Bessel Function with Linear Differential Operator

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Abstract

In this paper, a class of analytic functions \(f\) defined on the open unit disc satisfying

\[
\text{Re} \left( e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right) \right)^2 + \beta > \left| 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right|^2
\]

is studied, among other results, inclusion relations and applications involving a certain class of integral operator are also considered.

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1 INTRODUCTION

Let \(A\) denote the class of all analytic function of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

in the open unit disc \(U = \{ z : z \in \mathbb{C} : |z| < 1 \}\). Let \(S\) be the subclass of \(A\) consisting

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of univalent functions in $U$. We say that the function $f$ is convex when $f(U)$ is a convex set. Also, we say that a function $f$ is starlike with respect to the origin when $f(U)$ is a starlike set with respect to 0. By $K$ or $S$ we denote the subclasses of $A$ consisting of all functions which are convex or starlike respectively, while by $S(\delta)$ we denote the class of starlike functions of order $\delta$, $\delta \in [0,1]$.

In 1991, Goodman [7] introduced the class $UCV$ of uniformly convex functions. A function $f \in CV$ is the class $UCV$ if for every circular arc $\xi \subset U$, with center in $U$, the arc $f(\xi)$ is convex. A more useful characterization of class $UCV$ was given by Ma and Minda [11] (see also [17]) as:

$$f \in UCV \iff f \in A \text{ and } \text{Re} \left( 1 + \frac{|zf''(z)|}{|f'(z)|} \right) > 1, \quad (z \in U).$$

In 1999, Kanas and Wisniowska [8] (see also [9]) introduced the class of $k$-uniformly convex functions, $k \geq 0$, denoted by $k$-$UCV$ and the class $k$-$ST$ related to $k$-$UCV$ by Alexandar type relation, i.e.,

$$f \in k$-$UCV \iff zf' \in k$-$ST \iff f \in A \text{ and } \text{Re} \left( 1 + \frac{|zf''(z)|}{|f'(z)|} \right) > k, \quad (z \in U).$$

In [8] and [9] respectively, their geometric definitions and connections with the conic domains were also considered. For a fixed $k \geq 0$, the class $k$-$UCV$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $U$ with any disk centered at $\zeta$, $|\zeta| \leq k$, onto a convex domain. The notion of $k$-uniform convexity is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center $\zeta$ is the origin and the class $k$-$UCV$ reduces to the class $kV$.

Moreover for $k = 1$ it coincides with the class of uniformly convex functions $UCV$ introduced by Goodman [7] and studied extensively by Rønning [17] and independently by Ma and Minda [11]. The class $k$-$UCV$ started much earlier in papers [5, 6] with some additional conditions but without the geometric interpretation.

We say that a function $f \in A$ is in the $S^*_k$, $k \geq 0, \gamma \in C \setminus \{0\}$, if and only if

$$\text{Re} \left[ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] > k, \quad (z \in U).$$
A lot of authors investigated the properties of the class $S_{k,r}^*$ and their generalizations in several directions, e.g., see [1, 2, 6, 9, 15, 17, 19]. An analytic function $f$ is said to be subordinate to an analytic function $g$ (written as $f \prec g$) if and only if there exists an analytic function $\omega$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for } z \in U$$

such that

$$f(z) = g(\omega(z)) \text{ for } z \in U.$$ In particular, if $g$ is univalent in $U$, we have the following equivalence

$$f \prec g \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The convolution or Hadamard product of two functions of class $A$ is denoted and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where $f$ has the form (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad z \in U.$$ Let us consider the following second-order linear homogeneous differential equation (see for details [3] and [4]):

$$z^2 \omega''(z) + b z \omega'(z) + \left[ d z^2 - v^2 + (1-b)v \right] \omega(z) = 0 \quad (v,b,d \in \mathbb{C}). \quad (1.2)$$

The function $\omega_{v,b,d}(z)$, which is called the generalized Bessel function of the first kind of order $v$, it is defined as a particular solution of (1.2). The function $\omega_{v,b,d}(z)$ has the familiar representation as

$$\omega_{v,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n! \left( v + n + \frac{b+1}{2} \right) \left( \frac{z}{2} \right)^{2n+v}} \quad (z \in \mathbb{C}). \quad (1.3)$$

Here $\Gamma$ stands for the Euler gamma function. The series (1.3) permits the study of Bessel, modified Bessel, and spherical Bessel function altogether. It is worth mentioning that, in particular:

i) For $b = d = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order $u$. 

