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Generalized Operators on the Space of New Generalized Functions $A(E)$

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Abstract

In this paper we will define and study the main linear generalized operators on the space of new generalized functions $A(E)$ and their properties. We will show that any operator $B: E \to E$ can be generalized to some operator $\bar{B}: A(E) \to A(E\,)$, where B be linear bounded operator and $A(E)$ be the space of new generalized functions defined in **[1]**. Also we will show that the generalized operators $\bar{B}$ possess the main important properties of the operator B. Also we will define the generalized composite of two linear bounded operators $\overline{B \circ C}$ and study the composite of generalized differential operator $\bar{D}$ with the generalized fourier operator $\bar{F}$ as a special case.

Keywords: Generalized functions, New Generalized functions, Locally Convex Algebra, Distributions, Algebra.

1. Introduction

In order to define the associative multiplication $f(gh) = (fg)h$ in the space of distributions E' where E be the set of tests functions $D(\mathbb{R})$ or $S(\mathbb{R})$ **[5]** and following the general method of construction algebras of new generalized functions **[2]** the algebra $A(E)$ was defined as a factor algebra **[1]**

$A(E) = G_{\theta_1}(E) \setminus G_{\theta_2}(E)$.

Where the sets $G_{\theta_1}(E)$ and $G_{\theta_2}(E)$ are define in the following way:

Let $G(E\,)$ be the set of all sequences in E, where the set E be separated and complete locally convex algebra **[5]**, and the topology on E is defined by a collection of semi norms P_α where α be an element of some real interval I. And P_α satisfy the following conditions :

$$\forall\, \alpha \in I \;\; \exists \beta \in I \;, \qquad \exists C_\alpha > 0 \;\; P_\alpha(uv) \le \; P_\beta(u)P_\beta(v) \;\; \forall\, u, v \in E$$

Now define the following sets:

$$G_{\theta_1}(E) = \{(x_k) \in G(E) \colon \exists m \in \mathbb{N} \,, \forall\, \alpha \in I, \exists d_\alpha > 0 \,, p_\alpha(x_k) < \; d_\alpha e^{mk} \;\; \forall\, k \;\}$$

$$G_{\theta_2}(E) = \{(y_k) \in G(E) \colon \forall m \in \mathbb{N} \,, \forall\, \alpha \in I, \exists d_\alpha > 0 \,, p_\alpha(y_k)\; d_\alpha e^{-mk} \;\; \forall\, k \;\}$$

Now we define the algebra

$$A(E) = G_{\theta_1}(E) \setminus G_{\theta_2}(E)$$

Very important special case when $E = S\,(\mathbb{R})$, where $S\,(\mathbb{R}\,)$ be the space of the real functions of rapid decay with topology given by the following family of semi norms :

$$p_{n,l}(x) = \sup_{\substack{k \le n \\ m \le l}} \sup_{t \in \mathbb{R}} \left| t^k x^{(m)}(t) \right|$$

We get the algebra of new generalized functions

$$A(S\,(\mathbb{R})) = G_{\theta_1}(S\,(\mathbb{R})) \setminus G_{\theta_2}(S\,(\mathbb{R}))$$

The spaces $S\,(\mathbb{R})$ and $S'\,(\mathbb{R})$are embedded in the algebra $A(S\,(\mathbb{R}))$.

$$S\,(\mathbb{R}) \subset \; S'\,(\mathbb{R}) \subset \; A(S\,(\mathbb{R})).$$

So we can define the associate multiplication of distributions as elements of the algebra $A(S\,(\mathbb{R}))$ **[1]** .

2. Generalized Operators on the Space of New Generalized Functions $\boldsymbol{A(S\,(\mathbb{R}))}$

Now let $B \colon \boldsymbol{S}\,(\mathbb{R}) \to \boldsymbol{S}\,(\mathbb{R})$ be bounded linear operator, then **[5]** for each $x(t) \in \boldsymbol{S}\,(\mathbb{R})$ and for each $\alpha \in I$, there is a constant $d_\alpha > 0$ and $\beta \in I$, such that

$$p_\alpha(B(x(t)) \le d_\alpha \;\, p_\beta(x(t)) \qquad \textbf{(1)}$$

Now to generalize the operator B on the space $A(S\,(\mathbb{R}))$ we prove the following results:

Lemma 1 a) $B\big(G_{\theta_1}(S\,(\mathbb{R})\,\big) \subset G_{\theta_1}(S\,(\mathbb{R}))$,

b) $B\big(G_{\theta_2}(S\,(\mathbb{R})\,\big) \subset G_{\theta_2}(S\,(\mathbb{R}))$

Proof. a) Let $x = x_k \in G_{\theta_1}(S\,(\mathbb{R})\,)$, then $\exists m \in \mathbb{N}\,, \forall\, \alpha \in I, \exists d_\alpha > 0\,,\ p_\alpha(x_k) < d_\alpha e^{mk}\ \ \forall\, k$.

