Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 3 (2024), pp. 643-655
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http://www.ripublication.com/gjpam.htm

Using Banach Algebra to Get Certain New Fixed Point Results for Generalized Contractive Maps in Cone Metric Spaces

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Abstract

To identify one pair of generalized contractive maps using a Banach algebra in complete cone metric spaces, we investigate, elucidate, and prove some standard fixed point results. Our results extend and supplement those found in the literature.

Mathematics Subject Classification (2020): Primary: 47H10, Secondary: 54H25

Keywords and phrases: Banach algebra, complete cone metric spaces, generalized Lipschitz conditions, weakly compatible maps

1. INTRODUCTION

Huang and Zhang [3] recently implemented cone metric spaces(**CMSs**). This framework leads to a fixed point(**FP**) theorem by using an ordered Banach space instead of actual numbers. They also gave an additional requirement for this theorem to be true. Additionally, by using the normalcy of the cone, the author has demonstrated other findings for contractive mappings. We are going to talk about the convergence features of sequences. Furthermore, We give some **(FP)** theorems for two mappings that are contractive.

This study's goal is to demonstrate novel common fixed point(**CFP**) solutions for pairs of maps over Banach algebra(**BA**) that meet contractive criteria in full **(CMSs)**. Some **(FP)** theorems in the literature are generalized by the results ([10],[6],[11]).

Initially, we will examine the definitions and characteristics of **(CMSs)** and **(BA)**. Additionally, the subsequent lemmas and observations will prove to be useful in the later sections. The following properties apply when defining an operation.

Definition 1.1. *(See[1][11]) Consider $\mathcal{A}$ is always a real **(BA)**, which means that $\mathcal{A}$ is a Banach space whereby a multiplication operation has defined, and applied the subsequent characteristics for every $\zeta, \xi, \upsilon \in \mathcal{A}$ and α element of $\mathbb{R}$.*

(1) $\zeta(\xi\upsilon) = (\zeta\xi)\upsilon$;

(2) $\zeta\xi + \zeta\upsilon = \zeta(\xi + \upsilon)$ *and* $\zeta\upsilon + \xi\upsilon = (\zeta + \xi)\upsilon$;

(3) $(\alpha\zeta)\xi = \alpha(\zeta\xi) = \zeta(\alpha\xi)$;

(4) $||\zeta\xi|| \leq ||\zeta||.||\xi||$.

In the framework of a (BA), we postulate the presence of a unit (or multiplicative identity) denoted as e, satisfying the condition that $e\zeta = \zeta e = \zeta$ for every element $\zeta \in A$. An element ζ belonging to A is deemed invertible if there exists an element $\xi \in A$ that serves as its inverse, fulfilling the equation $\zeta\xi = \xi\zeta = e$. The inverse of ζ is represented as ζ^{-1}.

Proposition 1.2. *(See[1],[11]) Let $\mathcal{A}$ denote a **(BA)** equipped within a unit element e, and let ζ be an element of $\mathcal{A}$. If the spectral radius $\rho(\zeta)$ of the element $\zeta < 1$, that is,*

$$\rho(\zeta) = \lim_{n\to\infty} ||\zeta_n||^{\frac{1}{n}} = \inf ||\zeta^n||^{\frac{1}{n}} < 1$$

Consequently, the expression $(e-\zeta)$ is invertible, and its inverse is given by $(e-\zeta)^{-1} = \sum_{i=0}^{\infty} \zeta^i$.

A subset $\mathcal{P}$ of $\mathcal{A}$ is referred to as a cone if

(1) $\mathcal{P}$ is closed, non-empty, and $\{\theta, e\} \subset \mathcal{P}$, where θ is $\mathcal{A}$'s zero vector;

(2) $\mathcal{P}\mathcal{P} = \mathcal{P}^2 \subset \mathcal{P}$;

(3) For any non-negative real numbers β and α exists such that $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$,

(4) $(-\mathcal{P}) \cap (\mathcal{P}) = \{\theta\}$.

For a specified cone $\mathcal{P}$ subset of $\mathcal{A}$, a partial ordering $\preceq$ can be established in relation to $\mathcal{P}$ such that $\zeta \preceq \xi$ holds iff $\xi - \zeta \in \mathcal{P}$. The symbol $\zeta \ll \xi$ is used to indicate that $\xi - \zeta \in \mathcal{P}^o$, where $\mathcal{P}^o$ represents the interior of the cone $\mathcal{P}$.

