $n \to \infty$ in (2.1.2), we get

$$\lim_{n\to\infty} M(Sx_n, Tu, t) \leq \phi \lim_{n\to\infty} (\lambda\,(x_n, u)), \quad \textbf{(2.2)}$$

Where

$\lim_{n\to\infty} (\lambda\,(x_n, u)) =$
$\lim_{n\to\infty} (\max\{M(\ Ax_n, Bu, t), M(Ax_n, Sx_n, t), M(Bu, Tu, t), \frac{1}{2}[M(Ax_n, Tu, t) +$
$M(Bu, Sx_n, t)]\}).$

$= \max(M(w,w,t), M(w,w,t), M(w,Tu,t), \dfrac{1}{2}[M(w,Tu,t) + M(w,w,t)]) = M(w,Tu,t).$

Thus from (2.1.2), we get $M(w,Tu,t) \leq \phi\big(M(w,Tu,t)\big) < M(w,Tu,t)$,

a contradiction.

Therefore, $Tu = Bu = w$.

Since, B and T are weakly compatible, therefore $BTu = TBu$.

So, $TTu = TBu = BTu = BBu$.

Since, T(X) ⊆ A(X), there exists $l \in X$ such that

$Tu = Al$.

Now, we prove that $Al = Sl$.

Let us assume that $Al \neq Sl$.

Taking limit as $n \to \infty$ in (2.1.2), we get

$\lim_{n\to\infty} M(Sx_n, Ty_n, t) \leq \phi \lim_{n\to\infty} (\lambda\,(x_n, y_n)).$

$$M(Sl, Tl, t) \leq\ \phi\big(\lambda(l,u)\big), \quad \textbf{(2.3)}$$

where

$\lim_{n\to\infty} \lambda(l,u) = \max\{M(Al,Bu,t), M(Al,Sl,t), M(Bu,Tu,t), \frac{1}{2}[M(Al,Tu,t) + M(Bu,Sl,t)]\} = M(Al,Sl,t) = M(Tu,Sl,t)$.

Then using (2.3), we get

$M(Sl,Tu,t) \leq \phi\big(M(Tu,Sl,t)\big)$,

a contradiction.

Therefore, $Sl = Tu = Al$.

Thus we have $Tu = Bu = Sl = Al$.

Due to weak compatibility of A and S, we have

$ASl = SAl = SSl = AAl$.

Now, we claim that Tu is the common fixed point of A, B, S and T.

Suppose that, $TTu \neq Tu$.

From (2.1.2), we have

$$M(Tu,TTu,t) = M(Sl,TTu,t) \leq \phi\big(\lambda(l,Tu)\big), \qquad \textbf{(2.4)}$$

Where

$\lambda(l,Tu) = \max\Big\{M(Al,BTu,t), M(Al,Sl,t), M(BTu,TTu,t), \frac{1}{2}[M(Sl,BTu,t) + M(TTu,Al,t)]\Big\} = \max\{M(Tu,TTu,t), 0,0, M(Tu,TTu,t)\} = M(Tu,TTu,t)$.

Thus from (2.4), we have

$M(Tu,TTu,t) \leq \phi(M(Tu,TTu,t))$, which is a contradiction.

Therefore, $Tu = TTu = BTu$.

Hence, Tu is the common fixed point of B and T.

Similarly, we prove that Sl is the common fixed point of A and S.

Since, $Tu = Sl$.

Then Sl is the common fixed point of A, B, S and T.

The proof is similar when AX is assumed to be complete subspace of X are similar to the cases in which SX or BX respectively is complete subspace of X.

Since $A(X) \subset S(X)$ and $B(X) \subset T(X)$.

Now, we shall prove that common fixed point is unique.

Let r and s be two common fixed points of A, B, S and T such that $r \neq s$.