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defined by
\[
J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v+n+1)} \left( \frac{z}{2} \right)^{2n+v} (z \in \mathbb{C}).
\] (1.4)

ii) For \( b = 1 \) and \( d = -1 \) in (1.3), we obtain the modified Bessel function of the first kind of order \( v \) defined by
\[
l_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v+n+1)} \left( \frac{z}{2} \right)^{2n+v} (z \in \mathbb{C}).
\] (1.5)

iii) For \( b = 2 \) and \( d = 1 \) in (1.3), the function \( \omega_{v,b,d}(z) \) reduces to
\[
S_v(z) = \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v+n+\frac{3}{2})} \left( \frac{z}{2} \right)^{2n+v} (z \in \mathbb{C}).
\] (1.6)

Now, consider the function \( u_{v,b,d}(z) : \mathbb{C} \rightarrow \mathbb{C} \), defined by the transformation
\[
u_{v,b,d}(z) = 2^\nu \Gamma \left( \nu + \frac{b+1}{2} \right) \frac{z^\nu}{\sqrt{z}} \omega_{v,b,d}(\sqrt{z}).
\] (1.7)

By using the well-known Pochhammer symbol (or the shifted factorial) \( (\lambda)_\mu \) defined, for \( \lambda, \mu \in \mathbb{C} \) and in terms of the Euler \( \Gamma \) function, by
\[
(\lambda)_\mu : = \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} \frac{1}{\lambda(\lambda + 1) \cdots (\lambda + n - 1)} & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}
\]
and \( (\lambda)_0 = 1 \), we obtain for the function \( u_{v,b,d}(z) \) the following representation
\[
u_{v,b,d}(z) = \sum_{n \in \mathbb{Z}} \left( \frac{-d}{4} \right)^n \left( \frac{z}{\sqrt{z}} \right)^n \left( \frac{b+1}{2} \right)^n n!
\]
where \( k = \nu + \frac{b+1}{2} \neq 0, -1, -2, \ldots \). This function is analytic on \( \mathbb{C} \) and satisfies the
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second order linear differential equation
\[ 4z^2 u''(z) + 2(2v + b + 1)zu'(z) + du(z) = 0. \]

Now, we introduce the function \( \varphi_{v,b,d}(z) \) defined in terms of generalized Bessel function \( \omega_{v,b,d}(z) \), defined by
\[
\varphi_{v,b,d}(z) = zu_{v,b,d}(z)
\]
\[
= 2^n \Gamma \left( n + \frac{b + 1}{2} \right) \sqrt{\frac{z}{2}} \omega_{v,b,d}(\sqrt{z})
\]
\[
= z + \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{4^n n!(k)_n}, \text{ where } k = \left( n + \frac{b + 1}{2} \right)
\]
\[
= g(k, d, z)
\]

Motivated by Selvaraj and Karthikeyan [14], we define the following \( D^m_{\lambda}(k, d) f(z) : U \rightarrow U \) by
\[
D^1_{\lambda}(k, d) f(z) = f(z) \ast g(k, d, z)
\]
\[
D^1_{\lambda}(k, d) f(z) = (1 - \lambda)(f(z) \ast g(k, d, z)) + \lambda z (f(z) \ast g(k, d, z))'
\]
\[
D^m_{\lambda}(k, d) f(z) = D^1_{\lambda} \left( D^{m-1}_{\lambda}(k, d) f(z) \right)
\]

If \( f \in \mathbb{A} \), then from (1.9) and (1.10) we may easily deduce that
\[
D^m_{\lambda}(k, d) f(z) = z + \sum_{n=2}^{\infty} \frac{(1 + (n-1)\lambda)^m (-1)^m}{4^{n-1} (n-1)! (k)_n} \overline{a_{n-1}} z^n
\]
where \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( \lambda \geq 0 \).