The cone $\mathcal{P}$ is referred to as usually if there is a constant $\mathcal{K} > 0$ so that for any $\alpha, \beta \in \mathcal{A}$, the condition $\alpha \preceq \beta$ leads to the conclusion that $||\alpha|| \leq \mathcal{K}||\beta||$.

The smallest positive value of $\mathcal{K}$ that satisfies the aforementioned inequality is referred to as the normal constant (refer to [3]). It is important to note that for any normal cone P, the condition $\mathcal{K} \geq 1$ holds (see [4]). In the subsequent discussion, we will assume that $\mathcal{P}$ represents a cone within a real **(BA)**$\mathcal{A}$, where $\mathcal{P}^o \neq \phi$ (indicating that the cone $\mathcal{P}$ is solid) and that $\preceq$ denotes the partial ordering associated with $\mathcal{P}$.

Definition 1.3. *(See[2][3][5]) Let $\mathcal{U}$ represent a non-empty set. Assume that a function $d_c : \mathcal{U} \times \mathcal{U} \to \mathcal{A}$ fulfills the following conditions:*

(1) For all $\zeta, \xi \in \mathcal{X}$, $\theta \preceq d_c(\zeta, \xi)$ and $d_c(\zeta, \xi) = \theta$ only in the event that $\zeta = \xi$;

(2) $d_c(\zeta, \xi) = d_c(\xi, \zeta), \forall \zeta, \xi \in \mathcal{U}$;

(3) $d_c(\zeta, \xi) \preceq d_c(\zeta, \upsilon) + d_c(\upsilon, \xi)$ for each $\zeta, \xi, \upsilon \in \mathcal{X}$.

*A cone metric on the set $\mathcal{U}$ is denoted by d, and the pair $(\mathcal{U}, d_c)$ is referred to as a **(CMSs)** over the **(BA)** $\mathcal{A}$ (abbreviated as **CMSBA**). It is important to observe that for every pair of elements $\zeta, \xi \in \mathcal{U}$, the value $d_c(\zeta, \xi)$ belongs to the set $\mathcal{P}$.*

Definition 1.4. *(See[3]) Let $(\mathcal{U}, d_c)$ represent a **(CMSs)**, where $\zeta \in \mathcal{U}$ and $\{\zeta_n\}$ denotes a sequence within $\mathcal{U}$. Consequently:*

(1) The sequence $\{\zeta_n\}$ is said to converge to ζ if, for every $c \in \mathcal{A}$ with $\theta \ll c$, there exists an integer $n_0 \in \mathbb{N}$ so that $c \gg d_c(\zeta_n, \zeta)$ holds for every $n_0 < n$. That is expressed as $\lim_{n\to\infty} \zeta_n = \zeta$ or $\zeta_n \to \zeta$ as $n \to \infty$.

(2) The sequence $\{\zeta_n\}$ is classified as a Cauchy sequence if, for every $c \in \mathcal{A}$ where $c \gg \theta$, there exists an integer $n_0 \in \mathbb{N}$ so that $d_c(\zeta_n, \zeta_m) \ll c$ holds true for all $n_0 < n, m$.

*(3) A complete **(CMSs)** is defined as $(\mathcal{U}, d_c)$ if every Cauchy sequence contained in $\mathcal{U}$ converges..*

Example 1.5. *(See[2]) Let $\mathcal{A}$ denote the Banach space comprising all continuous real-valued functions $C(\mathcal{K})$ defined on a topological space $\mathcal{K}$, with multiplication performed pointwise. Consequently, $\mathcal{A}$ qualifies as a **BA**, where the function $f(\zeta) = 1$ serves as the identity element of $\mathcal{A}$.*

Define the set $P = \{f \in \mathcal{A} : 0 \leq f(\zeta)$for every$\zeta \in \mathcal{K}\}$. It follows that $\mathcal{P} \subset \mathcal{A}$ constitutes a normal cone with a normal constant $L = 1$. Let $\mathcal{U} = C(\mathcal{K})$, equipped with the metric $d_c : \mathcal{U} \times \mathcal{U} \to \mathcal{A}$ defined by the expression

$$d_c(f, g) = |f(\zeta) - g(\zeta)|$$

for each $\zeta \in \mathcal{K}$. Thus, $(\mathcal{U}, d_c)$ forms a (CMSs) over the (BA) $\mathcal{A}$.

