Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 4 (2024), pp. 729-745

http://ripublication.com/gjpam23/GJPAMv20n4_22.pdf

Chebyshev's Second Function and a Dynamical Zeta Function

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Abstract

An exponentially-weighted Dirichlet series is used to map Chebyshev's second function to a circle. The imaginary part of this circle is used to derive a dynamical zeta function. A related dynamical zeta function is derived using a logarithmically-weighted Dirichlet series. This is relevant to the Riemann hypothesis.

Keywords: Chebyshev's second function, exponentially weighted Dirichlet series, Euler product formula, dynamical zeta functions, Riemann hypothesis.

1. INTRODUCTION

Chebyshev's second function is the summatory Mangoldt function, that is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n), x > 0. \tag{1}$$

$\Lambda(n)$ equals $\log(p)$ if $n = p^m$ for some prime p and some $m \geq 1$ or 0 otherwise. The prime number theorem is equivalent to the asymptotic formula

$$\sum_{n \leq x} \Lambda(n) \sim x, x \to \infty \tag{2}$$

This asymptotic formula states that

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1. \tag{3}$$

Dynamical zeta functions are generating functions for the lengths of closed orbits of a map f that sends a set M to itself. An example is the map $x \to 1 - \mu x^2$ of the interval $[-1, 1]$ to itself. For a special value of $\mu \approx 1.40155...$ (the Feiningenbaum value), this map has one periodic order of 2^n for every integer $n \geq 0$. This ζ then satisfies the functional equation $\zeta(z^2) = (1 - z)\zeta(z)$. See Kargin [1] for more details on dynamical zeta functions.

2. EXPONENTIALLY WEIGHTED DIRICHLET SERIES

A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{4}$$

where $s = (a, b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1 - e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \cos(kb)$ and the imaginary part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \sin(kb)$. A partial sum of this function will be denoted by $D(n, a, b)$. See the Methods section for the C code for generating a variant dynamical zeta function using this transformation.
A plot of the imaginary parts of $D(n, a, b)$ versus the real parts for $a = \frac{1}{2}$ and b the first thousand ψ values is

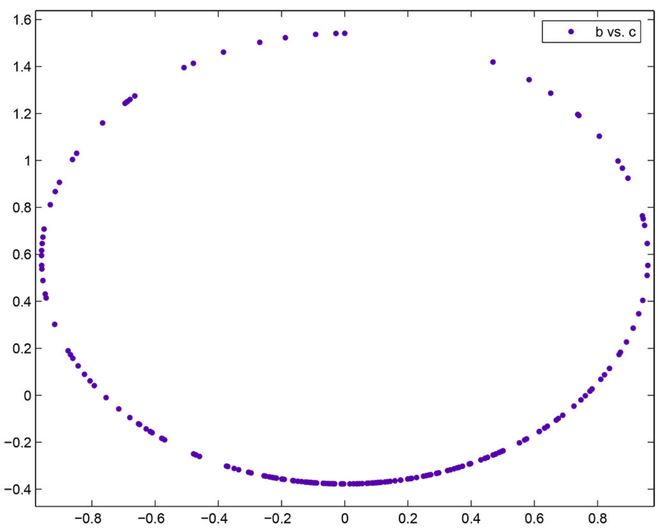

Figure 1:

3. VARIANT DYNAMICAL ZETA FUNCTIONS

The variant dynamical zeta functions corresponding to $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ satisfy functional relationships similar to $\zeta(z^2) = (1 - z)\zeta(z)$. The real parts of the z values are set to the reciprocals of $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and the imaginary parts are set to the imaginary parts of the corresponding $D(n, a, b)$ values. For the linear least-squares fits of the real parts of $\zeta(z^2)$ versus the real parts of $(1 - z)\zeta(z)$, the slopes are 1. The y-intercepts are $\Re z \cdot (\Re z * 2 - 1)$. For the linear least-squares fits of the imaginary parts of $\zeta(z^2)$ versus the imaginary parts of $(1 - z)\zeta(z)$, the slopes are $2 \cdot \Re z$. The imaginary parts are almost 0. For a quadratic least-squares fit of the real parts of $\zeta(z^2)$

versus the imaginary parts, the curve is an upside-down parabola with parameters of -1, 0 (almost), and $-\Re z \cdot (\Re z - 1)$. For a quadratic least-squares fit of the real parts of $(1 - z) \cdot \zeta(z)$ versus the imaginary parts, the curve is an upside-down parabola with parameters of $-(\frac{1}{2 \cdot \Re z})^2$, 0 (almost), and $\Re z^2$.