It can be easily verified from definition of (1.11) that
\[
\lambda z \left( D^m_{\lambda}(k, d) f(z) \right)' = D^{m+1}_{\lambda}(k, d) f(z) - (1 - \lambda) D^m_{\lambda}(k, d) f(z)
\]
In the special cases of the $D^n_k(k,d)f(z)$, we obtain the following operators related to the Bessel function:

i) Choosing $m=0$ in (1.11) we get the Deniz operator

$$B_k^c = z + \sum_{n=2}^{\infty} \frac{(-d)^{n-1}a_n z^n}{4^{n-1}(n-1)!(k)_{n-1}}.$$ 

ii) Choosing $m=0$ and $b=d=1$ in (1.11) we obtain the operator $J_v : A \rightarrow A$ related with Bessel function, defined by

$$J_v f(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}a_n z^n}{4^{n-1}(n-1)!(v+1)_{n-1}}.$$ 

iii) Choosing $m=0$, $b=1$ and $d=-1$ in (1.11) we obtain the operator $J_v : A \rightarrow A$ related with modified Bessel function, defined by

$$J_v f(z) = z + \sum_{n=2}^{\infty} \frac{(1)^{n-1}a_n z^n}{4^{n-1}(n-1)!(v+1)_{n-1}}.$$ 

iv) Choosing $m=0$, $b=2$ and $d=1$ in (1.11) we obtain the operator $S_v : A \rightarrow A$ related with spherical Bessel function, defined by

$$S_v f(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}a_n z^n}{4^{n-1}(n-1)!\left\{\frac{3}{2}\right\}_{n-1}}.$$ 

**Definition 1.1** A function $f(z) \in A$ is in the class $S^*(\alpha, \beta, \gamma)$, $\gamma \in \mathbb{C}\{0\}$ if and only if

$$\text{Re} \left[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right) \right]^2 + \beta > \left| \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right|^2. \quad (1.13)$$
Remark 1  The above differential inequality can be equivalently written as

\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) e^{-i\alpha} (h(z) \cos \alpha + i \sin \alpha) = K_\alpha(z),
\]

where \( h(z) = 1 - \frac{\beta}{\cos^2 \alpha} + \frac{2}{\pi} \log \left( \frac{1 + \sqrt{1 + \theta^2}}{\sqrt{1 + \theta^2}} \right)^2, \theta = \left( \frac{e^\mu - 1}{e^\mu + 1} \right)^2 \) and \( \mu = \frac{\sqrt{\beta \pi}}{2 \cos \alpha}. \)

Definition 1.2  A function \( f(z) \in A \) is in the class \( Q^m_\alpha(k,d,\alpha,\beta,\gamma) \) if and only if

\[
\Re \left\{ e^{i\alpha} \left( 1 + \frac{1}{\gamma} (J(\lambda,m,k,d,z) - 1) \right) \right\}^2 + \beta > \left| \frac{1}{\gamma} (J(\lambda,m,k,d,z) - 1) \right|^2,
\]

where

\[
J(\lambda,m,k,d,z) = \frac{\left( \frac{D^m_\alpha(k,d)f(z)}{D^m_\alpha(k,d)f(z)} \right)'}{D^m_\alpha(k,d)f(z)}.
\]  \hspace{1cm} (1.14)

Since \( K_\alpha(z) = e^{i\alpha} (h(z) \cos \alpha + i \sin \alpha) \) is a convex univalent function, we can write Definition 1.1 in subordination form

\[
f \in Q^m_\alpha(k,d,\alpha,\beta,\gamma) \iff f \in A \quad \text{and} \quad J(\lambda,m,k,d,z) < K_\alpha(z) \quad (z \in U).
\]

Special Cases

For \( \lambda = 0 \) and \( f \in Q^m_\alpha(k,d,\alpha,\beta,\gamma) \) we have \( D^m_\gamma(k,d)f(z) \in S'(\alpha,\beta,\gamma). \)

2  PRELIMINARY RESULTS

Lemma 1 [13] Let \( h \) be a convex univalent function in \( U \) with \( \Re\left( \lambda h(z) + \mu \right) > 0 \), where \( \mu \in \mathbb{C}, \lambda \in \mathbb{C} \setminus \{0\}, z \in U \). If \( p \) is analytic in \( U \) with \( p(0) = h(0) \), then

\[
p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} < h(z).
\]
Lemma 2 [12] Let \( d \) be a complex number with \( \Re d > 0 \). Suppose that \( \Psi : \mathbb{C}^2 \to \mathbb{C} \) is continuous and satisfies the conditions \( \Re \Psi(ix, y) \leq 0 \), when \( x \) is real and
\[
y \leq (-|d - ix|^2)/(2\Re d).
\]
If \( p \) is analytic in \( U \) with \( p(0) = d \) and \( \Re \left[ \Psi\left(p(z), zp'(z)\right)\right] > 0 \) for \( z \in U \), then \( \Re p(z) > 0 \) in \( U \).