Lemma 1.6. *(See[3]) Let $(\mathcal{U}, d_c)$ represent a **(CMSs)**, and let $\mathcal{P}$ denote a normal cone characterized by a usual constant $\mathcal{K}$. Consider the sequence ζ_n within the space $\mathcal{U}$. Then.*

(1) The sequence ζ_n is said to converge to ζ iff the distance $d_c(\zeta_n, \zeta)$ approaches 0 as n approaches infinity.

(2) A sequence ζ_n is classified as a Cauchy sequence iff the distance $d_c(\zeta_n, \zeta_m)$ approaches zero as both m and n tend to infinity.

Definition 1.7. *(See[3]) Let $(\mathcal{U}, d_c)$ represent a **(CMSs)**. If every Cauchy sequence within $\mathcal{U}$ converges, then $\mathcal{U}$ is referred to as a complete **(CMSs)**.*

Lemma 1.8. *(See[3])Let $(\mathcal{U}, d_c)$ represent a **(CMSs)**, and let $\mathcal{P}$ denote a normal cone characterized by a normal constant $\mathcal{K}$. Consider the sequences ζ_n and ζ_n within the space $\mathcal{U}$.*

(1) If the sequence ζ_n approaches the value ζ and simultaneously converges to the value ξ, it follows that ζ must equal ξ. This indicates that the limit of the sequence ζ_n is unique, and it is evident that the limit of the sequence ξ_n is also unique.

(2) The sequences ζ_n and ξ_n converge to ζ and ξ, respectively as n approaches infinity, then the distance $d_c(\zeta_n, \zeta_m)$ converges to $d_c(\zeta, \xi)$ as n approaches infinity.

Lemma 1.9. *(See[9]) Consider $\mathcal{A}$ represent a **(BA)**, and let k be a vector within $\mathcal{A}$. If the spectral radius $r(k)$ meets the criterion $0 \leq r(k) < 1$, it can be concluded that*

$$r((e-k)^{-1}) < (1 - r(k))^{-1}.$$

Lemma 1.10. *(See[7]) Consider $\mathcal{A}$ represent a **(BA)**, and let ζ and ξ denote vectors within $\mathcal{A}$. If the vectors ζ and ξ commute, the statement is valid:*

(1) $r(\zeta)r(\xi) \geq r(\zeta\xi)$;

(2) $r(\zeta) + r(\xi)r(\zeta + \xi) \geq r(\zeta + \xi)$;

(3) $r(\zeta - \xi) \geq |r(\zeta) - r(\xi)|$.

Lemma 1.11. *(See[8] [7]) If $\mathcal{A}$ represents a real **(BA)** equipped with a cone $\mathcal{P}$, and let $\{\zeta_n\}$ denote a sequence within $\mathcal{A}$. Assume that $||\zeta_n||$ approaches 0 as n tends to infinity for any $c \gg \theta$. It follows that $c \gg \zeta_n$ for all $N^1 < n$, where N^1 is a natural number.*

Lemma 1.12. *(See[8] [7]) If E is a real Banach space equipped with a solid cone $\mathcal{P}$, and if the norm $||\zeta_n||$ approaches 0 as n tends to infinity, then for any θ that is significantly less than c, there exists a natural number N such that for all n greater than N, it follows that ζ_n is also significantly less than c.*

Lemma 1.13. *(See[7]) Consider $\mathcal{A}$ represent a (BA) and let k be an element of $\mathcal{A}$. If it holds that $\rho(k) < 1$, then it follows that*

$$\lim_{n\to\infty} ||k^n|| = 0.$$

2. MAIN RESULTS

In this section, we will illustrate the **(FP)** and **(CFP)** for a set of contractive mappings by employing the normality of the cone in the framework of **(BA)**.