These values have normal probability distributions. The means and standard deviations of the normal probability fits determine the above relationships. For example, for $\Re(s) = \frac{1}{2}$ and the first thousand ψ values, the mean of the normal probability fit of the real parts of $(1 - z)\zeta(z)$ is -2.3092 with a 95% confidence interval of (-2.3287, -2.2897) and a standard deviation of 0.3147 with a 95% confidence interval of (0.3014, 0.3291). For a normal probablity fit of the real parts of $\zeta(z^2)$, the mean is 3.6908 with a 95% confidence interval of (3.6713, 3.7103) and a standard deviation of 0.3147 with a 95% confidence interval of (0.3013, 0.3291). The slope of 1 in the above is due to the standard deviations being equal. The y-intercept (6) equals the difference in means (3.6908+2.3092).

For a normal probability fit of the imaginary parts of $(1 - z)\zeta(s)$, the mean is 0.0125 with a 95% confidence interval of (-0.0220, 0.0470) and a standard deviation of 0.5562 with a 95% confidence interval of (0.5328, 0.5817). For a normal probability fit of the imaginary parts of $\zeta(z^2)$, the mean is 0.0499 with a 95% confidence interval of -0.0882, 0.1879) and a standard deviation of 2.2247 with a 95% confidence interval of (2.1313, 2.3268). The slope of 4 is due to the ratio of standard deviations (2.2247/0.5562) being almost 4.

Note that the standard deviations of the normal probability fits increase as $\Re(s)$ decreases. This is the motivation for considering large $\Re(s)$ values such as 4.0. For $\Re(s) = 4.0$ and the first ten thousand ψ values, the mean of the normal probability fit of the real parts of $(1 - z)\zeta(z)$ is 0.1873 with a 95% confidence interval of (0.1873, 0.1873) and a standard deviation of $1.8554 \cdot 10^{-4}$ with a 95% confidence interval of (0.0001172, 0.0001205). A plot of the sorted values versus $x = 1, 2, 3, \ldots, 10000$ is

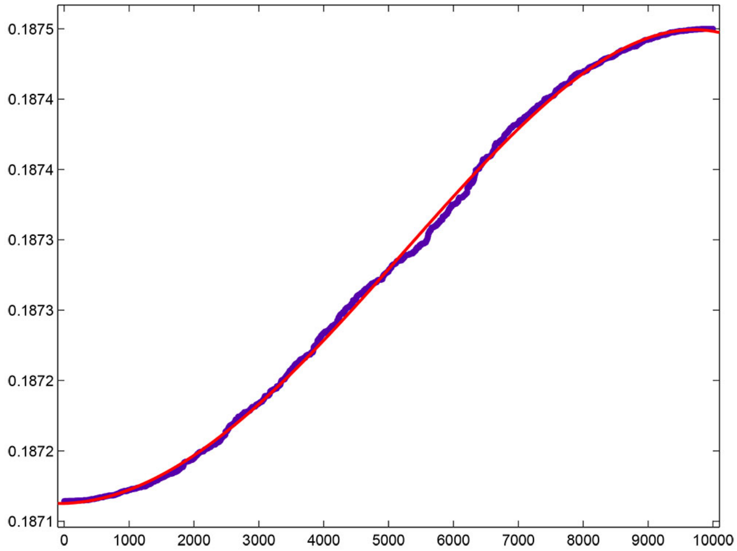

Figure 2:

For a quartic least-squares fit of the curve, $p_1 = -2.028 \cdot 10^{-20}$ with a 95% confidence interval of $(-2.159 \cdot 10^{-20}, -1.897 \cdot 10^{-20})$, $p_2 = -3.255 \cdot 10^{-16}$ with a 95% confidence interval of $(-3.519 \cdot 10^{-16}, -2.991 \cdot 10^{-16})$, $p_3 = 8.486 \cdot 10^{-12}$ with a 95% confidence interval of $(8.31 \cdot 10^{-12}, 8.662 \cdot 10^{-12})$, $p_4 = 1.533 \cdot 10^{-9}$ with a 95% confidence interval of $(1.101 \cdot 10^{-9}, 1.966 \cdot 10^{-9})$, $p_5 = 0.1872$ with a 95% confidence interval of (0.1872, 0.1872), SSE=$1.013 \cdot 10^{-7}$, R-squared=0.9993, and RMSE=$3.184 \cdot 10^{-6}$. For a larger number of ψ values such as 500000, the curve becomes smooth. This may be useful in an alternate proof of the prime number theorem.

