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Minimax and Shrinkage Estimation of Scale Parameter of Finite Range Distribution

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Abstract

In this paper we have studied the performance of the Bayes Shrinkage estimators for the scale parameter of the Finite Range Failure Time Distribution under the Squared Error Loss and the LINEX loss functions in the presence of a prior point information of the scale parameter when Type-II censored data are available. The properties of the minimax estimators are also discussed.

Keywords: Bayes estimate, Finite Range distribution, Squared Error loss function, Linex Loss Function, Shrinkage Estimator, Minimax Estimator

1. Introduction

The observed data time data, economic data, industrial data etc; can be considered with a sudden change or failure in life test will occur therein. It is very important to know when and where a change will occur as failure. The observed 'time of failure' and 'average life' of a component, measured from some specified time until it fails, is represented by a continuous random variable. Extensively in recent years, one distribution that has been used as a model to deal with such problems for product life is the Finite Range Failure time distribution. Its applications in life-testing problems and survival analysis have been widely advocated.

Mukherjee and Islam (1983) have proposed a finite range failure time distribution. For use in life testing problem the probability density function is given by

$$f(x;p,\sigma)=\frac{p}{x}\left(\frac{x}{\sigma}\right)^{p} \quad ;p,\sigma>0\,,\ 0<x\leq\sigma \tag{1.1}$$

Where $p\ and\ \sigma$ are scale and shape parameters.

The Probability density function may be re-parameterized by taking $p=\frac{1}{\theta}$ and then the pdf of re-parameterized finite range distribution be written as

$$f(x;\sigma,\theta) = \frac{1}{\theta x}\left(\frac{x}{\sigma}\right)^{\frac{1}{\theta}} \;;\theta > 0, \sigma > 0\,,\; 0 < x \le \sigma \qquad (1.2)$$

Its distribution function is given by

$$F(x;\sigma,\theta) = \left(\frac{x}{\sigma}\right)^{\frac{1}{\theta}} \;;\; \theta > 0,\; \sigma > 0,\; 0 < x < \sigma \qquad (1.3)$$

For the finite range distribution the rth raw moment is given by

$$\mu_r' = \frac{\sigma^r}{(1+r\theta)} \qquad (1.4)$$

Therefore, the mean and variance are given by

$$\mu_1' = E(X) = \frac{\sigma}{1+\theta} \qquad (1.5)$$

and

$$\mu_2' = E(X^2) = \frac{\sigma^2}{1+2\theta} \qquad (1.6)$$

Therefore,

$$\mu_2 = V(X) = \frac{(\sigma\theta)^2}{(1+2\theta)(1+\theta)^2} \qquad (1.7)$$

Let us suppose that n items are put to life test and terminate the experiment when $\boldsymbol{r}(< \boldsymbol{n})$ items have failed. If $x_1,\ldots\ldots\ldots\ldots\ldots,x_r$ denote the first r observations having a common density function is given by

$$f(\underline{x}|\theta) = \frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r \left(\prod_{i=1}^{r}\frac{1}{x_i}\right) e^{\left(\frac{T_r}{\theta}\right)} \qquad (1.8)$$

Where

$$T_r = \left[\sum_{i=1}^{r} log\left(\frac{x_i}{\sigma}\right) + (n-r)log\left(\frac{x_{(r)}}{\sigma}\right)\right] \qquad (1.9)$$

The maximum likelihood estimator (MLE) $\hat{\theta}$ of θ is given by

$$\hat{\theta} = \frac{T_r}{r} \qquad (1.10)$$

The pdf of $\boldsymbol{\hat{\theta}}$ is given by

$$f(\hat{\theta}) = \frac{\left(\frac{r}{\theta}\right)^r}{\Gamma(r)}\left(\hat{\theta}\right)^{r-1} e^{-r\frac{\hat{\theta}}{\theta}} \;;\; \hat{\theta} > 0 \qquad (1.11)$$

In many situations, the experimenter has some prior information about the parameter in the form of a point guess value. To utilize this guess value, the shrinkage estimators have been discussed by a number of authors Prakash & Singh (2006, 2008). The shrinkage estimator performs better than the usual estimator when a guess value is approximately the true value of the parameter and sample size is small. A shrinkage estimator (Thompson, 1968) for the parameter θ when prior point guess value θ_0 of θ is available, is defined as

$$S = k\,\hat{\theta} + (1-k)\theta_0 \quad ; \quad 0 \le k \le 1 \qquad (1.12)$$

Here $\hat{\theta}$ is any usual estimator of the parameter p The shrinkage procedure has been applied in numerous problems, including mean survival time in epidemio- logical studies (Harries & Shakarki, 1979), forecasting of the money supply and improved estimation in sample surveys (Wooff, 1985).