Lemma 3 [18] Let \( f \) and \( g \) be in the class \( kV \) and \( S^* \) respectively. Then, for every function \( F \) analytic in \( U \), we have
\[
\frac{f(z) \ast g(z)F(z)}{f(z) \ast g(z)} \in \overline{co}[F(U)], \quad z \in U,
\]
where \( \overline{co}[F(U)] \), denotes the closed convex hull of the set \( F(U) \).

Lemma 4 [16] Let the function \( \phi(z) \) given by
\[
\phi(z) = \sum_{n=1}^{\infty} B_n z^n
\]
be convex in \( U \). Suppose also that the function \( h(z) \) given by
\[
h(z) = \sum_{n=1}^{\infty} h_n z^n
\]
is holomorphic in \( U \). If \( h(z) \prec \phi(z), \ z \in U \), then \( |h_n| \leq |B_1|, \ n \in \mathbb{N} = \{1, 2, 3, \cdots\} \).

3 COEFFICIENT INEQUALITIES

A function \( f(z) \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Let \( \sum \) denote the class of bi-univalent functions defined in the unit disk \( U \).

Definition 3.1 Let \( h : U \to \mathbb{C} \) be a convex univalent function such that \( h(0) = 1 \) and \( h(\bar{z}) = 
\overline{h(z)} \), for \( z \in U \) and \( \Re(h(z)) > 0 \). A function \( f \in \sum \) is said to be in the class
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\[ S^*(\alpha, \gamma), \gamma \in \mathbb{C} \setminus \{0\} \text{ if the following conditions are satisfied:} \]

\[ f \in \sum, \quad e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right) < h(z) \cos \alpha + i \sin \alpha, \quad z \in U \quad (3.1) \]

and

\[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zg'(\omega)}{g(\omega)} - 1 \right) \right) < h(\omega) \cos \alpha + i \sin \alpha, \quad \omega \in U, \quad (3.2) \]

where

\[ g(\omega) = f^{-1}(\omega) = \omega - a_2 \omega^2 + (2a_2^2 - a_3) \omega^3 - (5a_2^3 - 5a_2a_3 + a_4) \omega^4 + \cdots, \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right). \]

**Definition 3.2** A function \( f \in \sum \) given by (1.1) is said to be in the class \( S^m(\alpha, \gamma, k, d, h) \) if the following conditions are satisfied:

\[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{z\left(D^m_\lambda(k, d) f(z)\right)^{\prime}}{D^m_\lambda(k, d) f(z)} - 1 \right) \right) < h(z) \cos \alpha + i \sin \alpha, \quad z \in U \quad (3.3) \]

and

\[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{\omega\left(D^m_\lambda(k, d) g(\omega)\right)^{\prime}}{D^m_\lambda(k, d) g(\omega)} - 1 \right) \right) < h(\omega) \cos \alpha + i \sin \alpha, \quad \omega \in U, \quad (3.4) \]

where \( g(\omega) = f^{-1}(\omega), \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, z, \omega \in U \) on specializing the parameter \( \lambda \).

**Theorem 1** Let \( f \) given by (1.1) be in the class \( S^*(\alpha, \gamma), \) then

\[ |a_2| \leq \sqrt{\gamma \| B_1 \| \cos \alpha} \]

and

\[ |a_3| \leq 4 \gamma \| B_1 \| \cos \alpha. \]
Theorem 2 Let \( f \) given by (1.1) be in the class \( S_{\lambda,m}(\alpha,\gamma,k,d,h) \), then

\[
|a_2| \leq \frac{2|\gamma| B_1 |\cos \alpha|}{(1+2\lambda)^m d^2} \left( \frac{(1+\lambda)^{2m} d^2}{8(k)_2} - \frac{(1+\lambda)^{2m} d^2}{8(k)_1} \right)
\]

and

\[
|a_3| \leq \frac{2|\gamma|^2 |B_1| |\cos \alpha|}{(1+2\lambda)^m d^2} \left( \frac{(1+\lambda)^{2m} d^2}{8(k)_2} - \frac{(1+\lambda)^{2m} d^2}{8(k)_1} \right).
\]