4. ANALOGUE OF EULER'S PRODUCT FORMULA FOR THE RIEMANN ZETA FUNCTION

If f is a flow on M, that is, a map $MxR^+ \to M$, then the zeta function of this flow is defined as

$$\zeta(s) = \prod_{\omega} (1 - e^{-sl(\omega)})^{-1} \tag{5}$$

where ω denotes a periodic orbit of f and $l(\omega)$ is its length. If it is assumed that prime numbers correspond to periodic orbits of a flow and the lengths of the orbit indexed by p is given by $\log p$, then the zeta function of the flow will be similar to Riemann's zeta function. See the Methods section for C code that computes the above equation with the ψ values as input. A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 10000 ψ values where $a = 1/2$ is

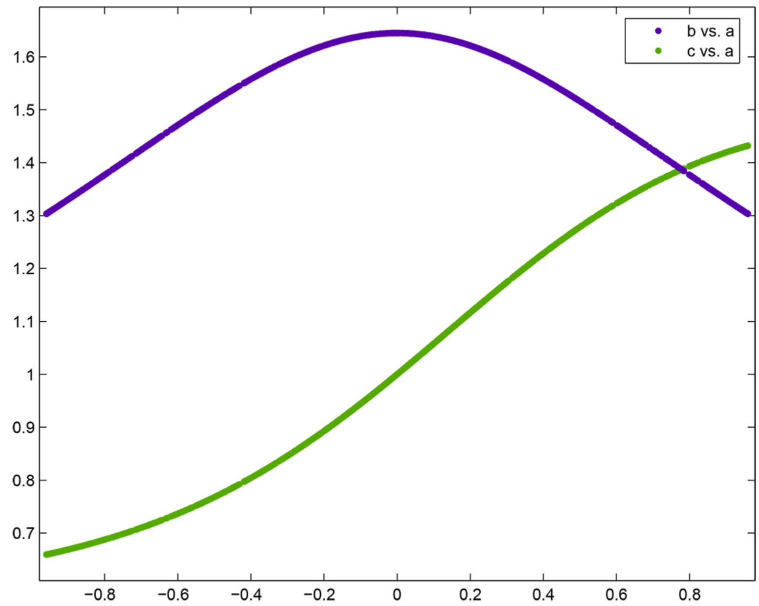

Figure 3:

The curves have good quartic least-squares fits. As will be shown, these are small portions of sinusoidal curves. Note that the range is from -1.0 to 1.0.

A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 10000 ψ values where $a = 1/3$ is

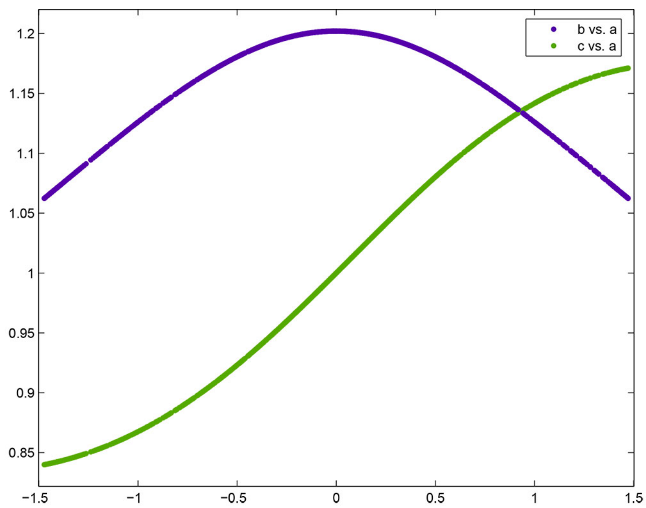

Figure 4:

The range is from -1.5 to 1.5 so that more of the curves are shown. Also, the amplitudes decrease.

A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 10000 ψ values where $a = 1/4$ is

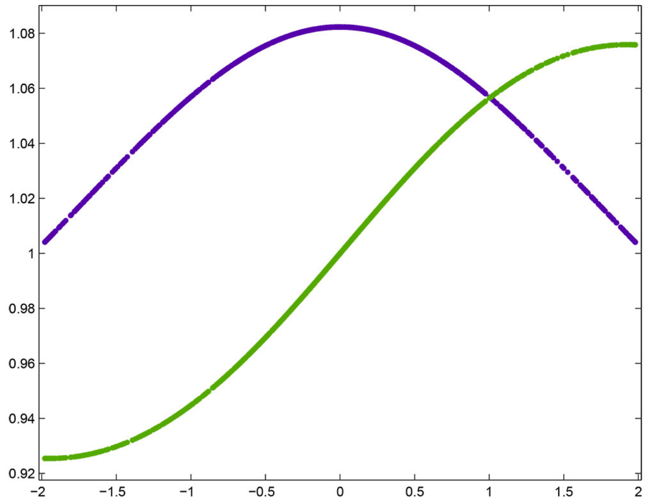

Figure 5:

A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 10000 ψ values where $a = 1/5$ is

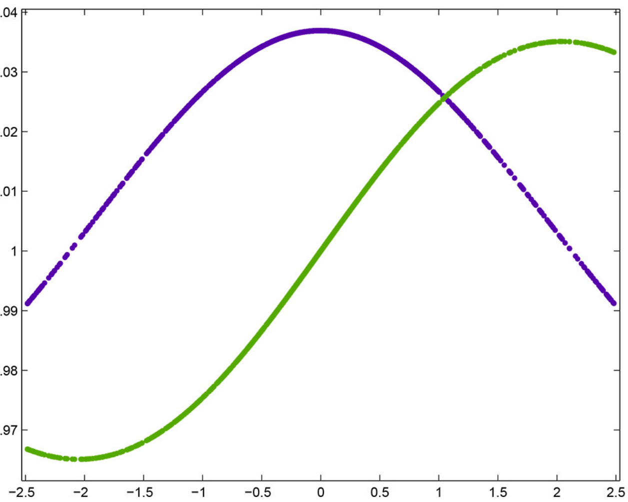

Figure 6:

A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 10000 ψ values where $a = 1/10$ is

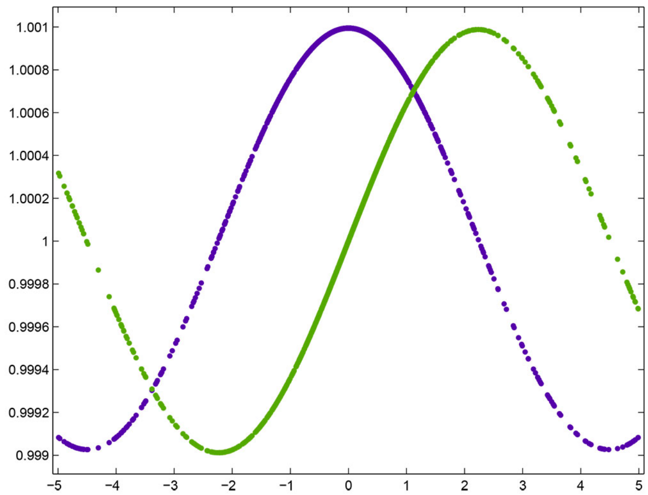

Figure 7:

The curves are now good approximations of the sine and cosine functions.

The Euler product formula for the Riemann zeta function is

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, (\Re s > 1) \tag{6}$$

5. THE RIEMANN HYPOTHESIS AND LOGARITHMICALLY WEIGHTED DIRICHLET SERIES

In this section, the real part of $(1-z)\zeta(z)$ is input to the imaginary part of a logarithmically-weighted Dirichlet series and the real part of $\zeta(z^2)$ is input into the imaginary part of a logarithmically-weighted Dirichlet series (see the C code in the Methods section). The real parts of the Dirichlet series are set to $R(z)$. This gives another dynamical zeta function denoted by ζ_1. A plot of the real part of $\zeta_1(z^2)$ versus the real part of $(1-z)\zeta_1(z)$, the imaginary part of $\zeta_1(z^2)$ versus the imaginary part of $(1-z)\zeta_1(z)$, the real part of $(1-z)\zeta_1(z)$ versus the imaginary part, and the real part of $\zeta_1(z^2)$ versus the imaginary part where $\Re(z) = 1.5$ and $n = 10000$ is

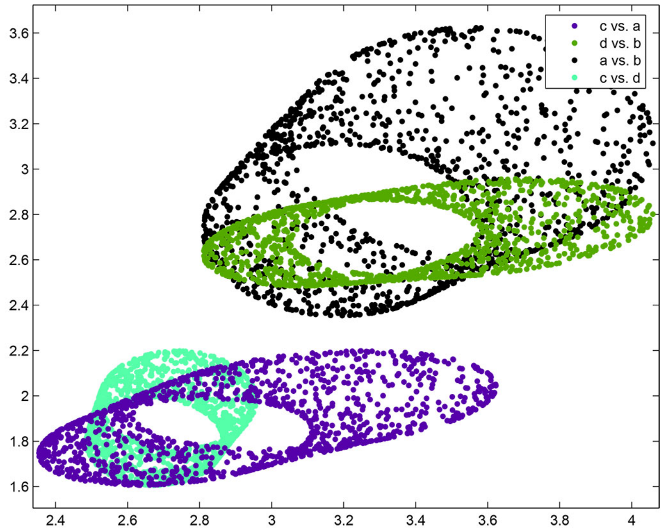

Figure 8:

The real and imaginary parts of $\zeta_1(z^2)$ are normally distributed and the real and imaginary parts of $(1-z)\zeta_1(z)$ are normally distributed. In this plot, $\Re((1-z)\zeta_1(z))$, $\Im((1-z)\zeta_1(z))$, $\Re(\zeta_1(z^2))$, and $\Im(\zeta_1(z^2))$ are denoted by a, b, c, and d respectively.

A plot of these values where $\Re(z) = 2.0$ and $n = 10000$ is

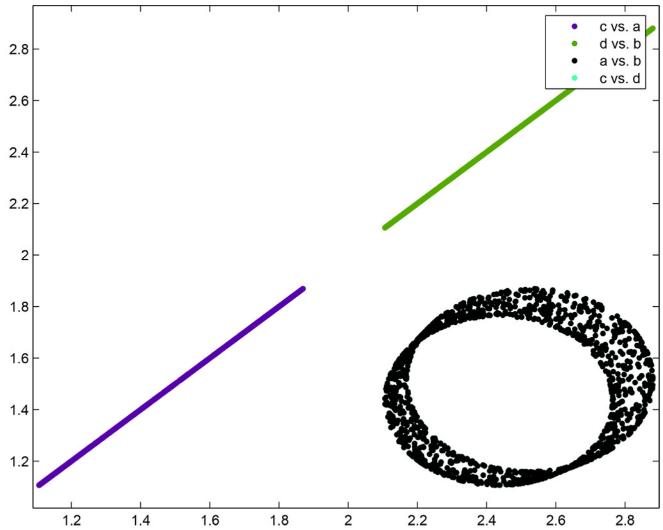

Figure 9:

Taking into account the precision of the floating point arithmetic and truncation of infinite sums, the linear least-squares fits of the real part of $\zeta_1(z^2)$ versus the real part of

$(1-z)\zeta_1(z)$ and the imaginary part of $\zeta_1(z^2)$ versus the imaginary part of $(1-z)\zeta_1(z)$ are apparently perfect and the slopes are 1. In this case, $\zeta_1(z^2) = (1-z)\zeta_1(z)$. Also, the curve of the real part of $\zeta_1(z^2)$ versus the imaginary part and the curve of the real part of $(1-z)\zeta_1(z)$ versus the imaginary part overlap.

A plot of these values where $\Re(z) = 2.5$ and $n = 10000$ is

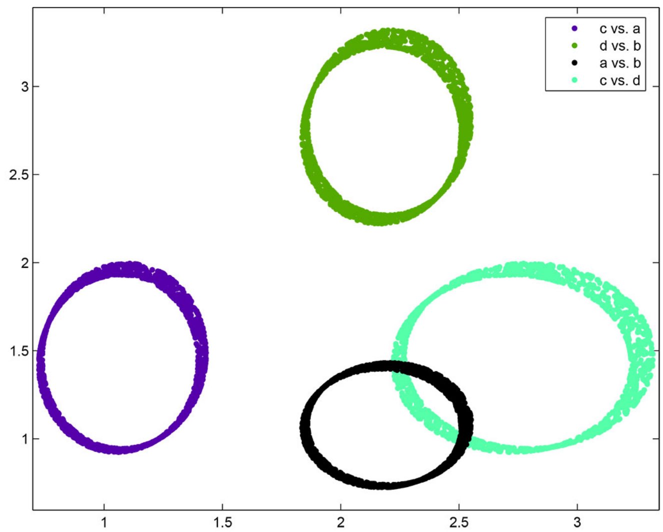

Figure 10:

A plot of these values where $\Re(z) = 3.0$ and $n = 1000$ is

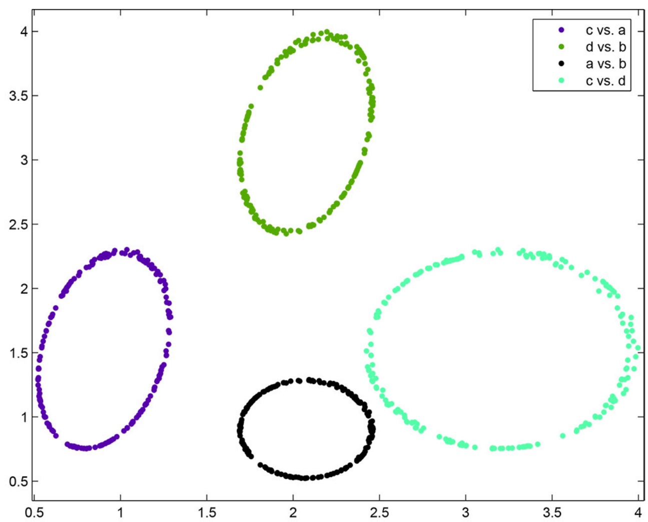

Figure 11:

The curves are becoming more elliptical. A plot of these values where $\Re(z) = 4.0$ and $n = 1000$ is

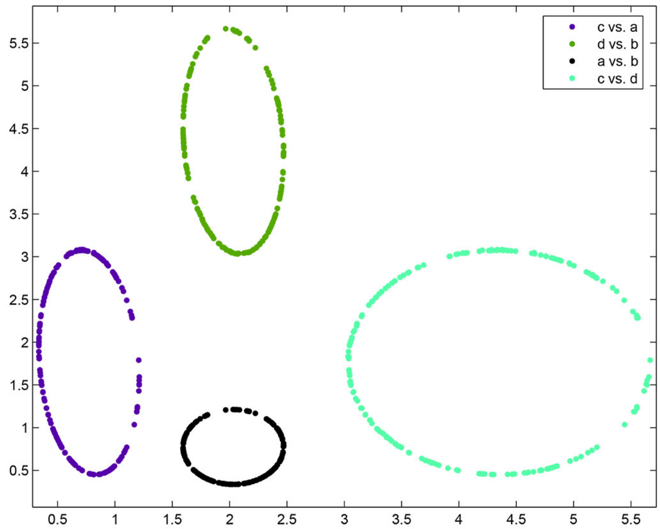

Figure 12:

A plot of these values where $\Re(z) = 5.0$ and $n = 1000$ is

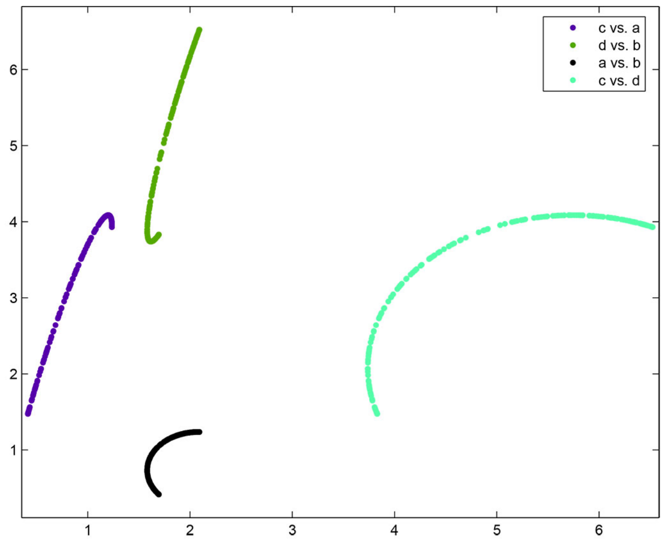

Figure 13:

A plot of these values where $\Re(z) = 6.4$ and $n = 1000$ is

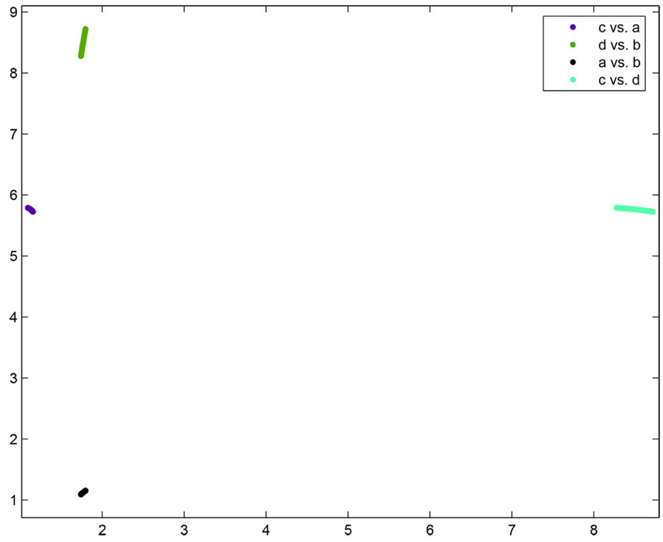

Figure 14:

The curves are quadratic. A plot of the real part of $\zeta_1(z^2)$ versus the real part of $(1-z)\zeta_1(z)$ is

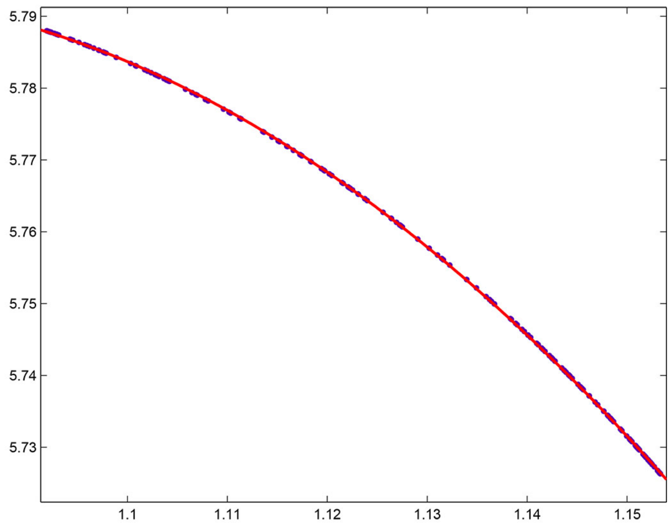

Figure 15:

For a quadratic least-squares fit of the curve, $p_1 = -9.113$ with a 95% confidence interval of $(-9.131, -9.095)$, $p_2 = 19.46$ with a 95% confidence interval of (19.42, 19.5), $p_3 = -4.598$ with a 95% confidence interval of $(-4.62, -4.575)$, SSE=$8.686 \cdot 10^{-6}$, R-squared=1, and RMSE=$9.334 \cdot 10^{-5}$

For $\Re(z) = 15.0$ and $n = 1000$, the curves are linear but the linear least-squares fit of the real part of $\zeta_1(z^2)$ versus the real part of $(1-z)\zeta_1(z)$ is not perfect (and doesn't have a slope of 1) and the linear least-squares fit of the imaginary part of $\zeta_1(z^2)$ versus the imaginary part of $(1-z)\zeta_1(z)$ is not perfect (and doesn't have a slope of 1). A plot of the real part of $\zeta_1(z^2)$ versus the real part of $(1-z)\zeta_1(z)$ is

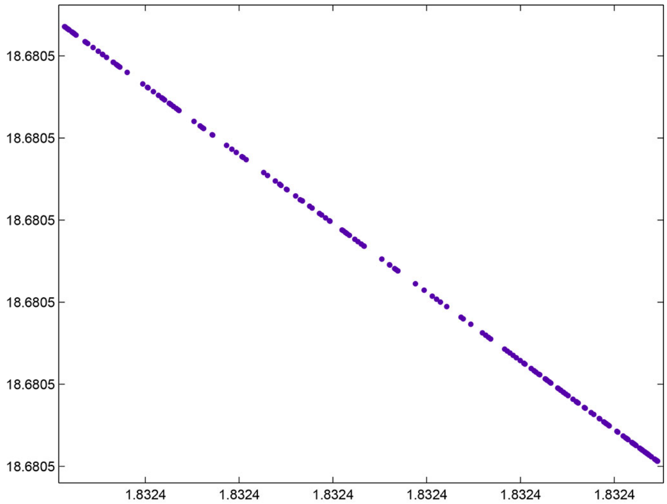

Figure 16:

For a linear least-squares fit of the curve, $p_1 = -16.75$ with a 95% confidence interval of $(-16.75, -16.75)$, $p_2 = 49.37$ with a 95% confidence interval of (49.37, 49.37), SSE=$4.825 \cdot 10^{-20}$, R-squared=1, and RMSE=$6.953 \cdot 10^{-12}$.