When positive and negative errors have different consequences, the use of squared error loss function (SELF) in Bayesian estimation may not be appropriate. To overcome this difficulty, Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as the LINEX loss function (LLF). The invariant version of LLF for any parameter p is given by

$$L(\Delta) = e^{\alpha\Delta} - \alpha\Delta - 1, \ \alpha \neq 0 \text{ and } \Delta = \frac{\hat{\theta}-\theta}{\theta} \qquad (1.13)$$

The sign and magnitude of ‘α’ represents the direction and degree of asymmetry respectively. The positive (negative) value of ‘α’ is used when overestimation is more (less) serious than underestimation. The loss function (1.13) is approximately square error and almost symmetric if |α | is near to zero. A number of authors have discussed the estimation procedures under LLF criterion. A Few recent works under the Bayesian and/or the LLF criterions are Nigam et al. (2003), Bellhouse (2004), Xu & Shi (2004), Ahmadi et al. (2005), Prakash & Singh (2006), Singh et al. (2007), Ahmad et al. (2007), Prakash & Singh (2008), among others.

2. The Estimators

Let $x_1, x_2, \dots\dots\dots. x_n$ be the life times of n items put to test under model (1.1) are recorded lives.

The maximum likelihood estimate of θ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{n} \boldsymbol{log}\left(\frac{x_i}{\sigma}\right)}{n} \qquad (2.1)$$

In Type-II censored sampling, where the test terminates as soon as the rth item fails $r \leq n$. Let $x_1, x_2, \dots\dots. x_r$ be the observed failure times for the first r components. Then the likelihood function for the r failure items is

$$\boldsymbol{L}(\underline{x}|\theta) = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{r})!}\left(\frac{\boldsymbol{1}}{\boldsymbol{\theta}}\right)^{\boldsymbol{r}}\left(\prod_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{r}} \frac{1}{\boldsymbol{x_i}}\right) \boldsymbol{e}^{\left(\frac{T_r}{\theta}\right)} \qquad (2.2)$$

Where

$$T_r = \left[\sum_{i=1}^{r} log\left(\frac{x_i}{\sigma}\right) + (n-r) log\left(\frac{x_{(r)}}{\sigma}\right)\right]$$

which gives

$$\hat{\theta} = \frac{T_r}{r} \qquad (2.3)$$

The pdf of $\hat{\theta}$ is gamma distribution as

$$f(\hat{\theta}) = \frac{\left(\frac{r}{\theta}\right)^r}{\Gamma(r)} (\hat{\theta})^{r-1} e^{-r\frac{\hat{\theta}}{\theta}} \tag{2.4}$$

Then

$$f(T_r) = \frac{r^r}{\theta^r \Gamma r} {T_r}^{r-1} \exp\left(-\frac{r\,T_r}{\theta}\right) \tag{2.5}$$

The risks under the SELF is given by

$$R_s(T_r) = \frac{\theta^2}{r} \tag{2.6}$$

where, suffix *S* denote the risk taken under the SELF criterions.

The risks under the LLF

$$R_L(T_r) = e^{-\alpha}\left(1 - \frac{\alpha}{r}\right)^{-r} - 1 \tag{2.7}$$

where, suffix L denote the risk taken under the LLF criterions.

If parameter σ is known, the natural family of conjugate prior of θ is taken as the inverted Gamma distribution with probability density function

$$g_1(\theta) = \frac{b^a}{\Gamma a} \theta^{-(a+1)} e^{-\frac{b}{\theta}} \quad , \quad a > 0 \,, \; b > 0 \tag{2.8}$$

In the situation where the researchers have no prior information about the parameter θ, one may use the uniform, quasi or improper prior. A family of priors is given by

$$g_2(\theta) = \theta^{-d} e^{-\frac{cd}{\theta}} \quad , \qquad d > 0 \,, \; c > 0 \tag{2.9}$$

If $d = 0$, we get a diffuse prior and if d = 1, c = 0 a non-informative prior is obtained. For a set of values of d and c, that satisfies the equality

$\Gamma(d-1) = (cd)^{(d-1)}$ makes $g_2(\theta)$ as a proper prior.

