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"Companion" Riemann Zeta Function Zeros

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Abstract

An exponentially weighted Dirichlet series is used to map the non-trivial Riemann zeta function zeros to a circle. The imaginary parts of this circle are taken to be the imaginary parts of companion Riemann zeta function zeros. This results in a dynamical zeta function.

Keywords : Riemann zeta function zeros, exponentially weighted Dirichlet series, Euler product formula, dynamical zeta functions

1. INTRODUCTION

Dynamical zeta functions are generating functions for the lengths of closed orbits of a map f that sends a set M to itself. An example is the map $x \to 1 - \mu x^2$ of the interval $[-1, 1]$ to itself. For a special value of $\mu \approx 1.40155...$ (the Feiningenbaum value), this map has one periodic order of 2^n for every integer $n \geq 0$. This ζ then satisfies the functional equation $\zeta(z^2) = (1 - z)\zeta(z)$. The companion zeta function zeros to be empirically derived satisfy this functional equation up to a scaling factor and an offset. For a linear least-squares fit of the real parts of $\zeta(z^2)$ versus the real parts of $(1-z)\zeta(z)$, the slope is 1 and the y-intercept is 6 (a perfect fit). For a linear least-squares fit of the imaginary parts of $\zeta(z^2)$ versus the imaginary parts of $(1-z)\zeta(z)$, the slope is 4 and the y-intercept is 0 (a perfect fit). The real part of z is set to 2.0. These linear least-squares fits were for 100000 companion zeta function zeros. See Kargin [1] for more details on dynamical zeta functions.

2. EXPONENTIALLY WEIGHTED DIRICHLET SERIES

A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{1}$$

where $s = (a, b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1 - e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \cos(kb)$ and the imaginary part can be expressed as

$\sum_{k=1}^{\infty} e^{-ka}\sin(kb)$. A partial sum of this function will be denoted by $D(n, a, b)$. See the Methods section for the C code.

3. NON-TRIVIAL RIEMANN ZETA FUNCTION ZEROS

A plot of the imaginary parts of the first hundred thousand non-trivial zeta function zeros is

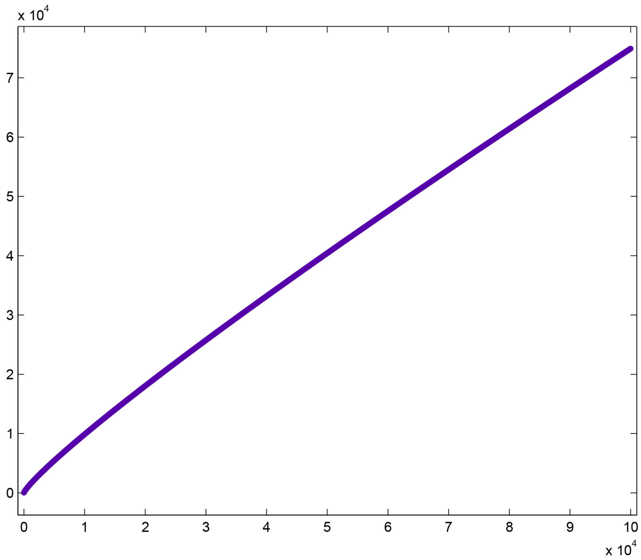

Figure 1

A plot of the zeta function zeros versus $\sqrt{x}, x = 1, 2, 3, \ldots, 100000$ is

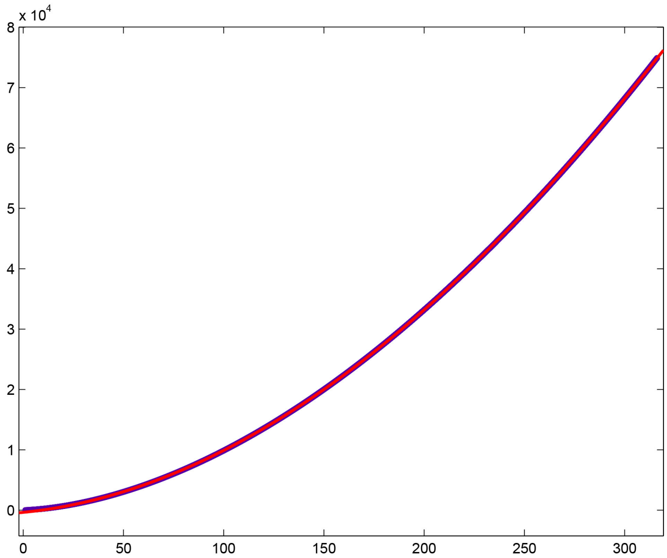

Figure 2

For a cubic least-squares fit of the curve, $p_1 = -0.0002153$ with a 95% confidence interval of $(-0.0002155, -0.000215)$, $p_2 = 0.7156$ with a 95% confidence interval of (0.7155, 0.7157), $p_3 = 33.21$ with a 95% confidence interval of (33.19, 33.23), $p_4 = -365.2$ with a 95% confidence interval of $(-366.4, -364)$, SSE=$3.653 \cdot 10^7$, R-squared=1, and RMSE=19.11.