Proof. It follows from (3.3) and (3.4) that

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{z D^m_{\lambda}(k,d) f(z)}{D^m_{\lambda}(k,d) f(z)} - 1 \right) \right) = p(z) \cos \alpha + i \sin \alpha, \quad z \in U \tag{3.5}
\]

and

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{\omega D^m_{\lambda}(k,d) g(\omega)}{D^m_{\lambda}(k,d) g(\omega)} - 1 \right) \right) = q(\omega) \cos \alpha + i \sin \alpha, \quad \omega \in U, \tag{3.6}
\]

where \( p(z) \prec h(z) \) and \( q(\omega) \prec h(\omega) \) have the forms

\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots
\]

and

\[
q(\omega) = 1 + q_1 \omega + q_2 \omega^2 + \cdots
\]

respectively. It follows from (3.5) and (3.6) that

\[
\frac{e^{ia}}{\gamma} \frac{(1+\lambda)^m(-d)}{4(1)!(k)_1} a_2 = p_1 \cos \alpha \tag{3.7}
\]

\[
\frac{e^{ia}}{\gamma} \left( \frac{(2)(1+2\lambda)^m d^2}{4^2(2)!(k)_2} a_3 - \left( \frac{(1+\lambda)^m(-d)}{4(1)!(k)_1} \right)^2 \right) = p_2 \cos \alpha \tag{3.8}
\]

\[
-\frac{e^{ia}}{\gamma} \frac{(1+\lambda)^m(-d)}{4(1)!(k)_1} a_2 = q_1 \cos \alpha \tag{3.9}
\]
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and

\[
\frac{e^{i\alpha}}{\gamma} \left( \frac{(4)(1+2\lambda)^m d^2}{4^2(2)!}(k_2)^2 \alpha \right) - \frac{(2)(1+2\lambda)^m d^2}{4^2(2)!}(k_2)^2 \alpha\left( \frac{(1+\lambda)^m (-d)}{4(1)!}(k_1) \right)^2 \right) = q_2 \cos \alpha. \tag{3.10}
\]

From (3.7) and (3.9), we obtain

\[ p_1 = -q_1. \]

Adding (3.8) and (3.10), we get

\[ \frac{e^{i\alpha}}{\gamma} \left( \frac{(1+2\lambda)^m d^2}{8(k_2)} - \frac{(1+\lambda)^m d^2}{8(k_1)^2} \right) a_2 = (p_2 + q_2) \cos \alpha. \tag{3.11}\]

Since \( p, q \in h(U) \), applying Lemma 4, we have

\[ |p_m| = \frac{|p^{(m)}(0)|}{m!} \leq |B_1|, \quad m \in N \tag{3.12}\]

\[ |q_m| = \frac{|q^{(m)}(0)|}{m!} \leq |B_1|, \quad m \in N. \tag{3.13}\]

Applying (3.12), (3.13) and Lemma 4 for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we get

\[ \left| a_2 \right| \leq \sqrt{\frac{2 |\gamma| |B_1| \cos \alpha}{(1+2\lambda)^m d^2 (1+\lambda)^m d^2 \frac{8(k_2)}{8(k_1)^2}}}. \tag{3.14}\]

Subtracting (3.10) from (3.8), we get

\[ \frac{e^{i\alpha}}{\gamma} \left( \frac{(1+2\lambda)^m d^2}{8(k_2)} a_3 - \frac{(1+2\lambda)^m d^2}{8(k_2)} a_2 \right) = (p_2 - q_2) \cos \alpha \tag{3.15}\]

or, equivalently

\[ a_3 = \left( \frac{\gamma}{e^{i\alpha}} \right) \left( \frac{8(k_2)^2 (p_2 - q_2) \cos \alpha}{(1+2\lambda)^m d^2} \right) + \left( \frac{\gamma}{e^{i\alpha}} \right)^2 \left( \frac{(p_2 + q_2) \cos \alpha}{(1+2\lambda)^m d^2 (1+\lambda)^m d^2} \frac{8(k_2)}{8(k_1)^2} \right). \]
Applying (3.12), (3.13) and Lemma 4 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we get

\[
|a_3| \leq \frac{2|\gamma|^2|B_1| \cos \alpha}{(1+2\lambda)^{m+\frac{3}{2}}} \frac{1}{8(k_2) - (1+\lambda)^{m+\frac{3}{2}}} \frac{1}{8(k_2)^2}.
\]

This complete the proof of Theorem 2.