From (2.1.2), we have

$$M(r,s,t) = M(Sr,Ts,t) \leq \phi\big(\lambda(r,s)\big), \qquad \textbf{(2.5)}$$

Where

$\lambda(r,s) = \max\left\{M(Ar,Bs,t), M(Ar,Sr,t), M(Bs,Ts,t), \frac{1}{2}[M(Ar,Ts,t) + M(Bs,Sr,t)]\right\} = \max\{M(r,s,t), 0,0, M(r,s,t)\} = M(r,s,t).$

Thus from (2.5), we get

$M(r,s,t) \leq \phi(M(r,s,t)),$

which is a contradiction.

Therefore, $r = s$.

Thus A, B, S and T have a common unique fixed point.

Theorem 2.2. Let A, B, S and T be self-mappings of a fuzzy metric space $(X, M, *)$ with $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$ satisfying

2.2.1: $M(Sx,Ty,t) \leq \phi(\lambda(x,y))$,

where ϕ is an upper semi continuous, contractive modulus and $\lambda(x,y) = \max\{M(Ax,By,t), M(Ax,Sx,t), M(By,Ty,t), \frac{(M(Ax,Ty,t)+M(By,Sx,t))}{2}\}$.

2.2.2: Pairs (S,A) and (T,B) are weakly compatible.

2.2.3: Pairs (S,A) and (T,B) satisfy (CLR_S) and (CLR_T) properties respectively.

Then A, B, S and T have a unique common fixed point.

Proof: we have $S(X) \subseteq B(X)$ and pairs (S,A) satisfy (CLR_S) property then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = Sx$, for some x in X.

Since $S(X) \subseteq B(X)$, then there exists a sequence $\{y_n\}$ in X such that $Ax_n = By_n$.

Hence $\lim_{n\to\infty} By_n = Sx$.

We will show that $\lim_{n\to\infty} Ty_n = Sx$.

Let $\lim_{n\to\infty} Ty_n = u \neq Sx$.

From (2.2.1), we have

$\mathrm{M}(\mathrm{Sx_n}, \mathrm{Ty_n}, \mathrm{t}) \leq \phi(\lambda(\mathrm{x_n}, \mathrm{y_n}))$.

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(Sx_n, Ty_n, t) \leq \phi \lim_{n\to\infty} \lambda(x_n, y_n) \quad \textbf{(2.6)}$$

Where

$\lim_{n\to\infty} \lambda(x_n, y_n) =$

$\lim_{n\to\infty}(\max\{\ M(\mathrm{Ax_n},\mathrm{By_n},\mathrm{t}),\mathrm{M}(\mathrm{Ax_n},\mathrm{Sx_n},\mathrm{t}),\mathrm{M}(\mathrm{By_n},\mathrm{Ty_n},\mathrm{t}),\frac{1}{2}[\mathrm{M}(\mathrm{Ax_n},\mathrm{Ty_n},\mathrm{t})+$
$\mathrm{M}(\mathrm{By_n},\mathrm{Sx_n},\mathrm{t})]\}) = \max\{M(Sx,Sx,t),M(Sx,Sx,t),M(Sx,u,t),\frac{1}{2}[M(Sx,Sx,t)+$
$M(Sx,u,t)]\} = M(Sx,u,t)$.

Then from (1), we get

$\mathrm{M(Sx,u,t)} \leq\ \phi\mathrm{M(Sx,u,t)} < \mathrm{M(Sx,u,t)}$,

a contradiction.

Let $Sx \neq u$, then $M(Sx,u,t) > 0$ and therefore ϕ is a contractive modulus.

$\phi\big(\mathrm{M(Sx,u,t)}\big) < \mathrm{M(Sx,u,t)}$.

Therefore, $Sx = u$, that is $\lim_{n\to\infty} Ty_n = Sx$.

Now, we have

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sx = u$.

Now, we will prove that $\mathrm{Ax} = \mathrm{u}$.