A plot of the imaginary parts of $D(n, a, b)$ versus the real parts for these Riemann zeta function zeros is

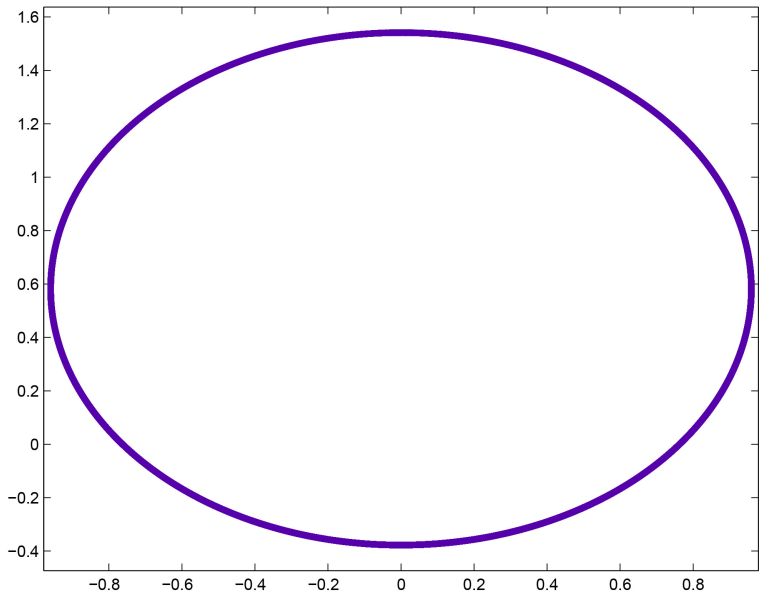

Figure 3

4. COMPANION ZETA FUNCTION ZEROS

The imaginary parts of the above circle are taken to be the imaginary parts of the companion Riemann zeta function zeros. The real part is set to 2.0 (the reciprocal of 1/2).

A plot of the sorted imaginary parts of the companion zeta function zeros is

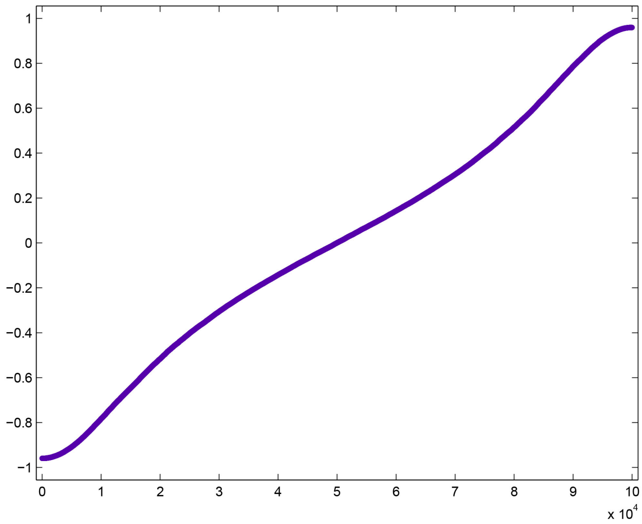

Figure 4

The values are between -1.0 and 1.0. The curve is symmetrical, so only the first fifty thousand companion zeta function zeros (those less than 0) will be considered. A plot of these values versus $\sqrt{x}$, $x = 1, 2, 3, \ldots, 50000$, is

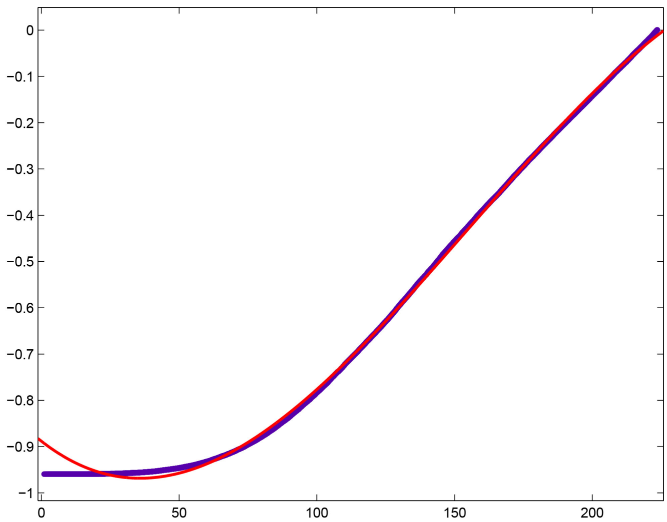

Figure 5

For a cubic least-squares fit of the curve, $p_1 = -1.576 \cdot 10^{-7}$ with a 95% confidence interval of $(-1.579 \cdot 10^{-7}, -1.572 \cdot 10^{-7})$, $p_2 = 7.367 \cdot 10^{-5}$ with a 95% confidence interval of $(7.354 \cdot 10^{-5}, 7.379 \cdot 10^{-5})$, $p_3 = -0.004674$ with a 95% confidence interval of $(-0.004688, -0.004659)$, $p_4 = -0.8883$ with a 95% confidence interval of $(-0.8888, -0.8878)$, SSE=1.732, R-squared=0.9996, and RMSE=0.005886.

