Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 16, Number 5 (2020), pp. 695-709

http://www.ripublication.com/gjpam.htm

On Bessel Functions of Two Complex Variables and Differential Operators

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Abstract

In this paper we presented a study of some recurrence relations of Bessel function and modified Bessel function using some differential operators in two complex variables. This study is an application of Bessel functions which give their solutions these functions a partial differential equations using these operators.

Keywords: Bessel function, modified Bessel function, generalized Bessel function, differential operator, recurrence relations.

2010 Mathematics Subject Classification: Primary 33C10, 34B30; Secondary 33C05.

1. INTRODUCTION

We previous results in this paper, and the formality we take advantage of, showing how new results can be obtained by applying the same formalism. In particular we see that these different Bessel function families are directly related to Gossa-like transformations.

Several researchers studied Bessel functions in different forms and used many applications in particular to solve many science and engineering problems as well as to solve other equations, such as Schrödinger equation, thermal equation, wave equation, Laplace equation and Helmholtz equation in cylindrical or spherical coordinates in [1 , 2, 3, 4 , 5 , 6].

The Bessel and Gaussian functions are different manifestations of the same function, as was recently demonstrated using concepts borrowed from parachute theory [2, 6].

The practical result of this determination is a significant simplification of the formalities associated with the processing of Bessel functions, and thus it is reduced to direct applications of the rules of basic calculus.

Bessel functions are orthogonal and appear in solving some partial differential equations. The function shown in the solution depends on geometry, physics (differential equation form), and boundary conditions.

Now, we demonstrate how new theoretical elements arise from pure algebraic manipulations, as the possibility of trigonometric framing.

We also see that numerous Bessel function families are connected by simple Gauussian-like transformations (see e.g.[7]).

The Bessel equation we give is as follows:

$$z^2 w'' + z w' + (z^2 - n^2) w = 0 \tag{1.1}$$

for $n \in \mathrm{N}$, which the separation of the wave equation in cylindrical or polar coordinates can be obtained, for example. Since this is a differential equation of second order we can have two linearly independent solutions. Now we are going to start developing a first solution. We are taking $w(z)$ a power series approach to solution. Writing would be useful

$$w(z) = z^n \sum_{k=0}^{\infty} b_k z^k$$

we get

$$z w'(z) = n z^n \sum_{k=0}^{\infty} b_k z^k + z^n \sum_{k=0}^{\infty} k b_k z^k$$

and

$$z^2 w''(z) = n(n-1) z^n \sum_{k=0}^{\infty} b_k z^k + 2nz^n \sum_{k=0}^{\infty} k b_k z^k + z^n \sum_{k=0}^{\infty} k(k-1) z^k \,. \tag{1.2}$$

We would like to create a complex order solution n. We need to exchange the factorial with the Gamma function here. Instead we receive [8]:

$$J_n(z) = \frac{\left(\frac{z}{2}\right)^n}{\Gamma(1+n)}\, {}_0F_1\left(-;1+n;-\frac{z^2}{4}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k)\Gamma(1+n+k)} \left(\frac{z}{2}\right)^{2k+n} \tag{1.3}$$

where $\Gamma(n)$ is the gamma function and

$$J_{-n}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+k)\Gamma(1-n+k)}\left(\frac{z}{2}\right)^{2k+n}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^{k+n}}{\Gamma(1+k)\Gamma(1+n+k)}\left(\frac{z}{2}\right)^{2k+n}=(-1)^n J_n(z). \tag{1.4}$$

The relations (1.3) and (1.4) are called the Bessel function of order n, from the first kind [9].

It can be shown that the Wronskian of $J_n(z)$ and $J_{-n}(z)$ are given by (see, e.g. [10]):

$$\Psi(J_n,J_{-n})=\frac{2\sin n\pi}{\pi z} \tag{1.5}$$

where n is not a negative integer, ${}_0F_1\left(-;1+n;-\frac{z^2}{4}\right)$ is the hypergeometric function of single complex variable, since

$${}_0F_1\left(-;1+n;-\frac{z^2}{4}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(1+n)_k}\left(\frac{z}{2}\right)^{2k},$$

where

$$(1+n)_k=(1+n)(2+n).....[(k-1)+(1+n)]\ ; \qquad n>1$$

and $\qquad (1+n)_0=1.$

The series (1.3) converges for all z, as the ratio test shows. Hence $J_n(z)$ is defined for all z. The series converges very rapidly because of the factorials in the denominator

$$J_n(z)\simeq\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right).$$

This shows that the $J_n(z)$ and $J_{-n}(z)$ forms a fundamental set of solutions when n is not equal to an integer.