**Corollary 1** Let \( f \in A \) be bi-convex function of order \( \beta \) then \( |a_2| \leq \sqrt{1-\beta} \) and \( |a_3| \leq 1-\beta \).

**Corollary 2** Let \( f \in A \) satisfy the condition \( 1+\frac{zf''(z)}{f'(z)} \leq h(z) \) and \( 1+\frac{zg''(w)}{g'(w)} \leq h(w) \) then \( |a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2B_1^2 + 2B_1 - 2B_2}} \) and \( |a_3| \leq \frac{1}{2}(B_1 + |B_2 - B_1|) \).

**Corollary 3** Let \( f \) be given by (1.1) and \( g = f^{-1} \). If \( f \) and \( g \) satisfies the condition \( (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda \left( 1+\frac{zf''(z)}{f'(z)} \right) \leq h(z) \)

and \( (1-\lambda)\frac{wg'(w)}{g(w)} + \lambda \left( 1+\frac{wg''(w)}{g'(w)} \right) \leq g(w) \),

then

\[
|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+\lambda)|B_1^2 + (1+\lambda)(B_1 - B_2)|}}
\] (3.16)

and

\[
|a_3| \leq \frac{B_1 + |B_2 - B_1|}{1+\lambda}
\] (3.17)
Remark 2 Put $\lambda = 1$ in corollary 3, we get the result $|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{2} |B_1^2 + 2B_1 - 2B_2|}$ and $|a_1| \leq \frac{1}{2} (B_1 + |B_2 - B_1|)$ of corollary 2.

4 CLOSURE PROPERTY

Theorem 3 Let

$$F(\lambda, m, k, d, f(z)) = D^m_{\lambda}(k, d) f(z).$$

Then $f \in Q^m_{\lambda}(k, d, \alpha, \beta, \gamma)$ if and only if $F(\lambda, m, k, d, f(z)) \in \mathcal{S}'(\alpha, \beta, \gamma)$.

Proof. Let $F(\lambda, m, k, d, f(z)) \in \mathcal{S}'(\alpha, \beta, \gamma)$, then

$$\text{Re} \left[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zF'(\lambda, m, k, d, f(z))(z)}{F(\lambda, m, k, d, f(z))} - 1 \right) \right) \right]^2 + \beta > \left| \frac{1}{\gamma} \left( \frac{zF'(\lambda, m, k, d, f(z))(z)}{F(\lambda, m, k, d, f(z))} - 1 \right) \right|^2.$$

(4.2)

Thus (4.1) together with (4.2) implies

$$\text{Re} \left[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} (J(\lambda, m, k, d, z) - 1) \right) \right]^2 + \beta > \left| \frac{1}{\gamma} (J(\lambda, m, k, d, z) - 1) \right|^2,$$

where $J(\lambda, m, k, d, z)$ is given by (1.14). Therefore $f(z) \in Q^m_{\lambda}(k, d, \alpha, \beta, \gamma)$.

Converse is immediate.

Theorem 4 For $m \geq 1$, $Q^m_{\lambda}(k, d, \alpha, \beta, \gamma) \subset Q^m_{\lambda}(k, d, \alpha, \beta, \gamma)$.