The companion zeta function zeros corresponding to $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ also satisfy the functional equation $\zeta(z^2) = (1-z)\zeta(z)$. The real parts of the z values are set to the reciprocals of $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. For the linear least-squares fits of the real parts of $\zeta(z^2)$ versus the real parts of $(1-z)\zeta(z)$, the slopes are still 1. The y-intercepts are $\Re z \cdot (\Re z * 2 - 1)$. For the linear least-squares fits of the imaginary parts of $\zeta(z^2)$ versus the imaginary parts of $(1-z)\zeta(z)$, the slopes are $2 \cdot \Re z$. The imaginary parts are still 0. For a quadratic least-squares fit of the real parts of $\zeta(z^2)$ versus the imaginary parts, the curve is an upside-down parabola with parameters of -1, 0, and $-\Re z \cdot (\Re z - 1)$. For a quadratic least-squares fit of the real parts of $(1-z) \cdot \zeta(z)$ versus the imaginary parts, the curve is an upside-down parabola with parameters of $-(\frac{1}{2 \cdot \Re z})^2$, 0, and $\Re z^2$.

The intervals for the companion zeta function zeros corresponding to $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{8}$ and the first fifty thousand Riemann zeta function zeros are $(-0.9595172056, 0.9595173744)$, $(-1.4725780883, 1.4725781317)$, $(-1.9793175247, 1.9793173936)$, $(-2.4834104762, 2.4834107811)$, $(-2.9861546050, 2.9861559470)$, $(-3.4881234967, 3.4881233674)$, and $(-3.9896009694, 3.9896016896)$. These intervals are contained in $(-1.0, 1.0)$, $(-1.5, 1.5)$, $(-2.0, 2.0)$, etc.

5. ANALOGUE OF EULER'S PRODUCT FORMULA FOR THE RIEMANN ZETA FUNCTION

If f is a flow on M, that is, a map $MxR^+ \to M$, then the zeta function of this flow is defined as

$$\zeta(s) = \prod_{\omega} (1 - e^{-sl(\omega)})^{-1} \tag{2}$$

where ω denotes a periodic orbit of f and $l(\omega)$ is its length. If it is assumed that prime numbers correspond to periodic orbits of a flow and the lengths of the orbit indexed by p is given by $\log p$, then the zeta function of the flow will be similar to Riemann's zeta function. See the Methods section for C code that computes the above equation with the companion Riemann zeta function zeros as input s values with a real part of 2. A plot of the real parts of the resulting values versus the real parts of the companion Riemann zeta function zeros for the first 50000 values is

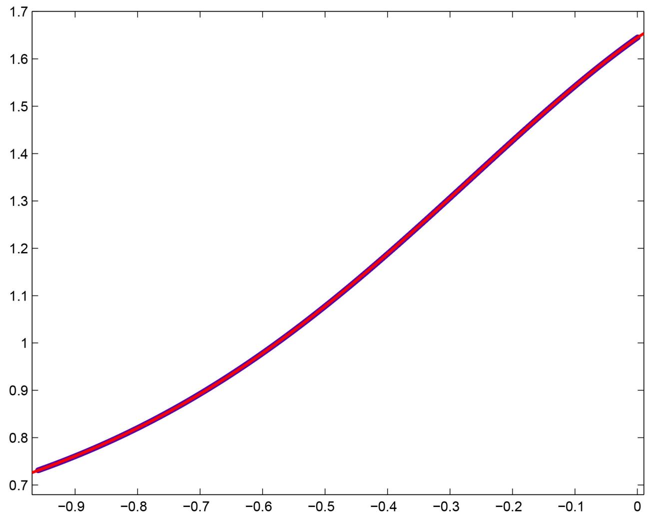

Figure 6

For a quartic least-squares fit of the curve, $p_1 = -0.6808$ with a 95% confidence interval of $(-0.6811, -0.6805)$, $p_2 = -1.846$ with a 95% confidence interval of $(-1.847, -1.845)$, $p_3 = -1.175$ with a 95% confidence interval of $(-1.176, -1.175)$, $p_4 = 0.9228$ with a 95% confidence interval of (0.9227, 0.9229), $p_5 = 1.645$ with a 95% confidence interval of (1.645, 1.645), SSE=0.001017, R-squared=1, and RMSE=0.0001426.