Now, we give some recurrence formula for $J_n(z)$ we consider arbitrary complex z to have

$$\frac{d}{dz}z^n J_n(z)=\frac{d}{dz}z^n\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+k)\Gamma(1+n+k)}\left(\frac{z}{2}\right)^{2k+n}=z^n J_{n-1}(z)$$

and

$$z\,J_n'(z)+n\,J_n(z)=z\,J_{n-1}(z).$$

Similarly

$$\frac{d}{dz}z^{-nr}J_n(z)=-z^{-n}J_{n+1}(z)$$

and

$$z\,J_n'(z)-n\,J_n(z)=-z\,J_{n+1}(z).$$

Subtracting and adding the above recurrence formula yield

$$J_{n-1}(z)+J_{n+1}(z)=\frac{2n}{z}J_n(z)$$

or

$$J_{n-1}(z)+J_{n+1}(z)=2J_n'(z).$$

The integral formula for the Bessel function is given as follows (see, e.g. [11]):

$$J_n(z)=\frac{\left(\frac{z}{2}\right)^n}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{n-\frac{1}{2}}\cos ztdt,\quad R(n)>-\frac{1}{2},|\arg z|<\pi$$

or equivalently, when $t=\cos\theta$ we get:

$$J_n(z)=\frac{\left(\frac{z}{2}\right)^n}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\int_{0}^{\pi}\cos\left(z\cos\theta\right)\sin^{2n}\theta\,d\theta,\quad R(n)>-\frac{1}{2},|\arg z|<\pi.$$

However, the Bessel function of two complex variables z and w can be defined for non-negative integers, since C represents the field of complex variables and the space C^2 of the two complex variables z and w, the successive monomial $1,z,w,z^2,zw,w^2,\ldots.$ is arranged in such a way that the enumeration number of the monomial $z^m w^n$; $m,n \geq 0$; in [12,13] is:

$$J_{m,n}(z,w) = \frac{\left(\frac{z}{2}\right)^m \left(\frac{w}{2}\right)^n}{\Gamma(1+m)\Gamma(1+n)} {}_0F_1\left(-;1+m,1+n;-\frac{z^2}{4},-\frac{w^2}{4}\right)$$

$$= \sum_{(h,k)=0}^{\infty} \frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

where m,n are not negative integers and

$$\Gamma(1+m+n+h+k) = (1+m+n)_{h+k}\,\Gamma(1+m+n)$$

and

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

Also, when m,n are not negative integers, then the Bessel function is

$$J_{-m,-n}(z,w) = \sum_{(h,k)=0}^{\infty} \frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h-m}\left(\frac{w}{2}\right)^{2k-n}}{h!k!\Gamma(1-m+h)\Gamma(1-n+k)}.$$

It follows directly that

$$J_{-m,-n}(z,w) = (-1)^{m+n} \sum_{(h,k)=0}^{\infty} \frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(1-m+h)\Gamma(1-n+k)}. \tag{1.6}$$

Then, we have the following theorem:

Theorem:

When m,n are both integers (positive or negative) we have:

$$J_{-m,-n}(z,w) = (-1)^{m+n} J_{m,n}(z,w).$$

Now, a function $I_n(z)$ that is one of the solutions to the modified differential equation and is closely related to of the Bessel function of first kind $J_n(z)$. The modified Bessel function of the first kind $I_n(z)$ can be defined by

$$I_n(z) = i^{-n} J_n(z).$$

For a real number n the function can be computed using

$$I_n(z)=i^{-n}J_n(z)=\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(1+n+k)} \qquad (1.7)$$

where $\Gamma(n)$ is the gamma function.