If both of the parameters θ and σ are unknown in model (1.2), the joint prior distribution (Sinha, 1986) is considered as

$$g_3(\theta,\sigma) = g_1(\theta).h(\sigma), \quad h(\sigma) = \frac{1}{v}\,, \;\; ; \quad 0 < v \tag{2.10}$$

3 Bayesian Shrinkage Estimator of θ when σ is known

The posterior of θ using $g_1(\theta)$ is

$$\rho_1(\theta) = \frac{L(\underline{x}\,|\theta) g_1(\theta)}{\int_0^\infty L(\underline{x}\,|\theta) g_1(\theta)\, d\theta} \tag{3.1}$$

$$= \frac{\frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r \left(\prod_{i=1}^{r}\frac{1}{x_i}\right) e^{\left(\frac{T_r}{\theta}\right)} \frac{b^a}{\Gamma a}\theta^{-(a+1)} e^{-\frac{b}{\theta}}}{\int_0^\infty \frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r \left(\prod_{i=1}^{r}\frac{1}{x_i}\right) e^{\left(\frac{T_r}{\theta}\right)} \frac{b^a}{\Gamma a}\theta^{-(a+1)} e^{-\frac{b}{\theta}} d\theta}$$

$$\rho_1(\theta) = \frac{(rT_r+b)^{(a+r)} e^{-(rT_r+b)/\theta}\,\theta^{-(a+r+1)}}{\Gamma(a+r)} \quad (3.2)$$

again which is inverted Gamma with (a+r) and $(rT_r + b)$ parameters.

The Bayes estimator under squared error loss function (SELF) is

$\hat{\theta} = \frac{(rT_r+b)}{(a+r-1)}$;

Which can be written as

$$\hat{\theta} = \emptyset.(rT_r + b) \quad (3.3)$$

where, $\emptyset = (a + r + 1)^{-1}$

We choose the parameters of prior density $g_1(\theta)$ s.t. $E(\hat{\theta}) = \theta_0$,when θ_0 is point guess value , which gives

$E(\hat{\theta}_1) = \theta_0$

$$b = \theta_0(a - 1) \quad (3.4)$$

Substitute ‘b’ in eqn. (3.3), we get the Bayes estimator as

$\theta_1 = \frac{rT_r+\theta_0(a-1)}{(a+r-1)}$

Again taking $k_1 = r\,\emptyset_1$, gives

$\theta_1 = k_1T_r + \frac{k_1}{r}(a - 1)\theta_0$

Some Shrinkage estimator $\tilde{\theta}_1$ is proposed as

$$\tilde{\theta}_1 = k_1T_r + (1 - k_1)\theta_0 \quad (3.5)$$

which is similar to shrinkage estimator ;

$T_r = k\,\hat{\theta} + (1 - k)\theta_0$

Again the Bayes estimator under L.L.F. for Natural Conjugate Prior is

$$E_{\rho_1}\left(\frac{1}{\theta}\, e^{-\frac{a\,\hat{\theta}_2}{\theta}}\right) = e^{\alpha}\, E_{\rho}\left(\frac{1}{\theta}\right) \quad (3.6)$$

$$\hat{\theta}_2 = \emptyset_2\,(rT_r + b) \quad (3.7)$$

Where,

$\emptyset_2 = \frac{1}{\alpha}\left(1 - \exp\left(-\frac{\alpha}{(\alpha+r+1)}\right)\right)$

again, $E(\hat{\theta}_2) = \theta_0$, gives

$$b = \theta_0 \, (1 - r\emptyset_2)\emptyset_2^{-1} \tag{3.8}$$

Hence the Bayes shrinkage under LLF is obtain by replacing 'b' in (3.7) we get,

$$\tilde{\theta}_2 = k_2 T_r + (1 - k_2)\theta_0, \qquad k_2 = r\emptyset_2 \tag{3.9}$$

The equation of risks function under SELF as

$$R_{(S)}(\hat{\theta}_i) = r\theta^2 \, \emptyset_i^{\,2} + (\theta(\emptyset_i - 1) + b\emptyset_i)^2 \tag{3.10}$$

For i =1, 2

$$R_{(L)}(\hat{\theta}_i) = \exp\left(a\left(\frac{\emptyset_i \, b}{\theta} - 1\right)\right)(1 - a\emptyset_i)^{-r} - 1 - a\left(r\frac{b}{\theta} - 1\right) \tag{3.11}$$

$$R_{(S)}(\tilde{\theta}_i) = \theta^2 \left[k_i^{\,2}\left(\frac{(r+1)}{r} + \delta(\delta - 1\right) + (1 - \delta)^2 \, (1 - 2k_i)\right] \tag{3.12}$$