From (2.2.1), we have

$$\mathrm{M(Sx_n,Ty_n,t)} \neq \phi\big(\lambda(\mathrm{x_n},\mathrm{y_n})\big).$$

Letting limit as $n \to \infty$, we get

$\lim_{n\to\infty} M(Sx_n,Ty_n,t) \leq\ \phi \lim_{n\to\infty} \lambda(x_n,y_n)$, **(2.7)**

Where

$$\lim_{n\to\infty}\lambda(x_n,y_n) = \lim_{n\to\infty}(\max\{\ M(Ax,By_n,t),M(Ax,Sx,t),M(By_n,Ty_n,t),\frac{1}{2}[M(Ax,Ty_n,t) + M(By_n,Sx,t)\}\})$$
$$= \max\{M(Ax,u,t),M(Ax,Ax,t),M(Ax,u,t),\frac{1}{2}[M(Ax,Ax,t) + M(Ax,u,t)]\} = M(Ax,u,t)$$

Thus from (2.7), we have

$M(Ax,u,t) \leq M(Ax,u,t)$,

a contradiction.

Therefore, $\mathrm{Ax} = \mathrm{u} = \mathrm{Sx}$.

Since pair (S,A) is weakly compatible therefore $Au = Su$.

Also, $S(X) \subseteq B(X)$, then there exists y in Xsuch that $Sx = By$ that follows $By = u$.

Now we will prove that $By = u$.

Let $By \neq u$.

From (2.2.1), we have

$M(Sx_n, Ty_n, t) \le \phi(\lambda(x_n, y_n))$.

Letting limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(Sx_n, Ty_n, t) \le \phi \lim_{n\to\infty} \lambda(x_n, y_n), \quad \textbf{(2.8)}$$

where

$\lim_{n\to\infty} \lambda(x_n, y_n) =$
$\lim_{n\to\infty}(\max\{ M(Ax_n, By_n, t), M(Ax_n, Sx_n, t), M(By_n, Ty_n, t), \frac{1}{2}[M(Ax_n, Ty_n, t) +$
$M(By_n, Sx_n, t)\}) = \max\{M(u,u,t), M(u,By,t), M(u,u,t), \frac{1}{2}[M(u,u,t) + M(u,By,t)]\} =$
$M(u,By,t)$.

Thus from (2.8), we get

$M(u, By, t) \le \phi\big(M(u, By, t)\big) < M(u, By, t)$,

a contradiction.

Hence $u = By = Ty$.

Since pair (T, B) is weakly compatible then we get $Tu = Bu$.

Let us suppose that $Tu = Bu$.

From (2.2.1), we have

$$M(Su, Tu, t) \le \phi(\lambda(u, u)), \quad \textbf{(2.9)}$$

where

$\lambda(u,u) = \max\Big\{M(Au, Bu, t), M(Au, Su, t), M(Bu, Tu, t), \frac{1}{2}[M(Au, Tu, t) +$
$M(Bu, Su, t)]\Big\} = M(Su, Tu, t)$.

Thus, from (2.9), we get $M(Su, Tu, t) \le \phi\big(M(Su, Tu, t)\big)$,

a contradiction.

Therefore, $Au = Su$.

Hence, $Au = Su = Tu = Bu$.

Now we will prove that $z = Sz$.

Let us suppose that $z \ne Sz$.

From (2.2.1), we have

$M(Su, Tu, t) \le \phi(\lambda(x, u))$, where

$\lambda(x, u) =$
$\max\Big\{M(Ax, Bu, t), M(Ax, Sx, t), M(Bu, Tu, t), \frac{1}{2}[M(Ax, Tu, t), +M(Bu, Sx, t)]\Big\} =$
$$M(Sx, Tu, t) = M(u, Su, t). \quad \textbf{(2.10)}$$

Thus from (2.10), we get

$M(u, Su, t) \leq \phi(M(u, Su, t))$,

a contradiction.

Therefore, $u = Su = Au = Bu = Tu$.

Hence u is the common fixed point of A, B, S and T.