The function $I_\nu(x)$ is called modified Bessel function of the first kind of order ν and it is a solution of Bessel modified equation for all values n (e.g. [14]).

2- DIFFERENTIAL OPERATORS AND BESSEL FUNCTION

Here, we study the Bessel function of two complex variables by using the differential operators.

At first suppose that the Bessel function of the first kind of index as follows (e.g. [15, 16]):

$$J_n(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k+n}}{k!\Gamma(1+n+k)}$$

and

$$J_{m,n}(z,w)=\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}. \qquad (2.1)$$

In equation (2.1) multiply both numbers by $z^m w^n$ and then differentiate with respect to z and w by using the differential operator $\Xi=\left(\frac{1}{w}\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial w}\right)$ we see that (see, e.g. [13]):

$$\Xi\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}=\left(\frac{1}{w}\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial w}\right)\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}$$

$$=\frac{1}{4}(2h+m)\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}+\frac{1}{4}(2k+n)\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}$$

$$=\frac{1}{4}[2h+m+2k+n]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}=\frac{1}{4}[(m+n)+2(h+k)]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}.$$

Now, by using the operator Ξ to the Bessel function we get

$$\Xi J_{m,n}(z,w)=\left(\frac{1}{w}\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial w}\right)J_{m,n}(z,w)$$

$$=\sum_{(h,k)=0}^{\infty}\frac{\frac{1}{4}(-1)^{h+k}\ (m+n+2h+2k)\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}.$$

Also,

$$\Xi J_{m,n}(z,w)=\sum_{(h,k)=0}^{\infty}\frac{\frac{1}{4}(-1)^{h+k}\ \left[(m+n)+2(h+k)\right]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{4}(m+n)\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$+\frac{1}{4}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\ \left[2h+2k\right]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{4}(m+n)\left(\frac{2}{z}\right)\left(\frac{2}{w}\right)J_{m,n}(z,w)$$

$$+\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\ \left[h+k\right]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{4}(m+n)\left(\frac{2}{z}\right)\left(\frac{2}{w}\right)J_{m,n}(z,w)+\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{h(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$+\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{k(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{4}(m+n)\left(\frac{2}{z}\right)\left(\frac{2}{w}\right)J_{m,n}(z,w)$$

$$-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(h+1)(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m+1}\left(\frac{w}{2}\right)^{2k+n-1}}{(h+1)!k!\Gamma(2+m+h)\Gamma(1+n+k)}$$

$$-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(k+1)(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n+1}}{h!(k+1)!\Gamma(1+m+h)\Gamma(2+n+k)}$$

$$=(m+n)\left(\frac{1}{z}\right)\left(\frac{1}{w}\right)J_{m,n}(z,w)-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m+1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(2+m+h)\Gamma(1+n+k)}$$

$$-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n+1}}{h!k!\Gamma(1+m+h)\Gamma(2+n+k)}$$

$$=(m+n)\left(\frac{1}{z}\right)\left(\frac{1}{w}\right)J_{m,n}(z,w)-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m+1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma\left[1+(1+m)+h\right]\Gamma(1+n+k)}$$

$$-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n+1}}{h!k!\Gamma(1+m+h)\left[1+(1+n)+k\right]}.$$

Then we have

$$\Xi J_{m,n}(z,w)=\left(\frac{1}{zw}\right)(m+n)J_{m,n}(z,w)$$

$$-\left(\frac{1}{w}\right)\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m+1}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma\left[1+(1+m)+h\right]\Gamma(1+n+k)}$$

$$-\left(\frac{1}{z}\right)\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n+1}}{h!k!\Gamma(1+m+h)\left[1+(1+n)+k\right]}$$

$$\Xi J_{m,n}(z,w)=\left(\frac{1}{zw}\right)(m+n)J_{m,n}(z,w)-\left(\frac{1}{w}\right)J_{m+1,n}(z,w)-\left(\frac{1}{z}\right)J_{m,n+1}(z,w).$$