Where, $\delta = \frac{\theta}{\theta_0}$, $\qquad i = 1,2$

$$R_{(L)}(\tilde{\theta}_i) = e^{a((1-k_i)\delta - 1)} \, (1 - \frac{\alpha k_i}{r})^{-r} - 1 + a(1 - \delta)(1 - k_i) \tag{3.13}$$

The posterior density of $g_2(\theta)$ is given as

$$\rho_2(\theta) = \frac{(rT_r + cd)^{(r+d-1)}}{\Gamma(r+d-1)} \, e^{-\frac{(rT_r + cd)}{\theta}} \, \theta^{-(r+d)} \tag{3.14}$$

Equation (3.14) and equation (3.2) are same. Take

$d = (a + 1)$ and $c = \frac{b}{a+1}$ for calculation.

4. Bayesian Shrinkage estimator if both the parameter are unknown

For Finite Range p.d.f. equation. (1.2), the Joint posterior with respect to $g_3\ (\theta, \sigma)$ is given by

$$\rho_3(\theta, \sigma) = \frac{L\,(\underline{x}\,|\theta, \sigma) g_3\,(\theta,\sigma)}{\int_0^\infty L\,(\underline{x}\,|\theta, \sigma) g_3\,(\theta,\sigma)\, d\theta} \tag{4.1}$$

$$\rho_3(\theta, \sigma) = \frac{\frac{1}{\nu}\frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r\left(\prod_{i=1}^r \frac{1}{x_i}\right) e^{\left(\frac{T_r}{\theta}\right)} \frac{b^a}{\Gamma a}\, \theta^{-(a+1)}\, e^{-\frac{b}{\theta}}}{\int_0^\infty \int_0^\infty{}_{\theta,\sigma} \quad \frac{1}{\nu}\frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r\left(\prod_{i=1}^r \frac{1}{x_i}\right) e^{\left(\frac{T_r}{\theta}\right)} \frac{b^a}{\Gamma a}\, \theta^{-(a+1)}\, e^{-\frac{b}{\theta}} d\theta\, d\sigma}$$

Let, $w = \frac{1}{\nu}\prod_{i=1}^r \frac{1}{x_i}$

$$\rho_3(\theta, \sigma) = \frac{w\, \theta^{-(r+a+1)} \exp\left\{-\frac{T_r + b}{\theta}\right\}}{\int_\sigma w \frac{\Gamma r + a}{(T_r + b)^{(r+a)}} d\sigma} \tag{4.2}$$

The Marginal density of 'θ' is obtained as

$$\rho_4(\theta) = \frac{L\,(\underline{x}\,|\theta) g_3\,(\theta,\sigma)}{\int_0^\infty L\,(\underline{x}\,|\theta) g_3(\theta,\sigma)\, d\theta}$$

$$\rho_4(\theta) = \frac{p^{-(r+a+1)} \int_0^\nu w\, e^{(-\frac{T_r+b}{\theta})} d\sigma}{\int_0^\nu w \frac{\Gamma(r+a)}{(T_r+b)^{(r+a)}} d\sigma} \tag{4.3}$$

Hence Bayes Estimator of the parameter ‘θ’ under SELF

$$\hat{p}_3 = \frac{I(w,(r+a-1))}{(r+a+1) I(w,(r+a))} \tag{4.4}$$

Where, $I\,[t_1, t_2] = \int_0^\nu t_1\,(rT_r + b)^{-t_2}\, d\sigma$

Similarly the Bayes estimator of ‘θ’ under LLF is

$$I\,[w', (r + a - 1)] = e^\alpha I[w, (r + a)\,] \tag{4.5}$$

Where,

$$w' = w\left(1 - \frac{\alpha \hat{\theta}_4}{(rT_r+b}\right)^{-(r+a-1)}$$

5 Minimax Estimator

Let, $\tau = \{F_\theta : \theta \in \Theta\}$ be a family of distribution functions and ‘D’ be a class of estimators of the parameter θ. Suppose $d^* \in D$ is a Bayes estimator against a prior π (θ) on the parameter space ‘Θ’. Then the Bayes estimator d^* is said to be minimax estimator, If the risk function of d^* is independent on Θ.

When the shape parameter σ is considered to be known, the Bayes estimator for the parameter θ corresponding to the SELF and LLF are given respectively in the equations (3.3) and (3.7).

Further, the expressions of the risk for these Bayes estimators corresponding to the considered loss criterion are given in equation (3.10) and (3.11) respectively.