A plot of the imaginary parts of the resulting values versus the imaginary parts of the companion Riemann zeta function zeros for the first 50000 values is

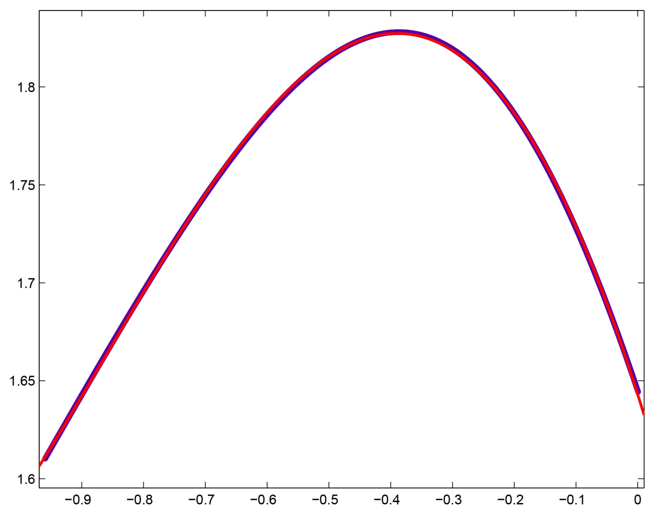

Figure 7

For a quartic least-squares fit of the curve, $p_1 = 0.2153$ with a 95% confidence interval of (0.214, 0.2166), $p_2 = -0.1978$ with a 95% confidence interval of (-0.2004, -0.1953), $p_3 = -1.467$ with a 95% confidence interval of (-1.468, -1.465), $p_4 = -1.002$ with a 95% confidence interval of (-1.002, -1.002), $p_5 = 1.643$ with a 95% confidence interval of (1.643, 1.643), SSE=0.0217, R-squared=0.9999, and RMSE=0.0006588.

Similar quartic curves are obtained for the companion zeta function zeros corresponding to $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

The Euler product formula for the Riemann zeta function is

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, (\Re s > 1) \tag{3}$$

6. APPLICATIONS OF THE MÖBIUS FUNCTION

Let v_n denote $\sum_{i|n}(c_{i+1} - c_i)\mu(i)$ where c_i are the above sorted companion zeta function zeros and μ_i denotes the Möbius function. A plot of v_n, $n = 1, 2, 3, \ldots 99999$, is

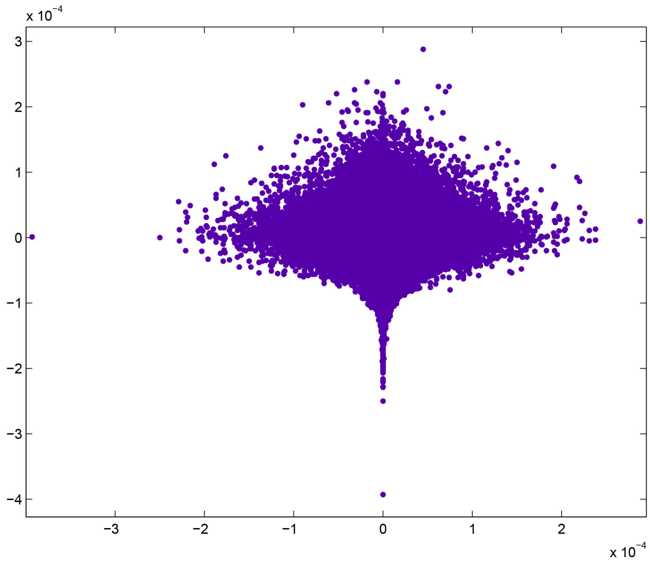

Figure 8

Minimum values (including negative values) in the sequence v_2, v_3, v_4,. . ., v_{50000} are considered. If the current value is smaller than that of all previous values, the index of the minimum is output. The indices are 2, 10, 13, 17, 29, 89, 101, 170, 241, 322, 443, 499, 577, 651, 663, 673, 769, 859, 1087, 1173, 1283, 1583, 1643, 2281, 2351, 2570, 2665, 3854, 4201, 4585, 4703, 4789, 6841, 9579, 10657, 17837, and 18235. These are square-free numbers (usually primes).

7. ANOTHER FUNCTION INVOLVING AN EXPONENTIALLY WEIGHTED DIRICHLET SERIES

The Riemann zeta function $\zeta(s)$ for $0 < Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{4}$$

Let $C(n, a, b)$ denote

$$\frac{2 \cdot n^{-a}}{1 - 2^{1-s}} \cdot (\sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j^s} \cdot \cos(b \cdot (\ln(\frac{n}{j}))) \cdot \sum e^{-js}) \tag{5}$$

where $s = (a, b)$. Note that n is the upper bound of the j values used. See the Methods section for the C code.