Consequently the Bessel function (2.1) is a solution for the partial differential equation:

$$\left[\Xi-\left(\frac{1}{zw}\right)(m+n)\right]J_{m,n}(z,w)+\left(\frac{1}{w}\right)J_{m+1,n}(z,w)+\left(\frac{1}{z}\right)J_{m,n+1}(z,w)=0. \qquad (2.2)$$

The following relations are obtained:

$$\left[\Xi-\left(\frac{1}{zw}\right)(m+n)\right]J_{m,n}(z,w)=\sum_{(h,k)=0}^{\infty}\frac{\frac{1}{4}(-1)^{h+k}\left[2(m+n)+2(h+k)\right]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left[(m+n)+(h+k)\right]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(m+h)(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}+\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(n+k)(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(m+h)\Gamma(1+n+k)}+\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(n+k)}$$

$$=\left(\frac{1}{w}\right)J_{m-1,n}(z,w)+\left(\frac{1}{z}\right)J_{m,n-1}(z,w) \qquad (2.3)$$

and

$$\Xi\left(\frac{z}{2}\right)^{m}\left(\frac{w}{2}\right)^{n}J_{m,n}(z,w)=\frac{1}{4}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left[2(m+n)+2(h+k)\right]\left(\frac{z}{2}\right)^{2h+2m-1}\left(\frac{w}{2}\right)^{2k+2n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=\frac{1}{2}\left(\frac{z}{2}\right)^{m}\left(\frac{w}{2}\right)^{n-1}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left[(m+n)\right]\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$+\frac{1}{2}\left(\frac{z}{2}\right)^{m-1}\left(\frac{w}{2}\right)^{n}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left[(h+k)\right]\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$\Xi\left(\frac{z}{2}\right)^{m}\left(\frac{w}{2}\right)^{n}J_{m,n}(z,w)=\frac{1}{4}\left(\frac{z}{2}\right)^{m-1}\left(\frac{w}{2}\right)^{n-1}\left[z\,J_{m-1,n}(z,w)+w\,J_{m,n-1}(z,w)\right]. \quad (2.4)$$

Then we see that

$$\Xi\left(\frac{z}{2}\right)^{-m}\left(\frac{w}{2}\right)^{-n}J_{m,n}(z,w)=$$

$$=\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\,h\left(\frac{z}{2}\right)^{2h-1}\left(\frac{w}{2}\right)^{2k-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}+\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\,k\left(\frac{z}{2}\right)^{2h-1}\left(\frac{w}{2}\right)^{2k-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

$$=-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+1}\left(\frac{w}{2}\right)^{2k-1}}{h!k!\Gamma(2+m+h)\Gamma(1+n+k)}-\frac{1}{2}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h-1}\left(\frac{w}{2}\right)^{2k+1}}{h!k!\Gamma(1+m+h)\Gamma(2+n+k)}$$

$$=-\frac{1}{2}\left(\frac{z}{2}\right)^{-m}\left(\frac{w}{2}\right)^{-n-1}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m+1}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(2+m+h)\Gamma(1+n+k)}$$

$$-\frac{1}{2}\left(\frac{z}{2}\right)^{-m-1}\left(\frac{w}{2}\right)^{-n}\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n+1}}{h!k!\Gamma(1+m+h)\Gamma(2+n+k)}$$

i.e.

$$\Xi\left(\frac{z}{2}\right)^{-m}\left(\frac{w}{2}\right)^{-n}J_{m,n}(z,w)=-\frac{1}{4}\left(\frac{z}{2}\right)^{-m-1}\left(\frac{w}{2}\right)^{-n-1}\left[z\,J_{m+1,n}(z,w)+w\,J_{m,n+1}(z,w)\right]. \quad (2.5)$$