Both expressions of the risk involves the parameter ‘θ’. Hence, the Bayes estimator $\hat{\theta}_1$ and $\hat{\theta}_2$ are not minimax estimators. These under the natural family of the conjugate prior the Minimax estimators do not exist.

Now the Bayes estimators corresponding to the posterior $\rho_2(\theta)$ given in equation (3.14) are obtained respectively under both loss criterion as

$$\hat{\theta}_5 = \emptyset_5 rT_r \tag{5.1}$$

Where,

$$\emptyset_5 = (r + d - 2)^{-1}$$

And $\hat{p}_6 = \emptyset_6 rT_r$ (5.2)

Where,

$$\emptyset_6 = \frac{1}{\alpha}\ (1 - \exp(-\frac{\alpha}{r+d}))$$

The risks of these Bayes estimators corresponding to SELF and the LLF are given respectively as

$$R_{(S)}\left(\hat{\theta}_i\right) = \theta^2\ (r(r+1)\ \emptyset_i^{\ 2} + \ -2r\ \emptyset_i) \tag{5.3}$$

And

$$R_{(L)}\left(\hat{\theta}_i\right) = e^{-\alpha}\ (1 - \alpha\emptyset_i)^{-r} - \ 1 - \ \alpha(r\emptyset_i - 1) \tag{5.4}$$

Where, $i = 5,6$

It is observed that the Bayes estimator $\hat{\theta}_5$ and $\hat{\theta}_6$ are not the minimax estimator corresponding to the loss criterion SELF. However, the risk of Bayes estimator $\hat{\theta}_5$ and $\hat{\theta}_6$ are independent of the parameter 'θ' under the LLF criterion. Hence both estimators $\hat{\theta}_5$ and $\hat{\theta}_6$ are minimax estimator under LLF loss function.

The following statistical problem (Minimax Estimation) is equivalent to some two person zero sum game between the Statistician (Player-II) and Nature (Players-I). Hence the pure strategies of nature are the different values of 'θ' in the interval $(0, \infty)$ and the mixed strategies of nature are the prior densities of θ in the interval $(0, \infty)$. The pure strategies of Statistician are all possible decision functions in the interval $(0, \infty)$.

The expected value of the loss function is the risk function and it is the gain of the Player-I. Further, the Bayes risk is defined as

$$R^*\left(\eta, \hat{\theta}_B\right) = \ E_\theta\ R(\hat{\theta}_B)$$

Here, the expectation has been taken under the prior density of parameter 'θ'. If the loss function is continuous in both the estimator $\hat{\theta}_B$ and the parameter 'θ' and convex in $\hat{\theta}_B$ for each value of θ then there exist measures η^* and $\hat{\theta}_B$ for all θ and $\hat{\theta}_B$ so that, the following relation holds:

$$R^*\left(\eta, \hat{\theta}_B\right) \leq R^*\ (\eta^*, \theta_B) \leq R^*\left(\eta^*, \hat{\theta}^*_B\right)$$

The number $R^*\left(\eta^*, \hat{\theta}^*_B\right)$ is known as the value of the game and η^* and $\hat{\theta}^*_B$ are the corresponding optimum strategies of the Player-I and Player-II.

In statistical terms η^* is the least favorable prior density of θ and the estimator $\hat{\theta}^*_B$ is the minimax estimator. In fact, the value of the game is the loss of the Player-II. Hence, the optimum strategy of Player-II and the value of game are given as.

Optimum Strategy	Corresponding Loss	Value of Game
$\hat{\theta}_5 = \emptyset_5 r T_r$	LLF	$e^{-\alpha}(1 - \alpha\emptyset_5)^{-r} - 1 - \alpha(r\emptyset_5 - 1)$
$\hat{\theta}_6 = \emptyset_6 r T_r$	LLF	$e^{-\alpha}(1 - \alpha\emptyset_6)^{-r} - 1 - \alpha(r\emptyset_6 - 1)$

6 NUMERICAL ANALYSIS

The relative efficiencies of the Bayes shrinkage estimator $\widetilde{\theta_\iota}$ $(i = 1,2)$ relative to the UMVU estimator T_r under the SELF and the LLF criterions are defined as

$$\mathrm{RE}_{(S)}\left(\widetilde{\theta_\iota}, \mathrm{T_r}\right) = \frac{\mathrm{R}_{(S)}(\mathrm{T_r})}{\mathrm{R}_{(S)}(\widetilde{\theta_\iota})}, \quad (i=1,2) \tag{6.1}$$

and

$$\mathrm{RE}_{(L)}\left(\widetilde{\theta_\iota}, \mathrm{T_r}\right) = \frac{\mathrm{R}_{(L)}(\mathrm{T_r})}{\mathrm{R}_{(L)}(\widetilde{\theta_\iota})}, \quad (i=1,2) \tag{6.2}$$

