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Fixed point Theorems for Generalized Geraghty (α, ψ, ϕ) weak contractions

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Abstract

The main objective of this paper is to introduce and study a class of mappings called Generalised Geraghty (α, ψ, ϕ)- Weak contractive mappings in complete metric space which are the generalization of α-ψ contractive mappings and to explore the existence and uniqueness of common fixed points for this new generalized contractive mappings using the concept of α-admissibility. These results extend and improve upon several previously established recent results in the literature and illustrative examples are provided to demonstrate the effectiveness of these results.

Keywords : Fixed point, α-admissible, Geraghty (α, ψ, ϕ)-Weak contraction, Complete Metric space.

2000 Mathematics Subject Classification : 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays a significant role in non-linear analysis as many real-world problems in applied science, economics, physics and engineering can be reformulated as a problem of finding fixed points of non-linear maps. Inspired by this fact, many authors have directed their attention to obtain fixed point results by generalizing the notion of Banach contraction Principle, which plays a vital role in the existence of fixed points. During the last three decades, the work on generalizing and introducing different types of contractive conditions for the existence of common fixed points has been a very active field of research. For detailed survey on various definitions of contractive maps, refer Rhoades [1].

In this regard, Micheal A. Geraghty [2] introduced an intriguing contraction called Geraghty Contraction. By taking this into account, he examined some auxiliary functions for the existence and uniqueness of mappings in any complete metric spaces. The idea of α- contractive and α-admissible mappings was first presented in 2012 by Samet et al. [3], who also produced a number of fixed-point results for mappings that satisfy such contraction conditions. Later in 2013, Karapinar et al. [4] introduced an idea of triangular α-admissible mapping, which extended the scope of the α-admissibile mappings. Cho et al. [5] introduced the idea of α-Geragthy contraction mappings, which generalizes the idea of α-admissible mappings. Chandok [6] state and proved some interesting fixed point results for (α, β)-admissible Geraghty contractive mappings in 2015.

The work of Samet et al. [3] introduced $\alpha - \psi$ contractive type mappings as a new category of contractive mappings. The fixed point results obtained by Samet et al. [3] extended and generalized several fixed point results that exist in the literature, including the Banach contraction principle. In a further development, Karapinar and Samet [11] generalized the notion of $\alpha - \psi$ contractive type mappings and obtained various fixed point theorems for these mappings. Recently, Raji M. [10] presented a class of contractive type mappings called generalized $\alpha - \psi$ contractive pair of mappings and introduced the notion of α-admissible with respect to g-mapping which in turn generalized the concept of g-monotone mapping. In [9], Raji M. studied various coincidence fixed point theorems for generalized $\alpha - \psi$ contractive type mappings in complete metric spaces. H. Qawaqneh et al.[7] presented the notion of generalised Geraghty (α, ψ, ϕ)-Quasi contraction in partially ordered metric space and proved some distinct common fixed point results in partially ordered metric space. For more work in this domain, refer [12-20].

The purpose of this paper is to introduce the generalized form of $\alpha - \psi$ contractive mapping called, Generalised Geraghty (α, ψ, ϕ)-Weak contractive mapping for a self map in complete metric space and to explore the existence and uniqueness of fixed point for this new generalized contractive mapping using the concept of α-admissibility. Also we extend this result for a pair of self mappings in a complete metric space and proved existence and uniqueness of common fixed point. These results unify and generalize the recent works of Raji M. et al. in [9] and [11], H.Qawaqneh et al. in [7] and other related results. Additionally, we provide illustrative examples to demonstrate the improved results obtained with our approach.

Throughout this article, the following standard notations and terminologies are used.

Let Ψ be the family of nondecreasing continuous functions $\psi : [0, \infty) \to [0, 1)$ such that $\sum \psi^n(t) < \infty$ for each $t > 0$ where ψ^n is the n^{th} iterate of ψ. Then we write $\psi \in \Psi$.

Let Φ be the family of continuous functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < \psi(t)$ for each $t > 0$ then $\phi(0) = 0$. Then we write $\phi \in \Phi$.

Let $\beta : [0, \infty) \to [0, 1)$ be functions such that $\{r_n\}$ is a sequence in $[0, \infty)$ with $\lim_{n\to\infty} \beta(r_n) = 1 \Rightarrow \lim_{n\to\infty} r_n = 0$. Then we write $\beta \in \mathfrak{B}$

The following are the basic definitions needed in the main results.