Theorem 2.1. *Consider $(\mathcal{U}, d_c)$ represent a complete* ***(CMSs)*** *equipped with a* ***(BA)*** *$\mathcal{A}$, and let $\mathcal{P}$ denote a nonnormal cone characterized by a usual constant $\mathcal{K}$. Consider that the mappings are f and $g : \mathcal{U} \to \mathcal{U}$ fulfill the contractive condition expressed as follows:*

$$d_c(f\zeta, g\xi) \preceq \mu d_c(\zeta, \xi)$$

for every $\zeta, \xi \in \mathcal{U}$, where μ has a constant within the interval $[0, 1)$. Under these conditions, the mappings f and g possess a unique ***(CFP)*** *in $\mathcal{U}$. Furthermore, for any element $\zeta \in \mathcal{U}$, the iterative sequences $\{f^{2n+1}\zeta\}$ and $\{g^{2n+2}\zeta\}$ will converge to this* ***(CFP)****.*

Proof. Select element ζ_0 from the set $\mathcal{U}$. Define ζ_1 as $f(\zeta_0)$, ζ_3 as $f(\zeta_1)$, which can also be expressed as $f^3(\zeta_0)$, and in a general sense, ζ_{2n+1} can be represented as $f(\zeta_{2n})$, or $f^{2n+1}(\zeta_0)$.

In a similar manner, we can express ζ_2 as $g(\zeta_1)$, which is equivalent to $g^2(\zeta_0)$, and ζ_4 as $g(\zeta_3)$, or $g^4(\zeta_0)$. Thus, in general, ζ_{2n+2} can be defined as $g(\zeta_{2n+1})$, or $g^{2n+2}(\zeta_0)$.

$$\begin{aligned} d_c(\zeta_{2n+1}, \zeta_{2n}) = d_c(f\zeta_{2n}, g\zeta_{2n-1}) &\preceq \mu d_c(\zeta_{2n}, \zeta_{2n-1}) \\ &\preceq \mu^2 d_c(\zeta_{2n-1}, \zeta_{2n-2}) \quad \preceq \ldots \preceq \mu^{2n} d_c(\zeta_1, \zeta_0). \end{aligned}$$

For values of n greater than m, we obtain.

$$\begin{aligned} d_c(\zeta_{2n}, \zeta_{2m}) &\preceq d_c(\zeta_{2n}, \zeta_{2n-1}) + d_c(\zeta_{2n-1}, \zeta_{2n-2}) + - - - - - - - + d_c(\zeta_{2m+1}, \zeta_{2m}) \\ &\preceq (\mu^{2n-1} + \mu^{2n-2} + - - - - - + \mu^{2m}) d_c(\zeta_1, \zeta_0) \\ &\preceq (\sum_{n=0}^{\infty} \mu^{2n}) \mu^{2m} d_c(\zeta_1, \zeta_0) \\ &\preceq \frac{\mu^{2m}}{e - \mu} d_c(\zeta_1, \zeta_0) \\ &\preceq \mu^{2m}(e - \mu)^{-1} d_c(\zeta_1, \zeta_0) \end{aligned}$$

According to Lemma 1.13, it follows that.

$$\begin{aligned} \|d_c(\zeta_{2n}, \zeta_{2m})\| &\preceq \|\mathcal{K}\mu^{2m}(e - \mu)^{-1} d_c(\zeta_1, \zeta_0)\| \\ &\preceq \mathcal{K}\|\mu^{2m}\|.\|(e - \mu)^{-1}\|.\|d_c(\zeta_1, \zeta_0)\| \to 0 (m \to \infty) \end{aligned}$$

This indicates that.

$$d_c(\zeta_{2n}, \zeta_{2m}) \preceq \mu^{2m}(e - \mu)^{-1} d_c(\zeta_1, \zeta_0) \to 0 (n, m \to \infty).$$

According to Lemma 1.12, for every θ that is significantly less than c, there exists a value N_1 such that the inequality

$$d_c(\zeta_{2n}, \zeta_{2m}) \ll c$$

holds true for all n greater than N_1.

Therefore, the sequence $\{\zeta_{2n}\}$ qualifies as a Cauchy sequence. Given that $f(\mathcal{U})$ is a complete subspace of $\mathcal{U}$, it exists an element $\zeta^* \in \mathcal{U}$ so that $\zeta_{2n} \to \zeta^*$ as n approaches infinity. Additionally, we have.

$$\begin{aligned} d_c(f\zeta^*, \zeta^*) &\preceq d_c(f\zeta_{2n}, f\zeta^*) + d_c(f\zeta_{2n}, \zeta^*) \\ &\preceq \mu d_c(\zeta_{2n}, \zeta^*) + d_c(\zeta_{2n+1}, \zeta^*) \end{aligned}$$