The expressions of relative efficiencies are the functions of $\mathrm{r}, \alpha, \delta$ and a whereas $\mathrm{RE}_{(S)}\left(\widetilde{\theta_\iota}, \mathrm{T_r}\right)$ is independent with 'a'. For the selected set of values $\mathrm{r} = (05(05)20)$; $\alpha = (0.25(0.25)1.00)$; $\delta = (0.10(0.10)0.70)$ and $\mathrm{a} = (1.00, 1.25, 1.50, 2.00, 3.00, 5.00)$ the relative efficiencies have been calculated in percentage and presented in Tables- (1)-(4), respectively. The numerical findings are presented here only for $\mathrm{r} = 05$ when risk criterion is the LLF.

Table- 1

Relative Efficiences of Shrinkage Estimators under SELF

r	δ	a					
		1.00	1.25	1.50	2.00	3.00	5.00
05	0.10	1.1189	1.2226	1.4359	1.2359	8.5241	0.63789
	0.20	1.142	1.2458	1.6569	2.0054	1.7422	1.40374
	0.30	1.1267	1.2606	1.8266	3.2005	4.6623	4.97794
	0.40	1.1268	1.2658	1.8904	4.0004	1.0568	3.30581
	0.50	1.1276	1.2608	1.8265	3.2022	4.6619	4.97765
	0.60	1.1243	1.2464	1.6573	2.0008	1.7426	1.40383
	0.70	1.1189	1.2224	1.4364	1.2307	8.522	0.63851
10	0.10	1.0788	1.1466	1.2904	1.1122	7.269	0.49829
	0.20	1.0822	1.1614	1.4276	1.6664	1.4277	1.08246
	0.30	1.0834	1.1704	1.5266	2.3809	3.3889	3.64698
	0.40	1.0845	1.1732	1.5622	2.7768	6.2504	1.73564
	0.50	1.0842	1.1705	1.5266	2.3809	3.3897	3.64673
	0.60	1.0822	1.1612	1.4292	1.6666	1.4284	1.08209
	0.70	1.0778	1.1465	1.2902	1.1109	0.7272	0.49823
15	0.10	1.0577	1.1098	1.2176	1.0587	0.6743	0.43219
	0.20	1.0617	1.1204	1.3169	1.5004	1.2779	0.92691
	0.30	1.0629	1.1270	1.3852	2.0008	2.7654	2.98193
	0.40	1.0629	1.1278	1.4103	2.2502	4.5148	1.13897
	0.50	1.0626	1.1264	1.3856	2.0003	2.7654	2.98249
	0.60	1.0612	1.1202	1.3574	1.5003	1.2786	0.92784
	0.70	1.0586	1.1092	1.2174	1.0586	0.6743	0.43168
20	0.10	1.0456	1.0869	1.1733	1.0314	0.6494	0.39483
	0.20	1.0478	1.0956	1.2523	1.4008	1.1936	0.83795

	0.30	1.0501	1.1005	1.3039	1.7816	2.3965	2.58227
	0.40	1.0502	1.1022	1.3221	1.9605	3.6102	8.41322
	0.50	1.0508	1.1006	1.3034	1.7816	2.3957	2.58249
	0.60	1.0489	1.0952	1.2524	1.4004	1.1932	0.83768
	0.70	1.0459	1.0868	1.1729	1.0314	0.6494	0.39464

Table- (1) shows that the Bayes shrinkage estimator $\widetilde{\theta_1}$ performs uniformly well for small a ≤ 3.00 with respect to the UMVU estimator T_r under the SELF. The effective interval (the interval in which the relative efficiency is more than one) decreases with the sample size r as well as $'a'$ increases under the SELF. The efficiency attains maximum at the point $\delta = 1.00$ and the gain in efficiency decreases as r increases for all considered values of δ when other parametric values are fixed. Further, the gains in efficiencies increase as α increases in the interval $0.10 \leq \delta \leq 0.70$ with other fixed parametric values.

On the other hand, when the risk criterion is the LLF Table- (2) the estimator $\widetilde{\theta_1}$ performs uniformly well with respect to T_r when sample size is small r (≤ 10) for all considered values of parametric space but for a large sample size, this property holds in the interval $0.20 \leq \delta \leq 0.60$. The gain in efficiency increases when 'a' increases for all considered values of δ with small sample size r (≤ 10) and in the interval $\delta \leq 0.70$ otherwise, under other fixed parametric values. Other properties are similar to the SELF-criterion.