Definition 1.1. [2] A self-mapping $T : X \to X$ where (X, d) is any metric space, is called a Geraghty contraction if $d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$ for all $x, y \in X$.

Definition 1.2. [3] Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to R^+$, we say that T is an α-admissible mapping if $\forall x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Definition 1.3. Let $S, T : X \to X$ be two self mappings and $\alpha : X \times X \to R^+$, we say that (S, T) is α-admissible if $\forall x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.

Definition 1.4. [5] A self-mapping $T : X \to X$ where (X, d) is any metric space, is called a α-Geraghty generalized contraction if there exists a mapping $\alpha : X \times X \to R$ and $\beta \in \mathfrak{B}$ such that for all $x, y \in X$, $\alpha(x, y)d(Tx, Ty) \leq \beta(M(x, y))M(x, y)$ where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$

Definition 1.5. [3] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha-\psi-$contractive mapping if there exist a (c)-comparison functions $\psi \in \Psi$ and a function $\alpha : X \times X \to R$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is a $\alpha - \psi$ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt, k \in (0, 1)$.

Karapinar and Samet [11] introduced the following concept of generalized $\alpha - \psi-$contractive type mappings:

Definition 1.6. [11] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalised $\alpha - \psi$ contractive type mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have $\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$ where $M(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$

Further, Karapinar and Samet [11] established fixed point theorems for this new class of contractive mappings. Also, they obtained fixed point theorems on metric spaces endowed with a partial order and fixed point theorems for cyclic contractive mappings.

The concept of generalized $\alpha - \psi$ contractive type mappings was defined as follows by Raji M. et al.[9].

Definition 1.7. [9] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. T is a generalised $\alpha{-}\psi$ contractive type mapping if there exist two functions $\alpha : X\times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have $\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$ where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{d(x,y)},\right. \\ \left.\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)}\right\}$$

H. Qawaqneh et al.[7] presented the notion of generalised Geraghty (α, ψ, ϕ)-Quasi contraction in partially ordered metric space and proved some distinct common fixed point results in partially ordered metric space.

Definition 1.8. [7] Let (X, σ) be a partially ordered metric-like space and $S, T : X \to X$ be two mappings. Then we consider that the pair (S, T) is generalised Geraghty (α, ψ, ϕ) quasi contraction self mapping if there exist $\alpha : X \times X \to [0, \infty)$, $\beta \in \mathfrak{B}$, $\psi \in \Psi$ and $\phi : [0, \infty) \to [0, \infty)$ are continuous functions with $\phi(t) \leq \psi(t)$ for all $t > 0$ such that $\alpha(x, y)\psi(\sigma(Sx, Ty)) \leq \lambda\beta(\psi(M_{x,y}))\phi(M_{x,y})$,
holds for all elements $x, y \in X$ and $0 \leq \lambda \leq 1$ where

$$M_{x,y} = \max\{\sigma(x, y), \sigma(x, Sx), \sigma(y, Ty), \sigma(Sx, y), \sigma(x, Ty)\}$$

2. MAIN RESULT

In this section, we present the notion of generalised Geraghty (α, ψ, ϕ)-weak contractive condition for self maps.

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a self mapping. Then T is called a generalised Geraghty (α, ψ, ϕ)-weak contractive mapping if there exist $\alpha : X \times X \to [0, \infty)$, $\beta \in \mathfrak{B}$, $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \lambda\beta(\psi(M(x, y)))\phi(M(x, y))$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$ where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}, \frac{d(y,Tx)d(x,Ty)}{d(x,y)+d(x,Tx)+d(y,Ty)}, \frac{d(x,Tx)d(y,Ty)+d(y,Tx)d(x,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}\right\} \tag{1}$$

Using this new contractive condition the following results are proved and are illustrated with examples.

Theorem 2.2. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalised Geraghty (α, ψ, ϕ)-weak contractive mapping and satisfying the following conditions:*

1. *T is $\alpha-$admissible*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
3. *T is continuous*

Then T has a fixed point.