As a result,

$$\|d_c(f\zeta^*, \zeta^*)\| \preceq \mathcal{K}(\|\mu\|\|d_c(\zeta_{2n}, \zeta^*)\| + \|d_c(\zeta_{2n+1}, \zeta^*)\|) \to 0 \quad (n \to \infty)$$

Therefore, it follows that $\|d_c(f\zeta^*, \zeta^*)\| = 0$. This indicates that $f\zeta^* = \zeta^*$. Consequently, ζ^* is identified as a fixed point of the function f. Furthermore, if ξ^* represents another **(FP)** of f, then we have

$$d_c(\zeta^*, \xi^*) = d_c(f\zeta^*, f\xi^*) \preceq \mu d_c(\zeta^*, \xi^*)$$

This can be expressed as

$$(e - \mu)d_c(\zeta^*, \xi^*) \preceq \theta$$

By multiplying both sides of the inequality by

$$(e - \mu)^{-1} = \left(\sum_{i=0}^{\infty} \mu^i \right) \succeq \theta.$$

Consequently, we have $\|d_c(\zeta^*, \xi^*)\| = \theta$, which indicates that $\zeta^* = \xi^*$, resulting in a contradiction. Therefore, the **(FP)** of the function f is unique.

In a similar manner, it can be demonstrated that $g\zeta^* = \zeta^*$. As a result, we find that $f\zeta^* = \zeta^* = g\zeta^*$.

Thus, ζ^* serves as the **(CFP)** for the mappings f and g. This concludes the proof. □

Corollary 2.2. *Consider a complete* ***(CMSs)*** $(\mathcal{U}, d_c)$ *equipped with a (BA)* $\mathcal{A}$ *and a non-normal cone* $\mathcal{P}$ *that possesses a usual constant* $\mathcal{K}$*. Assume that the maps* f *and* g *from* $\mathcal{U}$ *to itself fulfill the following 1. requirement for certain positive integer* n*:*

$$d_c(f^{2n+1}\zeta, g^{2n+2}\xi) \preceq \mu d_c(\zeta, \xi)$$

in favor of every $\zeta, \xi \in \mathcal{U}$*, where* $\mu \in [0, 1)$ *is a constant. Under these circumstances,* f *and* g *exhibit a unique* ***(CFP)*** *within the space* $\mathcal{U}$*.*

Proof. In the aforementioned theorem, it is asserted that the function f^{2n+1} possesses a unique **(FP)** denoted as ζ^*. Furthermore, it is demonstrated that $f^{2n+1}(f\zeta^*) = f(f^{2n+1}\zeta^*) = f\zeta^*$, indicating that $f\zeta^*$ is also a **(FP)** of f^{2n+1}. This results in the inference that $f\zeta^* = \zeta^*$, thereby establishing that ζ^* is a **(FP)** of f. Given that the (FP) of f is also a **(FP)** of f^{2n+1}, it can be inferred that the **(FP)** of f is indeed unique.

In a comparable trend, it can be demonstrated that $g\zeta^* = \zeta^*$. Consequently, we have $f\zeta^* = \zeta^* = g\zeta^*$. Thus, ζ^* serves as a **(CFP)** for both f and g. □

Theorem 2.3. *Let $(\mathcal{U}, d_c)$ denote a complete* ***(CMSs)*** *equipped with a* ***(BA)*** *$\mathcal{A}$. Consider $\mathcal{P}$ as a nonnormal cone characterized by a normal constant $\mathcal{K}$. If the functions f and $g : \mathcal{U} \to \mathcal{U}$ fulfill the condition of contraction given by*

$$d_c(f\zeta, g\xi) \preceq \mu[d_c(f\zeta, \zeta) + d_c(g\xi, \xi)]$$

in favor of every elements $\zeta, y \in \mathcal{U}$, where μ is a constant in the interval $[0, \frac{1}{2})$, it follows that f and g possess a unique ***(CFP)*** *within $\mathcal{U}$. Furthermore, for any $\zeta \in \mathcal{U}$, the iterative sequences $\{f^{2n+1}\zeta\}$ and $\{g^{2n+2}\zeta\}$ will converge to this* ***(CFP)****.*

Proof. Select $\zeta_0 \in \mathcal{U}$. Establish the sequence as follows: $\zeta_1 = f\zeta_0$, $\zeta_3 = f\zeta_1 = f^3\zeta_0$, and continuing in this manner, we have $\zeta_{2n+1} = f\zeta_{2n} = f^{2n+1}\zeta_0$.