Table- 2

Relative Efficiences of Shrinkage Estimators under LLF

$r = 05$		a					
α	δ	1.00	1.25	1.50	2.00	3.00	5.00
0.25	0.10	1.1364	1.2834	1.6459	1.4876	1.0429	1.01285
	0.20	1.1449	1.2955	1.8403	2.3579	2.0766	1.67652
	0.30	1.1504	1.2958	1.9556	3.6154	5.4009	5.76955
	0.40	1.1524	1.3010	1.9570	4.2401	1.1418	3.61787
	0.50	1.1287	1.2616	1.8308	3.2411	4.8408	5.27279
	0.60	1.1156	1.2310	1.6227	1.9924	1.7803	1.44353
	0.70	1.1036	1.1741	1.3643	1.2142	1.1520	1.13491
0.50	0.10	1.1539	1.3179	1.9504	1.8878	1.3369	1.01763
	0.20	1.1700	1.3504	2.0769	2.9134	2.6049	2.10555
	0.30	1.1844	1.3754	2.1063	4.2663	6.5836	7.04196
	0.40	1.1976	1.3912	2.1564	4.6859	1.2954	4.22397
	0.50	1.1359	1.2779	1.8865	3.4245	5.2806	5.871793
	0.60	1.11.9	1.2319	1.6338	2.0689	1.9079	1.56062
	0.70	1.1059	1.1809	1.3719	1.2478	1.1889	1.16219
0.75	0.10	1.1847	1.3825	2.3036	2.5673	1.8404	1.41078
	0.20	1.2102	1.4368	2.4287	3.8544	3.5119	2.84393
	0.30	1.2343	1.4858	2.4819	5.3705	8.6322	9.25591

	0.40	1.2579	1.5267	2.5246	5.5109	1.5687	5.17394
	0.50	1.1578	1.3222	2.0300	3.8549	6.1810	7.05526
	0.60	1.1305	1.2589	1.7205	2.2878	2.1929	1.81164
	0.70	1.1126	1.1942	1.4222	1.3604	1.1989	1.17373
1.00	0.10	1.2423	1.5049	2.7489	3.8943	2.8306	2.18447
	0.20	1.2789	1.5880	3.0770	5.6787	5.3005	4.30109
	0.30	1.3145	1.6667	3.2657	7.1632	1.2676	1.36488
	0.40	1.3511	1.7430	3.2841	7.4944	2.1476	7.19681
	0.50	1.2059	1.4209	2.3514	4.7977	8.0837	9.48784
	0.60	1.1592	1.3376	1.9508	1.7941	2.8118	2.35295
	0.70	1.1329	1.2539	1.5876	1.6378	1.2404	1.19202

The Bayes shrinkage estimator $\widetilde{\theta_2}$ performs well for all considered values of the parametric space when a ≤ 10.00 with respect to T_r under the SELF.

The gain in efficiency increases when 'α' increases in the interval $0.10 \leq \delta \leq 0.70$ for all considered parametric values when a ≤ 10.00. Other properties of the estimator $\widetilde{\theta_2}$ are similar to the estimator $\widetilde{\theta_1}$ under the SELF.

Table - 3

Relative Efficiences of Shrinkage Estimators under LLF

$r = 05$		a					
α	δ	1.00	1.25	1.50	2.00	3.00	5.00
0.25	0.10	1.4011	1.3789	1.2606	1.0114	1.0143	0.61751
	0.20	1.8487	1.9338	1.9975	1.9081	1.6314	1.36426
	0.30	2.4074	2.5482	3.0787	4.0784	4.8818	4.94667
	0.40	2.6426	2.8502	3.7558	6.5686	1.4541	3.98498
	0.50	2.4074	2.5491	3.0787	4.0779	4.8821	4.94599
	0.60	1.8992	1.9346	1.9979	1.9080	1.6309	1.36343
	0.70	1.4049	1.3800	1.2606	1.0114	1.0138	0.61802
0.50	0.10	1.3787	1.3504	1.2300	1.0924	1.0767	0.61609
	0.20	1.9355	1.9616	2.0005	1.8926	1.6204	1.36008
	0.30	2.5545	2.6920	3.2049	4.1530	4.8969	4.94266
	0.40	2.8590	3.0740	4.0114	6.9020	1.5027	4.06494
	0.50	2.5540	2.6920	3.2049	4.1530	4.8970	4.94263
	0.60	1.9350	1.9610	2.0002	1.8924	1.6204	1.35902
	0.70	1.3787	1.3506	1.2298	1.0928	1.0767	0.61587
0.75	0.10	1.3479	1.3200	1.1980	1.1740	1.1000	0.61409
	0.20	1.9627	1.9805	1.9980	1.8771	1.6100	1.35601
	0.30	2.7014	2.8350	3.3290	4.2241	4.9117	4.93805
	0.40	3.0889	3.3118	4.2800	7.2500	1.5530	4.14669
	0.50	2.7015	2.8350	3.3290	4.2242	4.9116	4.93802
	0.60	1.9627	1.9807	1.9980	1.8761	1.6098	1.35596