Proof. Define $\{x_n\} \in X$ by $x_{n+1} = Tx_n$ for all $n \in N$
If $x_{n+1} = x_n$ for some $n \in N$, then x_n is a fixed point and hence the theorem is proved.
If $x_{n+1} \neq x_n$ for all $n \in N$, then consider
$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_0) = \alpha(x_1, x_2) \geq 1$
By induction on n, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$. Consider

$$\begin{aligned}\psi(d(x_{n+1}, x_n) = \psi(d(Tx_n, Tx_{n-1})) &\leq \alpha(x_n, x_{n-1})\psi(d(Tx_n, Tx_{n-1}) \\ &\leq \lambda\beta(\psi(M(x_n, x_{n-1})))\phi(M(x_n, x_{n-1}))\end{aligned}$$

Where

$$\begin{aligned}M(x_n, x_{n-1}) = \\ \max\Bigg\{&d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \\ &\frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}, \frac{d(x_{n-1}, Tx_n)d(x_n, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}, \\ &\frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, Tx_n)d(x_n, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}\Bigg\}\end{aligned}$$

which implies $M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}$
If $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$ then

$\psi(d(x_{n+1}, x_n) \leq \lambda\beta(\psi(d(x_n, x_{n+1})))\phi(d(x_n, x_{n+1})) < \phi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1}))$,
a contradiction.
Hence $M(x_n, x_{n-1}) = d(x_n, x_{n-1})$.
Then

$$\psi(d(x_{n+1}, x_n) < \psi(d(x_n, x_{n-1})) \Rightarrow d(x_{n+1}, x_n) < d(x_n, x_{n-1}) \quad (2)$$

Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence and hence convergent.
Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Next we show that $r = 0$. Suppose $r > 0$, then letting $n \to \infty$ in (2) we get $\psi(r) < \psi(r)$, a contradiction. Hence $r = 0$.
Thus $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence.
Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Since T is Continuous, $Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$.
Hence u is a fixed point of T. □

To prove the uniqueness of the fixed point, include the following hypothesis to the above theorem:

$$\forall x, y \in Fix(T),\ \textit{there exists}\ z \in X\ \textit{such that}\ \alpha(x, z) \geq 1\ \textit{and}\ \alpha(y, z) \geq 1 \quad (3)$$

Theorem 2.3. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalised Geraghty (α, ψ, ϕ)-weak contractive mapping and satisfying the following conditions:*

1. *T is $\alpha-$admissible*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
3. *T is continuous*
4. *$\forall x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$*

Then T has a unique fixed point.

Proof. By Theorem 2.2, u is a fixed point of T. Suppose v is another fixed point of T, then there exists $z \in X$ such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$
Since T is $\alpha-$admissible, we get $\alpha(u, T^n z) \geq 1$ and $\alpha(v, T^n z) \geq 1 \quad \forall n$.
Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$ and $z_0 = z$. Then $\forall n$, we have $\psi(d(u, z_{n+1})) = \psi(d(Tu, Tz_n)) \leq \alpha(u, z_n)\psi(d(Tu, Tz_n)) \leq \lambda\beta(\psi(M(u, z_n)))\phi(M(u, z_n))$ where $M(u, z_n) = \max\{d(u, z_n), d(u, z_{n+1})\}$
If $M(u, z_n) = d(u, z_{n+1})$ then

$$\psi(d(u, z_{n+1})) \leq \lambda\beta(\psi(d(u, z_{n+1})))\phi(d(u, z_{n+1})) < \phi(d(u, z_{n+1})) < \psi(d(u, z_{n+1})),$$

which is a contradiction. Hence $M(u, z_n) = d(u, z_n)$ .This implies $\psi(d(u, z_{n+1})) < \psi(d(u, z_n))$. Hence the sequence $\{d(u, z_n)\}$ is decreasing and hence convergent. Let $\lim_{n\to\infty} d(u, z_n) = r$. It can be easily show that $r = 0$. That is $\lim_{n\to\infty} d(u, z_n) = 0$. Similarly $\lim_{n\to\infty} d(v, z_n) = r$ which implies $u = v$, hence the fixed point is unique. □