Now we will prove uniqueness of fixed point.

Let v be the another fixed common fixed point of A, B, S and T.

Let us suppose that $u \neq v$.

From (2.2.1), we have

$M(v, u, t) = M(Sv, Tu, t) \leq \phi(\lambda(v, u))$,

Where $\lambda(v, u) = \max\left\{M(Av, Bu, t), M(Av, Sv, t), M(Bu, Tu, t), \frac{1}{2}[M(Av, Tu, t), +M(Bu, Sv, t)]\right\} = M(v, u, t)$.

$M(v, u, t) \leq M(v, u, t)$,

a contradiction.

Thus, $v = u$ and hence there is unique fixed point of A, B, S and T.

Example 2.3. Let $X = [0,1]$ equipped with metric $d(x, y) = |x - y|$.

$M(x, y, t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, \text{for all } x, y \in X, t > 0 \\ 0, \text{for all } x, y \in X, t = 0 \end{cases}$ where * is defined by $a * b = ab$.

Clearly $(X, M, *)$ is fuzzy metric space on X.

Define A, B, S and T on $[0,1]$ as

$A(x) = \begin{cases} 1 \; if \; x \in [0,1] \cap Q \\ 0 \; if \; x \notin [0,1] \cap Q \end{cases}$ and $B(x) = \begin{cases} 1 \; if \; x \in [0,1] \cap Q \\ \frac{1}{2} \; if \; x \notin [0,1] \cap Q \end{cases}$, where Q is set of rationals.

$S(x) = \begin{cases} \frac{1}{2} \; if \; 0 \leq x < 1 \\ 1 \; if \; x = 1 \end{cases}$ and $T(x) = \begin{cases} \frac{1}{3} \; if \; 0 \leq x < 1 \\ 1 \; if \; x = 1 \end{cases}$.

Define ϕ: $[0,1] \to [0,1]$ as $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(s) = s$, $0 < s < 1$.

It satisfy $M(Sx, Ty, t) \leq \phi(\lambda(x, y))$,

Where

$$\lambda(x, y) = max\left\{M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), \frac{\left(M(Ax, Ty, t) + M(By, Sx, t)\right)}{2}\right\}.$$

$T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$.

Pair (S, A) and (T, B) satisfy E.A property and are weakly compatible.

AX, BX, SX and TX are complete subspace of X.

Then A, B, S and T have a unique common fixed point.

Example 2.4. Let $X = [1, \infty)$ equipped with metric $d(x, y) = |x - y|$.

$M(x,y,t)=\begin{cases} \frac{t}{t+|x-y|}, \text{for all } x, y \in X, t > 0 \\ 0, \text{for all } x, y \in X, t = 0 \end{cases}$ where * is defined by $a * b = ab$.

Clearly $(X, M,*)$ is fuzzy metric space on X.

Define A, B, S and T on [0,1] as

$A(x) = x^2$, $\mathrm{B}(x) = (2x - 1)$, $S(x) = (3x - 2)$ and $T(x) = x$.

Define ϕ: $[0,1] \to [0,1]$ as $\phi(s) = s$, $0 \le s \le 1$.

It satisfy $M(Sx, Ty, t) \le \phi(\lambda(x, y))$,

Where

$$\lambda(x, y) = \max\left\{M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), \frac{\big(M(Ax, Ty, t) + M(By, Sx, t)\big)}{2}\right\}.$$

Pair (S, A) and (T, B) satisfy E.A property and are weakly compatible.

Pair (S, A) and (T, B) satisfy $(CLR)_S$ and $(CLR)_T$ property.

Then A, B, S and T have a unique common fixed point.

Example 2.5. Let $X = [-2,1]$ equipped with metric $(x, y) = |x - y|$.

$M(x, y, t) = \frac{|x-y|}{t+|x-y|}$, where * is defined by $a * b = \min\{a, b\}$.

Clearly $(X, M,*)$ is fuzzy metric space on X.