In a same manner, we can express the even-indexed terms: $\zeta_2 = g\zeta_1 = g^2\zeta_0$, $v_4 = g\zeta_3 = g^4\zeta_0$, and thus, $\zeta_{2n+2} = g\zeta_{2n+1} = g^{2n+2}\zeta_0$. We possess

$$\begin{aligned} d_c(\zeta_{2n+1}, \zeta_{2n}) &= d_c(f\zeta_{2n}, g\zeta_{2n-1}) \\ &\preceq \mu[d_c(f\zeta_{2n}, \zeta_{2n}) + d_c(g\zeta_{2n-1}, \zeta_{2n-1})] \\ &= \mu[d_c(\zeta_{2n+1}, \zeta_{2n}) + d_c(\zeta_{2n}, \zeta_{2n-1})] \end{aligned}$$

Therefore

$$\begin{aligned} d_c(\zeta_{2n+1}, \zeta_{2n}) &\preceq \mu(e-\mu)^{-1} d_c(\zeta_{2n}, \zeta_{2n-1}) \\ &= \delta d_c(\zeta_{2n}, \zeta_{2n-1}) \end{aligned}$$

Where $\delta = \mu(e-\mu)^{-1}$. For $n > m$

$$\begin{aligned} d_c(\zeta_{2n}, \zeta_{2m}) &\preceq d_c(\zeta_{2n}, \zeta_{2n-1}) + d_c(\zeta_{2n-1}, \zeta_{2n-2}) + - - - - - - - + d_c(\zeta_{2m+1}, \zeta_{2m}) \\ &\preceq (\delta^{2n-1} + \delta^{2n-2} + - - - - + \delta^{2m}) d_c(\zeta_1, \zeta_0) \\ &\preceq (\sum_{n=0}^{\infty} \delta^{2n}) \delta^{2m} d_c(\zeta_1, \zeta_0) \\ &\preceq \frac{\delta^{2m}}{e-\delta} d_c(\zeta_1, \zeta_0) \\ &\preceq \delta^{2m}(e-\delta)^{-1} d_c(\zeta_1, \zeta_0) \end{aligned}$$

Consequently, according to lemma 1.13, we obtain.

$$\begin{aligned}\|d_c(\zeta_{2n},\zeta_{2m})\| &\preceq \|\mathcal{K}\delta^{2m}(e-\delta)^{-1}d_c(\zeta_1,\zeta_0)\| \\ &\preceq \mathcal{K}\|\delta^{2m}\|.\|(e-\delta)^{-1}\|.\|d_c(\zeta_1,\zeta_0)\| \to 0(m\to\infty)\end{aligned}$$

This indicates that

$$d_c(\zeta_{2n},\zeta_{2m}) \preceq \delta^{2m}(e-\delta)^{-1}d_c(\zeta_1,\zeta_0) \to 0 \text{ as } n,m\to\infty.$$

Therefore, according to Lemma 1.12, for every $c \gg \theta$, there is an integer N_1 so that.

$$c \gg d_c(\zeta_{2n},\zeta_{2m})$$

holds for every $N_1 < n$.

Therefore, the sequence $\{\zeta_{2n}\}$ constitutes a Cauchy sequence. Since $f(\mathcal{U})$ constitutes a complete subspace of $\mathcal{U}$, it follows that there exists an element $\zeta^* \in \mathcal{U}$ such that $\zeta_{2n} \to \zeta^*$ as n approaches infinity. Additionally, it follows that.

$$\begin{aligned}d_c(f\zeta^*,\zeta^*) &\preceq d_c(f\zeta_{2n},f\zeta^*) + d_c(f\zeta_{2n},\zeta^*) \\ &\preceq \delta[d_c(\zeta_{2n},\zeta_{2n}) + d_c(f\zeta^*,\zeta^*)] + d_c(f\zeta_{2n+1},\zeta^*)\end{aligned}$$

$$d_c(f\zeta^*,\zeta^*) \preceq (e-\delta)^{-1}[\delta d_c(\zeta_{2n},\zeta_{2n}) + d_c(f\zeta_{2n+1},\zeta^*)]$$

$$\|d_c(f\zeta^*,\zeta^*)\| \preceq \mathcal{K}(e-\delta)^{-1}[\|\delta\|\|d_c(\zeta_{2n+1},\zeta_{2n})\| + \|d_c(\zeta_{2n+1},\zeta^*)\|] \to 0 \ \ (n\to\infty)$$