For the companion Riemann zeta function zeros corresponding to the first two hundred and fifty Riemann zeta function zeros, the smallest imaginary part is -0.95928639 and the largest imaginary part is 0.95943008. A plot of the imaginary parts of $C(n, a, b)$ versus the real parts for $s = (2.0, -0.95928639)$ and $n = 251$ is

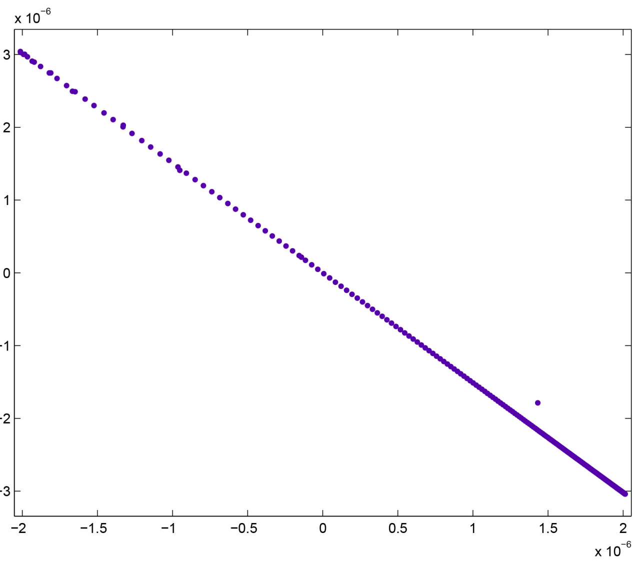

Figure 9

For a linear least-squares fit of the curve, $p_1 = -1.509$ with a 95% confidence interval of $(-1.511,\ -1.506)$, $p_2 = 9.481 \cdot 10^{-10}$ with a 95% confidence interval of $(-3.065 \cdot 10^{-9},\ 4.962 \cdot 10^{-9})$, SSE=$1.426 \cdot 10^{-13}$, R-squared=0.9998, and RMSE=$3.398 \cdot 10^{-8}$. The curve almost passes through the point (0, 0).

A plot of the imaginary parts of $C(n,a,b)$ versus the real parts for $s = (2.0, 0.95943008)$ and $n = 251$ is

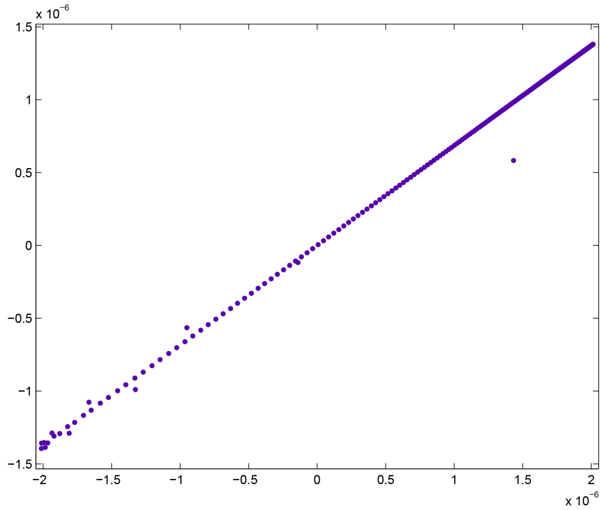

Figure 10

For a linear least-squares fit of the curve, $p_1 = 0.6847$ with a 95% confidence interval of (0.6818, 0.6876), $p_2 = -8.929 \cdot 10^{-10}$ with a 95% confidence interval of ($-5.446 \cdot 10^{-9}$, $3.66 \cdot 10^{-8}$), SSE=$1.836 \cdot 10^{-13}$, R-squared=0.9989, and RMSE=$2.721 \cdot 10^{-8}$. The curve almost passes through the point (0, 0).

The largest negative companion zeta function zero is -0.00027015876. A plot of the real parts and imaginary parts of $C(n,a,b)$ for this s value versus $j = 1, 2, 3, \ldots, 250$ is

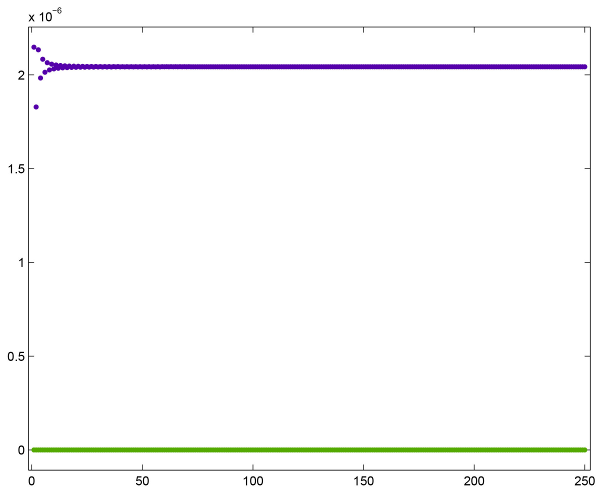

Figure 11

The smallest positive companion zeta function zero is 0.0028783225. A plot of the real parts and imaginary parts of $C(n, a, b)$ for this s value versus $j = 1, 2, 3, \ldots, 250$ is