Define A, B, S and T on [0,1] as follows

$$A(x) = B(x) = \begin{cases} \frac{1}{4}, 0 \le x < \frac{1}{3} \\ (\frac{2}{3} - x), \frac{1}{3} \le x \le 1 \end{cases}.$$

$$S(x) = T(x) = \begin{cases} \frac{1}{4}, 0 \le x < \frac{1}{3} \\ \frac{1}{3}, \frac{1}{3} \le x \le 1 \end{cases}.$$

Define ϕ: $[0,1] \to [0,1]$ as $\phi(s) = s$, $0 \le s \le 1$.

It satisfy $M(Sx, Ty, t) \le \phi(\lambda(x, y))$,where $\lambda(x, y) = max\left\{M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), \frac{\big(M(Ax,Ty,t)+M(By,Sx,t)\big)}{2}\right\}$.

Now $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$.

Now take $x_n = \frac{1}{3} + \frac{1}{n}$.

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T\left(\frac{1}{3} + \frac{1}{n}\right) = \frac{1}{3}.$$

$$\lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} B\left(\frac{1}{3} + \frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{2}{3} - \frac{1}{3} - \frac{1}{n}\right) = \frac{1}{3}.$$

Clearly (T, B) is weakly compatible mapping as $T(x) = B(x)$ when $x = \frac{1}{3}$.

Thus, (T, B) commute at coincident points.

Similarly (A, S) is weakly compatible mapping as $A(x) = S(x)$ when $x = \frac{1}{3}$.

Also pairs (A, S) and (T, B) satisfy E.A property.

Hence there is a unique common fixed point $x = \frac{1}{3}$ in $[-2,1]$.

References

[1] Aamri and Moutawakil D. EI, Some new common fixed point theorem under strict contractive coditions, J. Math. Anal. Appl., **270** (1) (2002), 181-188.

[2] Banach S., Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fundamental Mathematicae, **3** (7) (1922), 133-181.

[3] Deshpande B. and Sheikh A.S., Relation-theoretic fuzzy Banach contraction principle and fuzzy Eldestein contraction theorem, Global Journal of Pure and Applied Mathematics, **13** (9) (2017), 6929-6939.

[4] George A. and Veeramani P., On some results in fuzzy metric spaces, Fuzzy Sets and Systems, **64** (1994), 395-399.

[5] Grabiec M., Fixed points in fuzzy metric spaces, Fuzzy Sets and Systematic, **27** (1983), 385- 389.

[6] Jungck G., Compatible mappings and fixed points, Amer. Math. Monthly, **83** (1976), 261-263.

[7] Kramosil O. and Michalek J., Fuzzy metric and statistical metric spaces, Kybernetika, **11** (1975), 326-334.

[8] Kurdekar, A study of Fuzzy metric space, IJEDR, **7**(3)(2019).

[9] Manandhar K.B., Jha K. and Cho Y.J., Common fixed point theorem in intuitionistic fuzzy metric spaces using compatible mappings of type (K), Bull. Society for Math. Services and Standards, **3** (2014), 81-87.

[10] Mishra SN. and Singh SL., Common fixed point of maps on fuzzy metric spaces, Internat. J. Math. Math. Sci. **17**(1994), 253-258.

[11] Mohd. Imdad and Ali, Some common fixed point theorems in fuzzy metric spaces, Mathematical Communications, **11**(2006), 153-163.

[12] Nisha and Kumar M., Common fixed point theorems in metric space, Global Journal of Pure and Applied Mathematics, **17**(2)(2021), 265-274.

[13] Sessa S., On a weakly commutativity condition in a fixed point considerations, Publ. Inst. Math., **32** (46) (1986) 149-153.

[14] Sintunavart W., P. Kuman, Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space, J. Appl. Math., (2011) 637958, 1-14.

[15] Zadeh, L.A, Fuzzy Sets, Inform. and Control, **8** (1965), 338-353.