Therefore, it follows that $0 = \|d_c(f(\zeta^*),\zeta^*)\|$. This indicates that $f(\zeta^*) = \zeta^*$. Consequently, ζ^* is identified as a **(FP)** of the function f. Furthermore, if ξ^* represents an additional **(FP)** of f, then.

$$\begin{aligned}d_c(\zeta^*,\xi^*) &= d_c(f\zeta^*,f\xi^*) \\ &\preceq \delta[d_c(f\zeta^*,\zeta^*) + d_c(g\xi^*,\xi^*)] \\ &= \theta\end{aligned}$$

Consequently, we have $\|d_c(f(\zeta^*),\zeta^*)\| = \theta$, this results in the conclusion that $\zeta^* = \xi^*$, resulting in a contradiction. Hence, the **(FP)** of the function f is unique.

In a comparable manner, we can demonstrate that $g\zeta^* = \zeta^*$. Therefore, it can be concluded that. $f\zeta^* = \zeta^* = g\zeta^*$.

Thus, ζ^* serves as the **(CFP)** for the maps f and g. This concludes the result. □

Theorem 2.4. *Consider $(\mathcal{U}, d_c)$ represent a complete **(CMSs)** equipped with a **(BA)** $\mathcal{A}$. Consider $\mathcal{P}$ as a non-normal cone characterized by a normal constant $\mathcal{K}$. Assume that the maps f and $g : \mathcal{U} \to \mathcal{U}$ fulfill the condition of contraction given by*

$$d(f\zeta, g\xi) \preceq \mu[d_c(f\zeta, \xi) + d_c(g\xi, \zeta)]$$

*in favor of every elements $\zeta, \xi \in \mathcal{U}$, where μ is a constant within the interval $[0, \frac{1}{2})$. Under these circumstances, it can be concluded that f and g possess a unique **(CFP)** in $\mathcal{U}$. Furthermore, for all $\zeta \in \mathcal{U}$, the iterative sequences $\{f^{2n+1}\zeta\}$ and $\{g^{2n+2}\zeta\}$ will converge to this **(CFP)**.*

Proof. Select $\zeta_0 \in \mathcal{U}$. Establish the sequence as follows: $\zeta_1 = f\zeta_0$, $\zeta_3 = f\zeta_1 = f^3\zeta_0$, and in general, $\zeta_{2n+1} = f\zeta_{2n} = f^{2n+1}\zeta_0$.

In a similar manner, we can express the even-indexed terms: $\zeta_2 = g\zeta_1 = g^2\zeta_0$, $\zeta_4 = g\zeta_3 = g^4\zeta_0$, leading to the general form $\zeta_{2n+2} = g\zeta_{2n+1} = g^{2n+2}\zeta_0$.

Thus, we have established the sequences.

$$\begin{aligned} d_c(\zeta_{2n+1}, \zeta_{2n}) &= d_c(f\zeta_{2n}, g\zeta_{2n-1}) \\ &\preceq \mu[d_c(f\zeta_{2n}, \zeta_{2n-1}) + d_c(g\zeta_{2n-1}, \zeta_{2n})] \\ &= \mu[d_c(\zeta_{2n+1}, \zeta_{2n}) + d_c(\zeta_{2n}, \zeta_{2n-1})] \end{aligned}$$

Therfore

$$\begin{aligned} d_c(\zeta_{2n+1}, \zeta_{2n}) &\preceq \mu(e - \mu)^{-1} d_c(\zeta_{2n}, \zeta_{2n-1}) \\ &= \delta d_c(\zeta_{2n}, \zeta_{2n-1}) \end{aligned}$$

Where $\delta = \mu(e - \mu)^{-1}$. For $n > m$

$$\begin{aligned} d_c(\zeta_{2n}, \zeta_{2m}) &\preceq d_c(\zeta_{2n}, \zeta_{2n-1}) + d_c(\zeta_{2n-1}, \zeta_{2n-2}) + - - - - - - - + d_c(\zeta_{2m+1}, \zeta_{2m}) \\ &\preceq (\delta^{2n-1} + \delta^{2n-2} + - - - - + \delta^{2m}) d_c(\zeta_1, \zeta_0) \\ &\preceq (\sum_{n=0}^{\infty} \delta^{2n}) \delta^{2m} d_c(\zeta_1, \zeta_0) \\ &\preceq \frac{\delta^{2m}}{e - \delta} d_c(\zeta_1, \zeta_0) \\ &\preceq \delta^{2m}(e - \delta)^{-1} d(\zeta_1, \zeta_0) \end{aligned}$$