Now consider $B(x) = B(x_k)$. Now since the operator B is bounded, then by using the formula (1) we conclude

$p_\beta\big(B(x)\big) = \ p_\beta\big(B(x_k)\big) \le\ d_\beta\ p_\alpha(x_k)\ <\ d_\beta\ d_\alpha e^{mk}\ \ \forall\, k$.

which means $B(x) = B(x_k) \in\ \ G_{\theta_1}(S\,(\mathbb{R}))$.

b) Let $y = y_k \in G_{\theta_2}(S\,(\mathbb{R})\,)$, then $\forall m \in \mathbb{N}\,, \forall\, \alpha \in I, \exists d_\alpha > 0\,, p_\alpha(y_k) <\ d_\alpha e^{-mk}\ \ \forall\, k$.

In similar way

$p_\beta\big(B(y)\big) = \ p_\beta\big(B(y_k)\big) \le\ d_\beta\ p_\alpha(y_k)\ <\ d_\beta\ d_\alpha e^{-mk}\ \ \forall\, k$

which means $B(y) = B(y_k) \in\ \ G_{\theta_2}(S\,(\mathbb{R}))$.

Now since $G_{\theta_2}(S\,(\mathbb{R}))$ is ideal,then if $f \in A(S\,(\mathbb{R}))$,then if $f = (f_k) +\ G_{\theta_2}(S\,(\mathbb{R}))$.

Thus we can define the generalized operator $\bar{B}\colon A\big(S\,(\mathbb{R})\big) \to A\big(S\,(\mathbb{R})\big)$, where $\bar{B}\,(f) = B(\,f_k) +\ G_{\theta_2}(S\,(\mathbb{R}))$.

The following theorem shows that the operator $\bar{B}$ is correct defined:

Theorem: The operator B does not depend on the representative

Proof: Let $f \in \boldsymbol{A}(\boldsymbol{S}\,(\mathbb{R}))$ and let x_k , y_k be two representative of f, then $(x_k) -\ (y_k)\ \in\ G_{\theta_2}(S\,(\mathbb{R}))$. Which means then $\forall m \in \mathbb{N}\,, \forall\, \alpha \in I, \exists d_\alpha > 0\,, p_\alpha(\ x_k - y_k) <\ d_\alpha e^{-mk}\ \ \forall\, k$.

Now by using linearity of the operator B, and by formula (1)

$B(x_k) -\ B(y_k) = B\ (\ x_k -\ \ y_k)$

$p_\beta\big(B(y)\big) = \ p_\beta\big(B(x_k -\ \ y_k)\big) \le\ d_\beta\ p_\alpha(x_k -\ \ y_k)\ <\ d_\beta\ d_\alpha e^{-mk}\ \ \forall\, k$

Which means that $B(x_k -\ \ y_k)$ be an element of the ideal $G_{\theta_2}(S\,(\mathbb{R}))$.

Now let $C\colon S\,(\mathbb{R}) \to S\,(\mathbb{R})$ be bounded linear operator.

Now Since $B\colon S\,(\mathbb{R}) \to S\,(\mathbb{R})$ be bounded linear operator, then we define $(B \circ C)(x) = B(C(x))$, and $(C \circ B)(x) =\ C(B(x))$. It is easy to check that $B \circ C$, and $C \circ B$ are linear continuous operators.

So by lemma1 we conclude that

$(B \circ C)((G_{\theta_1}(S\ (\mathbb{R})\ )) \subset G_{\theta_1}(S\ (\mathbb{R}))$,

And

$(B \circ C)((G_{\theta_2}(S\ (\mathbb{R})\ )) \subset G_{\theta_2}(S\ (\mathbb{R}))$.

Thus we define the generalized operator:

$\overline{B \circ C}: A\big(S\ (\mathbb{R})\big) \to A\big(S\ (\mathbb{R})\big)$, where

$\overline{B \circ C}\ (f) = \overline{B \circ C}(\ f_k) +\ G_{\theta_2}(S\ (\mathbb{R}))$.

As we know the operator of differentiation $D: S\ (\mathbb{R}) \to S\ (\mathbb{R})$ is linear and continuous. Also the Fourier transform $F: S\ (\mathbb{R}) \to S\ (\mathbb{R})$ is linear and continuous.

So we can define the following generalized operators

$\bar{D}: A\big(S\ (\mathbb{R})\big) \to A(S\ (\mathbb{R}))$

and

$\bar{F}: A\big(S\ (\mathbb{R})\big) \to A(S\ (\mathbb{R}))$

Also we define the composite of these operators

$\overline{D \circ F}: A\big(S\ (\mathbb{R})\big) \to A(S\ (\mathbb{R}))$

and

$\overline{F \circ D}: A\big(S\ (\mathbb{R})\big) \to A(S\ (\mathbb{R}))$

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