	0.70	1.3481	1.3186	1.1990	1.1740	1.1000	0.61437
1.00	0.10	1.3160	1.2870	1.1697	1.2569	1.1534	0.61208
	0.20	1.9821	2.0000	1.9987	1.8600	1.6000	1.35003
	0.30	2.8478	2.9769	3.4500	4.2915	4.9250	4.94008
	0.40	3.3340	3.5640	4.5630	7.6085	1.6045	4.23009
	0.50	2.8482	2.9776	3.4500	4.3000	4.9245	4.93465
	0.60	1.9824	1.9939	2.0000	1.8592	1.5994	1.35208
	0.70	1.3200	1.2858	1.1689	1.2567	1.1528	0.61207

Under the LLF criterion Table (4), the estimator $\widetilde{\theta_2}$ also performs well for a ≤ 10.00 with respect to T_r and the gain in efficiency increases as 'α' increases for all considered values of parametric space. Other properties of $\widetilde{\theta_2}$ are similar to the Bayes shrinkage estimator $\widetilde{\theta_1}$ under the LLF criterion. The gain in efficiency is larger for the Bayes shrinkage estimator $\widetilde{\theta_2}$ under the LLF-criterion with respect to the SELF-criterion.

Table - 4

Relative Efficiences of Shrinkage Estimators under LLF

$r = 05$		a					
α	δ	1.00	1.25	1.50	2.00	3.00	5.00
0.25	0.10	1.00	1.50	2.00	2.50	5.00	10.00
	0.20	16700	1.6480	1.5228	1.2320	1.0480	0.76266
	0.30	21828	2.2390	2.3500	2.2700	1.9460	1.63009
	0.40	2.6500	2.8223	3.4669	4.7000	5.6628	5.72891
	0.50	2.7678	3.0000	3.9756	7.0420	1.5780	4.36696
	0.60	2.4215	2.5661	3.1151	4.1819	5.1067	5.24819
	0.70	1.8674	1.9050	1.9867	1.9300	1.6739	1.40217
0.50	0.10	1.3455	1.3239	1.2402	1.0010	1.0008	0.61407
	0.20	2.0561	2.0310	1.8868	1.5460	1.2074	0.98479
	0.30	2.7073	2.7747	2.9144	2.8200	2.4255	2.04227
	0.40	3.2547	3.4682	4.2748	5.7842	6.9390	6.98348
	0.50	3.2688	3.5345	4.7002	8.3264	1.8630	5.14354
	0.60	2.6880	2.8430	3.4325	4.5904	5.6419	5.85897
	0.70	1.9503	1.9879	2.0705	2.0349	1.7907	1.51209
0.75	0.10	1.3588	1.3400	1.2458	1.0309	1.0103	0.63663
	0.20	2.7212	2.6922	2.5132	2.0844	1.6523	1.36092
	0.30	3.6010	3.6928	3.8837	3.7635	3.2504	2.75193
	0.40	4.2806	4.5699	5.6578	7.6677	9.1513	9.16883
	0.50	4.3138	4.6537	5.9424	10.5646	23.6617	9.02684
	0.60	3.1689	3.3499	4.0388	5.4028	6.7077	7.0347
	0.70	2.1640	2.2056	2.3074	2.3005	2.0584	1.75096
1.00	0.10	1.4376	1.4205	1.3360	1.1285	1.0426	0.70677
	0.20	4.0366	4.0000	3.7496	3.1462	2.5257	2.10398

	0.30	5.36.00	5.5000	5.7926	5.6238	4.8784	4.15289
	0.40	6.28.44	6.7237	8.3744	11.3846	13.5362	1.36003
	0.50	6.74.86	6.8367	8.3884	1.4510	33.6700	92.9681
	0.60	4.13.59	4.3702	5.2878	7.1210	8.9348	9.49512
	0.70	2.65.89	2.7150	2.8616	2.9038	2.6446	2.26809

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