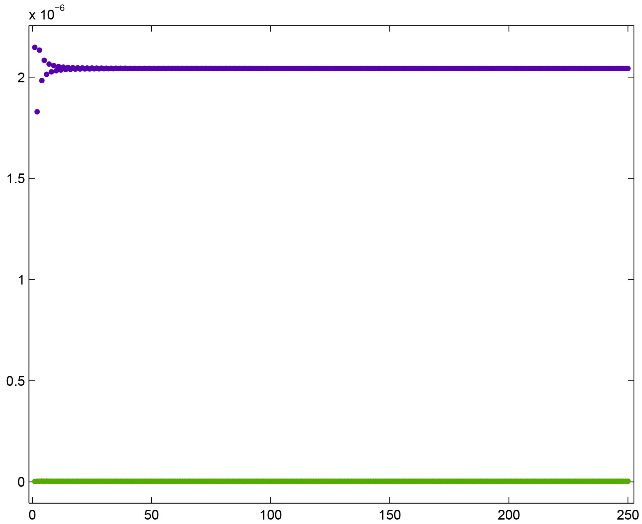

Figure 12

A plot of all two hundred and fifty $C(n, a, b)$ values (real and imaginary parts) for $j = 1, 2, 3, \ldots, 250$ is

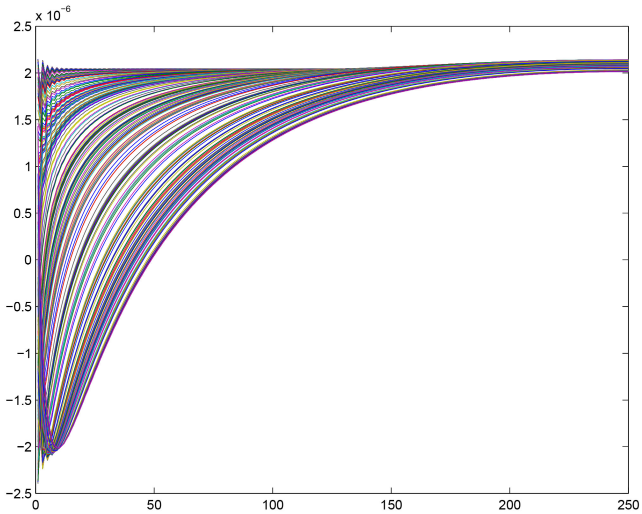

Figure 13

A Hermitian matrix is generated from this matrix by averaging it with its transpose. A plot of the eigenvalues of the Hermitian matrix is

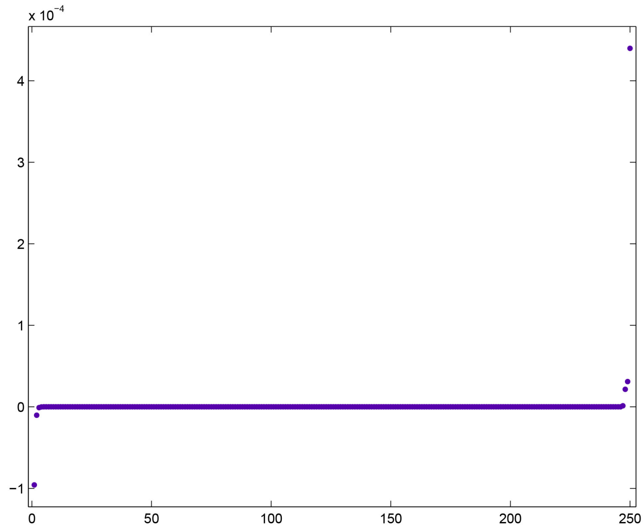

Figure 14

Let v_n denote $\sum_{i|n}(e_{i+1} - e_i)\mu(i)$ where e_i are these eigenvalues. A plot of the v_n values, $n = 1, 2, 3, \ldots, 249$ is