Example 1: Let $X = \{0, 1, 2\}$ endowed with the metric

$$d(0, 1) = 1, d(0, 2) = 2, d(1, 2) = 3, d(x, x) = 0 \qquad \forall x \in X.$$

Then (X, d) is a complete metric space.
Define the mapping $T : X \to X$ by $T(0) = 1, T(1) = 1, T(2) = 0$, then T is continuous.
Let $\lambda = \frac{1}{2}, \psi(t) = t, \phi(t) = \frac{2}{3}t \ \ and \ \ \beta(t) = \frac{e^t}{2}$. Define $\alpha : X \times X \to [0, \infty)$ as follows:

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \in \{0, 1, 2\} \\ 0 & \text{in all other cases .} \end{cases}$$

Note that $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, \forall x, y \in X$. Hence T is $\alpha-$admissible.
If $(x, y) = (0, 1)$ then

$$\alpha(0, 1)\psi(d(T0, T1)) \leq \lambda\beta(\psi(M(0, 1)))\phi(M(0, 1))$$

where $M(0, 1) = \max\{1, 1, 0, 0, 0, 0\} = 1$
Similarly, Generalised Geraghty (α, ψ, ϕ)-weak contractive condition is satisfied for all $x, y \in X$. Thus all the conditions of Theorem 2.3 are satisfied and 1 is the unique fixed point of T.

Definition 2.1 can be extended to a pair of self mappings as follows and the corresponding results are obtained.

Definition 2.4. Let (X, d) be a metric space and $S, T : X \to X$ be two self mappings. Then the pair (S, T) is called a generalised Geraghty (α, ψ, ϕ)-weak contractive mapping if there exist $\alpha : X \times X \to [0, \infty)$, $\beta \in \mathfrak{B}$, $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\alpha(x, y)\psi(d(Sx, Ty)) \leq \lambda\beta(\psi(M(x, y)))\phi(M(x, y)),$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$ where

$$M(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), d(Sx, y), d(x, Ty), \frac{d(x, Sx)d(y, Ty)}{d(Sx, y) + d(x, Ty)}, \frac{d(Sx, y)d(x, Ty)}{d(x, y) + d(x, Sx) + d(y, Ty)}, \frac{d(x, Sx)d(y, Ty) + d(Sx, y)d(x, Ty)}{d(x, y) + d(Sx, y) + d(y, Ty)}\right\} \tag{4}$$

Theorem 2.5. *Let (X, d) be a complete metric space. Suppose that $S, T : X \to X$ is a generalised Geraghty (α, ψ, ϕ)-weak contractive mapping and satisfying the following conditions:*

1. *(S, T) are $\alpha-$admissible*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$*
3. *S and T are continuous*

Then S and T have a common fixed point.

Proof. Define $\{x_n\} \in X$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n \geq 0$
If $x_{2n} = x_{2n+1}$ for some $n \in N$, then x_{2n} is a common fixed point of S and T and hence the theorem is proved.
If $x_{2n} \neq x_{2n+1}$ for all $n \in N$, then consider
$\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Sx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$ and
$\alpha(Tx_0, Sx_1) = \alpha(x_1, x_2) \geq 1$
By induction on n, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.
In particular $\alpha(x_{2n}, x_{2n+1}) \geq 1, \forall n \geq 0$. Consider

$$\begin{aligned}\psi(d(x_{2n+1}, x_{2n+2}) = \psi(d(Sx_{2n}, Tx_{2n+})) &\leq \alpha(x_{2n}, x_{2n+1})\psi(d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \lambda\beta(\psi(M(x_{2n}, x_{2n+1})))\phi(M(x_{2n}, x_{n+1}))\end{aligned}$$

Where

$$\begin{aligned}&M(x_{2n}, x_{2n+1}) = \\ &\quad \max\Bigg\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n+1}), \\ &\quad \frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{d(Sx_{2n}, x_{2n+1}) + d(x_{2n}, Tx_{2n+1})}, \frac{d(Sx_{2n}, x_{2n+1})d(x_{2n}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1})}, \\ &\quad \frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1}) + d(Sx_{2n}, x_{2n+1})d(x_{2n}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(Sx_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})}\Bigg\}\end{aligned}$$

which implies $M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}\}$
If $M(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$ then

$$\begin{aligned}\psi(d(x_{2n+1}, x_{2n+2}) \leq &\lambda\beta(\psi(d(x_{2n+1}, x_{2n+2}))\phi(d(x_{2n+1}, x_{2n+2}) \\ &< \phi(d(x_{2n+1}, x_{2n+2}) < \psi(d(x_{2n+1}, x_{2n+2}),\end{aligned}$$

which is a contradiction. Hence
$M(x_{2n+1}, x_{2n+2}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2})\}$