Consequently, according to lemma 1.13, we obtain

$$\begin{aligned}\|d_c(\zeta_{2n}, \zeta_{2m})\| &\preceq \|\mathcal{K}\delta^{2m}(e-\delta)^{-1}d_c(\zeta_1, \zeta_0)\| \\ &\preceq \mathcal{K}\|\delta^{2m}\|.\|(e-\delta)^{-1}\|.\|d_c(\zeta_1, \zeta_0)\| \to 0(m \to \infty)\end{aligned}$$

This indicates that

$$d_c(\zeta_{2n}, \zeta_{2m}) \preceq \delta^{2m}(e-\delta)^{-1}d_c(\zeta_1, \zeta_0) \to 0(n, m \to \infty)$$

According to the lemma 1.12 for each $c \gg \theta$ there is a number N_1 so that.

$$d_c(\zeta_{2n}, \zeta_{2m}) \ll c$$

holds for everyone $N_1 < n$.

Hence $\{\zeta_{2n}\}$ forms a Cauchy sequence. As $f(\mathcal{U})$ a complete subspace of $\mathcal{U}$ exists $\zeta^* \in \mathcal{U}$ so that $\zeta_{2n} \to \zeta^*$ as $n \to \infty$. Furthermore, one has

$$\begin{aligned}d_c(f\zeta^*, \zeta^*) &\preceq d_c(f\zeta_{2n}, f\zeta^*) + d_c(f\zeta_{2n}, \zeta^*) \\ &\preceq \mu[d_c(f\zeta^*, \zeta_{2n}) + d_c(f\zeta_{2n}, \zeta^*)] + d_c(f\zeta_{2n+1}, \zeta^*)\end{aligned}$$

$$d_c(f\zeta^*, \zeta^*) \preceq \mu[d_c(f\zeta^*, \zeta^*) + d_c(\zeta_{2n}, \zeta^*) + d_c(f\zeta_{2n+1}, \zeta^*)] + d_c(f\zeta_{2n+1}, \zeta^*)$$

$$d_c(f\zeta^*, \zeta^*) \preceq (e-\mu)^{-1}[\mu\{d_c(\zeta_{2n}, \zeta^*) + d_c(f\zeta_{2n+1}, \zeta^*)\} + d_c(f\zeta_{2n+1}, \zeta^*)]$$

$$\|d_c(f\zeta^*, \zeta^*)\| \preceq \mathcal{K}(e-\mu)^{-1}[\mu\{\|d_c(\zeta_{2n}, \zeta^*)\| + \|d_c(\zeta_{2n+1}, \zeta^*)\|\} + \|d_c(\zeta_{2n+1}, \zeta^*)\|] \to 0.$$

Hence $\|d_c(f\zeta^*, \zeta^*)\| = \theta$. This indicates $f\zeta^* = \zeta^*$. So ζ^* is a **(FP)** of f.

So, if y^* represents an alternative **(FP)** of f then

$$\begin{aligned}d_c(\zeta^*, \xi^*) &= d_c(f\zeta^*, f\xi^*) \\ &\preceq \mu[d_c(f\zeta^*, \xi^*) + d_c(g\xi^*, \zeta^*)] \\ &= 2\mu d_c(\zeta^*, \xi^*)\end{aligned}$$

Consequently, it follows that $\|d_c(\zeta^*, \xi^*)\| = \theta$, leading to the conclusion that $\zeta^* = \xi^*$, which presents a contradiction. Thus, the **(FP)** of f is established as unique.

Similarly, we can establish that $g\zeta^* = \zeta^*$. Therefore, $f\zeta^* = \zeta^* = g\zeta^*$.

Therefore, ζ^* is the **(CFP)** of the maps f and g. This completes the proof. □

3. CONCLUSION

In this research, we have established definite **(CFP)** theorems within the framework of complete **(CMSs)** using **(BA)**. We have also discussed the implications of our main findings. The results presented in this article build upon and broaden multiple aspects findings from existing literature.

4. ACKNOWLEDGEMENT

The author extends appreciation to the referee for their insightful recommendations that improved the overall presentation of the paper.

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