But equation (2.4) can be written in the form, by carrying out the differentiation of the product on left – side as follows:

$$\Xi\left(\frac{z}{2}\right)^m\left(\frac{w}{2}\right)^n J_{m,n}(z,w)=\frac{1}{4}(m+n)\left(\frac{z}{2}\right)^{m-1}\left(\frac{w}{2}\right)^{n-1} J_{m,n}(z,w)+\left(\frac{z}{2}\right)^m\left(\frac{w}{2}\right)^n J'_{m,n}(z,w)$$

$$=-\frac{1}{4}\left(\frac{z}{2}\right)^m\left(\frac{w}{2}\right)^n\left[z\, J_{m-1,n}(z,w)+w\, J_{m,n-1}(z,w)\right]+\left(\frac{z}{2}\right)^m\left(\frac{w}{2}\right)^n J'_{m,n}(z,w)$$

or

$$J'_{m,n}(z,w)+\frac{1}{4}\left[z\, J_{m-1,n}(z,w)+w\, J_{m,n-1}(z,w)\right]-(m+n)\frac{1}{zw}J_{m,n}(z,w)=0. \quad (2.6)$$

Similarly equation (2.5) given as follows

$$J'_{m,n}(z,w)-\frac{1}{4}\left[z\, J_{m+1,n}(z,w)+w\, J_{m,n+1}(z,w)\right]+(m+n)\frac{1}{zw}J_{m,n}(z,w)=0. \quad (2.7)$$

Hence the Bessel function (2.1) is a solution for partial differential equations (2.6) and (2.7).

Adding the two differential equations (2.6) and (2.7) we get:

$$2J'_{m,n}(z,w)=-\frac{1}{4}\left[z\, J_{m-1,n}(z,w)+w\, J_{m,n-1}(z,w)-z\, J_{m+1,n}(z,w)-w\, J_{m,n+1}(z,w)\right]$$

$$=-\frac{1}{4}\left[z\left\{J_{m-1,n}(z,w)-J_{m+1,n}(z,w)\right\}+w\left\{J_{m,n-1}(z,w)-J_{m,n+1}(z,w)\right\}\right]$$

i.e.

$$2J'_{m,n}(z,w)+\frac{1}{4}\left[z\left\{J_{m-1,n}(z,w)-J_{m+1,n}(z,w)\right\}+w\left\{J_{m,n-1}(z,w)-J_{m,n+1}(z,w)\right\}\right]=0.$$

Then the Bessel function (2.1) is a solution to the above partial differential equation.

If we subtract the equations (2.6) from the equation (2.7) we obtain:

$$2(m+n)\frac{1}{zw}J_{m,n}(z,w)=\frac{1}{4}\left[z\, J_{m-1,n}(z,w)+w\, J_{m,n-1}(z,w)\right]$$
$$+\frac{1}{4}\left[z\, J_{m+1,n}(z,w)+w\, J_{m,n+1}(z,w)\right]$$

from which we get:

$$2(m+n)\frac{1}{zw}J_{m,n}(z,w)-$$
$$\frac{1}{4}\left[z\left\{J_{m-1,n}(z,w)+J_{m+1,n}(z,w)\right\}+w\left\{J_{m,n-1}(z,w)+J_{m,n+1}(z,w)\right\}\right]=0.$$

The modified Bessel function $I_{m,n}(z,w)$ of the two complex variables z and w defined in the form:

$$I_{m,n}(z,w)=i^{-(m+n)}J_{m,n}(z,w)$$

where

$$I_{m,n}(z,w)=i^{-(m+n)}J_{m,n}(z,w)=\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\,i^{-(m+n)}\left(\frac{iz}{2}\right)^{2h+m}\left(\frac{iw}{2}\right)^{2k+n}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$
$$=\sum_{(h,k)=0}^{\infty}\frac{(-1)^{h+k}\,i^{-(m+n)}i^{(m+n)+2(k+k)}\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

and

$$I_{m,n}(z,w)=\sum_{(h,k)=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}. \tag{2.8}$$

For the modified Bessel function $I_{m,n}(z,w)$ of the two complex variables we obtain directly the following properties:

$$I'_{m,n}(z,w)=\frac{1}{4}\left[z\,I_{m-1,n}(z,w)+w\,I_{m,n-1}(z,w)\right]-(m+n)\frac{1}{zw}I_{m,n}(z,w),$$

$$I'_{m,n}(z,w)=-\frac{1}{4}\left[z\,I_{m+1,n}(z,w)+w\,I_{m,n+1}(z,w)\right]+(m+n)\frac{1}{zw}I_{m,n}(z,w),$$

$$2I'_{m,n}(z,w)=\frac{1}{4}\left[z\left\{I_{m-1,n}(z,w)-I_{m+1,n}(z,w)\right\}+w\left\{I_{m,n-1}(z,w)-I_{m,n+1}(z,w)\right\}\right]$$

thus we obtain:

$$2I'_{m,n}(z,w)-\frac{1}{4}\left[z\left\{I_{m-1,n}(z,w)-I_{m+1,n}(z,w)\right\}+w\left\{I_{m,n-1}(z,w)-I_{m,n+1}(z,w)\right\}\right]=0\ .$$