Put $\gamma = \max\{\lambda, \frac{\lambda}{1-\lambda}\}$. Then

$$\psi(d(x_{2n+1}, x_{2n+2}) < \gamma\beta(\psi(d(x_{2n}, x_{2n+1})))\phi(d(x_{2n}, x_{2n+1}) \qquad \forall n \in N_0 \quad (5)$$

Clearly $\gamma < 1$, thus $\{d(x_{2n}, x_{2n+1})\}$ is a decreasing sequence and hence convergent. Let $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = r$. Next we show that $r = 0$. Suppose $r > 0$, then letting $n \to \infty$ in (5) we get $\psi(r) < \psi(r)$, a contradiction. Hence $r = 0$.
Thus $d(x_n, x_m) \to 0$ as $n, m \to \infty$ Therefore $\{x_n\}$ is a Cauchy sequence.
Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Since S and T are Continuous, we get
$\lim_{n\to\infty} d(x_{n+1}, Tu) = d(Su, Tu)$ and $\lim_{n\to\infty} d(Su, x_{n+1}) = d(Su, Tu)$ and
$\lim_{n\to\infty} d(x_{n+1}, Tu) = d(u, Tu)$ and $\lim_{n\to\infty} d(Su, x_{n+1}) = d(Su, u)$
This implies $Su = Tu = u$. Hence u is a common fixed point of S and T. □

As in Theorem 2.3, include the hypothesis (3) to the Theorem 2.5 to obtain the following result.

Theorem 2.6. *Let (X, d) be a complete metric space. Suppose that $S, T : X \to X$ is a generalised Geraghty (α, ψ, ϕ)-weak contractive mapping and satisfying the following conditions:*

1. *(S, T) is $\alpha-$admissible*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$*
3. *S and T are continuous*
4. *$\forall x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$*

Then S and T have a unique common fixed point.

Proof. As in Theorem 2.3, one can easily prove this result following by the lines of proof of Theorem 2.2. and Theorem 2.3. □

Example 2: Let $X = \{0, 1, 2\}$ endowed with the metric

$$d(0, 1) = 1, d(0, 2) = 2, d(1, 2) = 3, d(x, x) = 0 \qquad \forall x \in X.$$

Then (X, d) is a complete metric space.
Define the mapping $S, T : X \to X$ by $T(0) = 1, T(1) = 1, T(2) = 2$ and
$S(0) = 2, S(1) = 1, S(2) = 1$, then S and T are continuous.

Let $\lambda = \frac{1}{2}, \psi(t) = t, \phi(t) = \frac{2}{3}t \ \ and \ \ \beta(t) = \frac{e^t}{2}$. Define $\alpha : X \times X \to [0, \infty)$ as follows:

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \in \{0, 1, 2\} \\ 0 & \text{in all other cases .} \end{cases}$$

Note that $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Ty) \geq 1 \quad and \quad \alpha(Tx, Sy) \geq 1, \forall x, y \in X.$. Hence (S, T) is $\alpha-$admissible.
If $(x, y) = (0, 1)$ then $\alpha(0, 1)\psi(d(S0, T1)) \leq \lambda\beta(\psi(M(0, 1)))\phi(M(0, 1))$
where $M(0, 1) = \max\{1, 2, 0, 3, 1, 0, 1, \frac{3}{4}\} = 3$
Similarly, (4) is satisfied for all $x, y \in X$. Thus all the conditions of Theorem 2.6 are satisfied and 1 is the unique common fixed point of S and T.

Remark: Theorem 3.2 and 3.5 of M.Raji et al.[9] and Theorem 11,13,15,18 of G. Durmaz et.al [21] are the direct consequence of our results Theorem 2.2 and 2.3. Also Theorem 2.1 and 2.4 in [10] are also the consequences of these results. Similarly Theorem 4 of [7] is the consequence of Theorems 2.5 and 2.6. Infact, by taking suitable values for $\alpha, \beta, \psi, \phi, \lambda$ and $M(x, y)$ in our results we get many corollaries which are the generalization and extension of results of Geraghty [2], Karapinar and Samet [11], Samet et al.[3] and other several known related results in the literature.

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