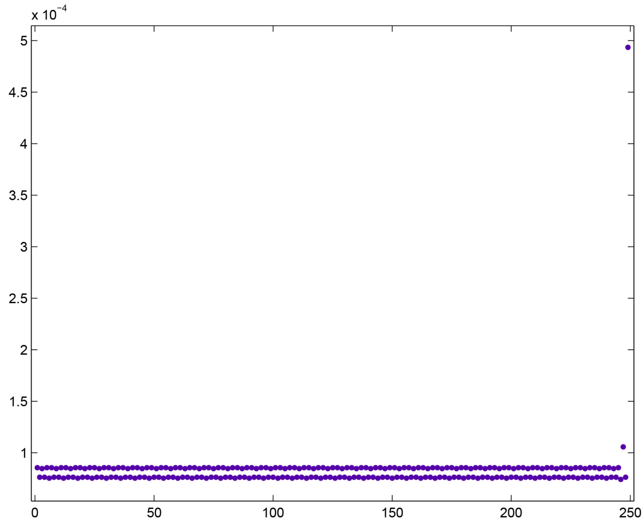

Figure 15

A table of the sorted v_n values and the corresponding n values is

249, 0.0004935179212640,

247, 0.0001057671688000,

1, 0.0000855627090000,
11, 0.0000855627090000,
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138, 0.0000753662992963,
144, 0.0000753662992963,
156, 0.0000753662992963,
162, 0.0000753662992963,
174, 0.0000753662992963,
186, 0.0000753662992963,
192, 0.0000753662992963,
198, 0.0000753662992963,
204, 0.0000753662992963,
216, 0.0000753662992963,
222, 0.0000753662992963,
228, 0.0000753662992963,
234, 0.0000753662992963,

42, 0.0000753662961752,
84, 0.0000753662961752,
126, 0.0000753662961752,
168, 0.0000753662961752,

30, 0.0000753607743628,
60, 0.0000753607743628,
90, 0.0000753607743628,
120, 0.0000753607743628,
150, 0.0000753607743628,
180, 0.0000753607743628,
240, 0.0000753607743628,

210, 0.0000753607712417,

246, 0.0000741240461263,

The smallest n value in a group of equal v_n values divides the other n values. Prime powers have the same v_n values. The first prime in the uppermost curve (neglecting the first two points) is 11. Note that 1, 121, 143, 187, 209, 169, and 221 are the only non-primes in this group. The v_n values for the primes 7, 5, 3, and 2 and their multiples are smaller.

A plot of the 250x250 matrix of $C(n, a, b)$ values where the real part of s is 3.0 and the imaginary parts are the imaginary parts of the companion Riemann zeta function zeros for $\Re s = \frac{1}{3}$ is

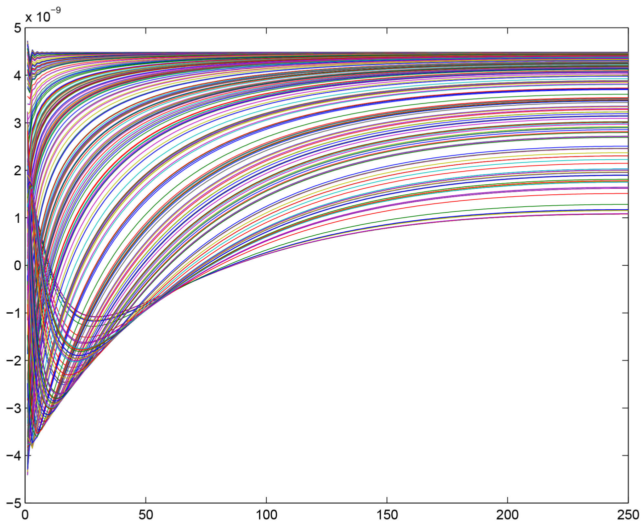

Figure 16

The v_n values generated from the eigenvalues of the corresponding Hermitian matrix have the same properties. Similar results are obtained for other companion Riemann zeta function zeros.