Then the modified Bessel function (2.8) is a solution to the above partial differential equation.

Now, some special cases of Bessel function $J_{m,n}(z,w)$ and modified Bessel function $I_{m,n}(z,w)$ are given as follows:

1- Acting by the differential operator $\Xi_z = \left(\frac{1}{w}\frac{d}{dz}\right)$ in equation (2.1) then we have

$$\Xi_z\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n} = \left(\frac{1}{w}\frac{d}{dz}\right)\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n} = \frac{1}{4}(2h+m)\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}$$

and

$$\Xi_z\, J_{m,n}(z,w) = \left(\frac{1}{w}\frac{d}{dz}\right)J_{m,n}(z,w)$$
$$= \sum_{(h,k)=0}^{\infty} \frac{\frac{1}{4}(-1)^{h+k}\,(m+2h)\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}}{h!k!\Gamma(1+m+h)\Gamma(1+n+k)}$$

thus,

$$\Xi_z\, J_{m,n}(z,w) = \left(\frac{1}{zw}\right)(m)J_{m,n}(z,w) - \left(\frac{1}{w}\right)\sum_{(h,k)=0}^{\infty} \frac{(-1)^{h+k}\left(\frac{z}{2}\right)^{2h+m+1}\left(\frac{w}{2}\right)^{2k+n}}{h!k!\Gamma\left[1+(1+m)+h\right]\Gamma(1+n+k)}$$

$$\Xi_z\, J_{m,n}(z,w) = \left(\frac{m}{zw}\right)J_{m,n}(z,w) - \left(\frac{1}{w}\right)J_{m+1,n}(z,w).$$

Therefore

$$\left[\Xi_z - \left(\frac{m}{zw}\right)\right]J_{m,n}(z,w) + \left(\frac{1}{w}\right)J_{m+1,n}(z,w) = 0. \tag{2.9}$$

Then the Bessel function (2.1) is a solution to the partial differential equation (2.9).

2- Similarly acting by the differential operator $\Xi_w = \left(\frac{1}{z}\frac{d}{dw}\right)$ in equation (2.1) then we have

$$\Xi_w\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n} = \left(\frac{1}{z}\frac{d}{\partial w}\right)\left(\frac{z}{2}\right)^{2h+m}\left(\frac{w}{2}\right)^{2k+n} = \frac{1}{4}(2k+n)\left(\frac{z}{2}\right)^{2h+m-1}\left(\frac{w}{2}\right)^{2k+n-1}$$

therefore

$$\Xi_w\, J_{m,n}(z,w) = \left(\frac{n}{zw}\right) J_{m,n}(z,w) - \left(\frac{1}{z}\right) J_{m,n+1}(z,w)$$

from which we obtain:

$$\left[\Xi_w - \left(\frac{n}{zw}\right)\right] J_{m,n}(z,w) + \left(\frac{1}{z}\right) J_{m,n+1}(z,w) = 0\,. \tag{2.10}$$

Then the Bessel function (2.1) is a solution to the partial differential equation (2.10).

The Bessel function contains properties that can be used to completely solve physical problems. It presents some of the most important ones in this section.

CONCLUSION

In addition to other equations such as Schrodinger equation, heat equation, wave equation, Laplace equation and Helmholtz equation, Bessel functions appear in cylindrical or spherical coordinates to solve other scientific and engineering problems [9]. For this paper, we have shown that a formalism of umbral nature can be exploited to greatly simplify the technicalities underlying Bessel function theory and its manipulations leading to combinations or the creation of new forms [17].

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