6. METHODS

variant dynamical zeta function

```
#include <math.h>
#include <stdio.h>
#include "cheby10k.h"
double a=1.5; // a>1 if out=1
unsigned int size=10000;
unsigned int max=50001;
unsigned int part=1;
unsigned int select=1;
unsigned int out=1;
void main() {
unsigned int x,i;
```

```
double esumr,esumi,tempr,ap,b,y,p,q,c,d,sum,out1,out2,out3,out4,bp;
FILE *Outfp;
Outfp = fopen(”weight1a.dat”,”w”);
ap=1.0/a;
y=1.0-ap;
for (i=1; <=size; i++) {
    b=zero[i-1];
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/exp((double)x*a);
        esumr=esumr+tempr*cos((double)x*b);
        esumi=esumi+tempr*sin((double)x*b);
        }
    if (part==1)
        sum=esumi;
    else
        sum=esumr;
    c=ap*y-sum*sum;
    d=ap*sum+sum*y;
    p=ap*ap-sum*sum;
    q=2*ap*sum;
    if (out==0) {
        fprintf(Outfp,” %.16llf, %.16llf, %.16llf, %.16llf \n”,c,d,p,q);
        printf(” %d %.10llf, %.10llf, %.10llf, %.10llf \n”,i,c,d,p,q);
        continue;
        }
    if (select==1)
        bp=c;
    else
        bp=d;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
```

```
    out1=esumr;
    out2=esumi;
    if (select==1)
        bp=p;
    else
        bp=q;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
    out3=esumr;
    out4=esumi;
    fprintf(Outfp," %.16llf, %.16llf, %.16llf, %.16llf, \n",out1,out2,out3,out4);
    if (i==(i/100)*100)
        printf(" %d, %.10llf, %.10llf, %.10llf, %.10llf \n",i,out1,out2,out3,out4);
    }
fclose(Outfp);
return;
}
```

analogue of Euler's product formula

```
#include <math.h>
#include <stdio.h>
#include "table5.h"
#include "cheby10k.h"
double a=1.0/2.0;
unsigned int size=10000;
unsigned int part=1;
unsigned int max=50001;
void main() {
unsigned int x,i,p;
double esumr,esumi,tempr,b,prodr,prodi,ap;
FILE *Outfp;
Outfp = fopen("weight2.dat","w");
```

```
ap=1.0/a;
for (i=1; i<=size; i++) {
    b=zero[i-1];
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/exp((double)x*a);
        esumr=esumr+tempr*cos((double)x*b);
        esumi=esumi+tempr*sin((double)x*b);
        }
    if (part==1)
        b=esumi;
    else
        b=esumr;
    prodr=1.0;
    prodi=1.0;
    for (x=1; x<=2000; x++) {
        p=table[x-1];
        tempr=1.0/(1.0-exp(ap*log((double)p)));
        prodr=prodr*(1.0-tempr*cos(log((double)p)*b));
        prodi=prodi*(1.0-tempr*sin(log((double)p)*b));
        }
    fprintf(Outfp," %.16llf, %.16llf, %.16llf, \n",b,prodr,prodi);
    printf(" %d %.10llf, %.10llf, %.10llf, \n",i,b,prodr,prodi);
    }
fclose(Outfp);
return;
}
```

second dynamical zeta function

```
#include <math.h>
#include <stdio.h>
#include "cheby10k.h"
double a=2.0; // a>1
unsigned int size=2000;
unsigned int max=50001;
unsigned int part=1; // usually set to 1
```

```
unsigned int select=1; // usually set to 1
unsigned int out=1;
void main() {
unsigned int x,i;
double esumr,esumi,tempr,ap,b,y,p,q,c,d,sum,out1,out2,out3,out4,bp;
FILE *Outfp;
Outfp = fopen("weight1a.dat","w");
ap=1.0/a;
y=1.0-ap;
for (i=1; i<=size; i++) {
    b=zero[i-1];
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/exp((double)x*a);
        esumr=esumr+tempr*cos((double)x*b);
            esumi=esumi+tempr*sin((double)x*b);
            }
    if (part==1)
        sum=esumi;
    else
        sum=esumr;
    c=ap*y-sum*sum;
    d=ap*sum+sum*y;
    p=ap*ap-sum*sum;
    q=2*ap*sum;
    if (out==0) {
        fprintf(Outfp," %.16llf, %.16llf, %.16llf, %.16llf \n",c,d,p,q);
        if (i==(i/100)*100)
            printf(" %d %.12llf, %.12llf, %.12llf, %.12llf\n",i,c,d,p,q);
        continue;
        }
    if (select==1)
        bp=c;
    else
        bp=d;
    esumr=0.0;
    esumi=0.0;
```

```
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
    out1=esumr;
    out2=esumi;
    if (select==1)
        bp=p;
    else
        bp=q;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
    out3=esumr;
    out4=esumi;
    fprintf(Outfp," %.16llf, %.16llf, %.16llf, %.16llf, \n",out1,out2,out3,out4);
    if (i==(i/100)*100)
        printf(" %d, %.14llf, %.14llf, %.14llf, %.14llf \n",i,out1,out2,out3,out4);
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] Vladislav Kargin, Statistical properties of zeta functions' zeros, Probability Surveys, Vol. 11 (2014) 121-160, ISSN: 1549-5787