8. METHODS

```
#include <math.h>
#include <stdio.h>
#include "zero1.h" // zeta function zeros
//
// D(n,a,b) 10/2/2024 (dkc)
// compute exponentially weighted series
// similar results are obtained for zeta function zeros
// and eigenvalues of Hermitian matrices
//
unsigned int max=200001;
double a=0.50; // real part
unsigned int size=100000;
void main() {
    unsigned int x,i;
    double esumr,esumi,tempr,b;
    FILE *Outfp;
    Outfp = fopen("weight.dat","w");
    for (i=1; i<=size; i++) { // number of input values
        b=zero[i-1]; // imaginary part
        esumr=0.0;
        esumi=0.0;
        for (x=1; x<=(max-1); x++) {
            tempr=1.0/exp((double)x*a);
            esumr=esumr+tempr*cos((double)x*b);
            esumi=esumi+tempr*sin((double)x*b);
            }
        fprintf(Outfp," %.10llf, %.10llf, %.10llf,\n",b,esumr,esumi);
        printf(" %d, %.10llf, %.10llf,%.10llf, \n",i,b,esumr,esumi);
        }
    fclose(Outfp);
    return;
    }
```

```
#include <math.h>
#include <stdio.h>
#include "table5.h"
#include "sorta.h"
//
// Euler product analogue
//
double a=2.0; // real part
unsigned int size=50000;
void main() {
unsigned int i,x,p;
double prodr,prodi,tempr,b,temp1,temp2;
FILE *Outfp;
Outfp = fopen("funct2.dat","w");
for (i=1; i<=size; i++) { // number of input values
    b=zero[i-1]; // imaginary part
    prodr=1.0;
    prodi=1.0;
    for (x=1; x<=1000; x++) {
        p=table[x-1];
        tempr=1.0/exp(a*log((double)p));
        temp1=tempr*cos(log((double)p)*b);
        temp2=tempr*sin(log((double)p)*b);
        temp1=1.0-temp1;
        tempr=temp1*temp1+temp2*temp2;
        temp1=temp1/tempr;
        temp2=-temp2/tempr;
        tempr=prodr*temp1-prodi*temp2;
        prodi=prodr*temp2+prodi*temp1;
        prodr=tempr;
        }
    fprintf(Outfp," %.10llf, %.10llf, %.10llf,\n",b,prodr,prodi);
    }
fclose(Outfp);
return;
}
```

```
#include <math.h>
#include <stdio.h>
double b=-9.5951592e-001;
//double b=-9.5928639e-001;
//double b=-9.5923819e-001;
//double b=-9.5898736e-001;
//double b=-9.5864776e-001;
//double b=-9.5810006e-001;
//double b=-9.5718336e-001;
//double b=-9.5707140e-001;
//double b=-9.5661458e-001;
//double b=-9.5633198e-001;

//double b= 9.5737171e-001;
//double b= 9.5799538e-001;
//double b= 9.5874571e-001;
//double b= 9.5877219e-001;
//double b= 9.5882628e-001;
//double b= 9.5887992e-001;
//double b= 9.5915584e-001;
//double b= 9.5932896e-001;
//double b= 9.5943008e-001;
//double b= 9.5951737e-001;

//double b=1.1;
//double b=–1.1;
//
// C(n,a,b)
//
unsigned int max=40001;
double a=2.0;
unsigned int xmin=0; // usually set to 0
unsigned int out=1; // usually 1, or 2 for inflection points
unsigned int out3p=1; // set to 1 for differences in j values >=2
unsigned int polar=0; // set to 1 for polar coordinates
void main() {
    unsigned int x,oldx;
```

```
double sumr,sumi,R,I,temp1,oldsumr,oldsumi,temp,tempa,tempb,y,e,f,g;
double esumr,esumi,tempr;
FILE *Outfp;
Outfp = fopen("c2nab2h.dat","w");
y=1.0-a;
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e=temp1*(cos(b*log(2)));
f=temp1*(sin(b*log(2)));
e=1.0-e;
f=-f;
y=-a;
if (y>=0.0)
    temp1=pow((double)max,y);
else {
    temp1=pow((double)max,-y);
    temp1=1.0/temp1;
    }
y=2.0*temp1;
esumr=0.0;
esumi=0.0;
oldx=0;
sumr=0.0;
sumi=0.0;
oldsumi=0.0;
oldsumr=0.0;
for (x=1; x<=(max-1); x++) {
    tempr=1.0/exp((double)x*a);
    esumr=esumr+tempr*cos((double)x*b);
    esumi=esumi+tempr*sin((double)x*b);
    temp1=pow((double)x,a);
    R=temp1*cos(b*log((double)x));
    I=temp1*sin(b*log((double)x));
    temp1=R*R+I*I;
```

```
        if (x!=(x/2)*2) {
            sumr=sumr+R/temp1;
            sumi=sumi-I/temp1;
            }
        else {
            sumr=sumr-R/temp1;
            sumi=sumi+I/temp1;
            }
        temp=cos(b*log((double)max/(double)x));
        tempa=sumr*temp;
        tempb=sumi*temp;
        tempa=tempa*y;
        tempb=tempb*y;
        g=tempa*e-tempb*f;
        tempb=tempa*f+tempb*e;
        tempa=g;
        g=tempa*esumr-tempb*esumi;
        tempb=tempa*esumi+tempb*esumr;
        tempa=g;
        if (x>xmin) {
            if (out==1) {
                if (polar==0)
                    fprintf(Outfp," %.10llf, %.10llf, \n",tempa,tempb);
                else
                    fprintf(Outfp,"          %.10llf,          %.10llf,
\n",sqrt(tempa*tempa+tempb*tempb),atan2(tempb,tempa));
                }
            if ((out==2)&&((oldsumr>0.0)&&(tempa<0.0))) {
                if (out3p==0)
                    fprintf(Outfp,"   %d   %.10llf   %.10llf   %.10llf   %.10llf
\n",x,tempa,tempb,oldsumr,oldsumi);
                if ((x-oldx)>2) {
                    if (out3p!=0)
                        fprintf(Outfp," %d, %d, \n",x,oldx);
                    }
                oldx=x;
                }
            oldsumr=tempa;
```

```
            oldsumi=tempb;
        }
    }
    fclose(Outfp);
    return;
}
```

REFERENCES

[1] Vladislav Kargin, Statistical properties of zeta functions’ zeros, Probability Surveys, Vol. 11 (2014) 121-160, ISSN: 1549-5787