Proof. Let $f \in Q^m_{\lambda}(k, d, \alpha, \beta, \gamma)$ and

$$\frac{z \left( D^m_{\lambda}(k, d) f(z) \right)'}{D^m_{\lambda}(k, d) f(z)} = p(z),$$

(4.3)

where $p$ is analytic in $U$ and $p(0) = 1$. 
From (1.12) and (4.3) and after some simplification, we obtain
\[
\frac{D_{\lambda}^{m+1}(k,d)f(z)}{D_{\lambda}^{m}(k,d)f(z)} = \lambda p(z) + (1 - \lambda). \tag{4.4}
\]

By logarithmic differentiation of (4.4), we have
\[
z\left(\frac{D_{\lambda}^{m+1}(k,d)f(z)}{D_{\lambda}^{m}(k,d)f(z)}\right)' = p(z) + \frac{\lambda zp'(z)}{(1 - \lambda) + \lambda p(z)}. \tag{4.5}
\]

Since \( f \in Q_{\lambda}^{m+1}(k,d,\alpha,\beta,\gamma) \), so
\[
p(z) + \frac{\lambda zp'(z)}{(1 - \lambda) + \lambda p(z)} \sim K_{\alpha}(z).
\]

Thus by using Lemma 1, \( p(z) \prec K_{\alpha}(z) \) and hence \( f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \).

Let us consider the Bernardi integral operator \( F_{\mu} \) given by
\[
F_{\mu}f(z) = \frac{\mu + 1}{z^\mu} \int_{0}^{z} t^{\mu - 1} f(t) dt. \tag{4.6}
\]

For \( \mu \) with \( \text{Re} \mu > -1 \) the operator has the property \( F_{\mu} : A \to A \), (see, for instance, [10], p.11).

**Theorem 5** Let \( \mu > -1 \) and \( f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \), then \( F_{\mu}f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \).

**Proof.** Let \( f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \) and
\[
z\left(\frac{D_{\lambda}^{m}(k,d)F_{\mu}(f)(z)}{D_{\lambda}^{m}(k,d)F_{\mu}(f)(z)}\right)' = p(z). \tag{4.7}
\]

where the function \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). From (4.6), we have
\[
z\left(F_{\mu}(f)\right)'(z) + \mu F_{\mu}(f)(z) = (\mu + 1) f(z)
\]
and so
\[
z\left(D_{\lambda}^{m}(k,d)F_{\mu}(f)(z)\right)' = (\mu + 1)(D_{\lambda}^{m}(k,d)f(z) - \mu D_{\lambda}^{m}(k,d)F_{\mu}(f)(z)). \tag{4.8}
\]
Then, by using equations (4.7) and (4.8), we obtain
\[
\frac{(\mu+1)(D_{\lambda}^m(k,d)f(z))'}{D_{\lambda}^m(k,d)f(z)} = p(z) + \mu
\] (4.9)

By logarithmic differentiation of (4.9), we have
\[
\frac{z\left(D_{\lambda}^m(k,d)f(z)\right)'}{D_{\lambda}^m(k,d)f(z)} - \frac{zp'}{p(z)+\mu}, \quad (z \in \mu).
\]

Hence by Lemma1, we conclude that \( p(z) \prec K_{\lambda}(z) \) in \( U \) which implies that \( F_{\mu}f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \).

**Theorem 6** Let \( f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \) and \( \psi \in kV \). If \( 0 < \gamma \leq 1 \), then \( \psi \ast f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \).

*Proof.* Let \( F = \psi \ast f \). If \( f \in Q_{\lambda}^{m}(k,d,\alpha,\beta,\gamma) \) then the condition (4.3) is satisfied with \( p \prec K_{\alpha}(z) \). Using the usual convolution properties and (4.3),
\[
p(z) = \frac{z\left(D_{\lambda}^m(k,d)f(z)\right)'}{D_{\lambda}^m(k,d)f(z)}
\]
\[
= \frac{z\left(D_{\lambda}^m(k,d)(\psi \ast f)\right)'}{D_{\lambda}^m(k,d)(\psi \ast f)}
\]
\[
= \frac{z\left(\psi \ast D_{\lambda}^m(k,d)f\right)'}{\psi \ast D_{\lambda}^m(k,d)f}
\]
\[
= \psi \ast \left(\frac{D_{\lambda}^m(k,d)f}{\psi \ast D_{\lambda}^m(k,d)f}\right)' = \psi \ast \left(\frac{P(z)f(z)}{\psi \ast f(z)}\right). \quad (4.10)
\]
By Theorem (3), the function $F(z) = D^m_\kappa(k,d)f(z) \in S^\ast(\alpha, \beta, \gamma)$.

Hence, by Lemma (3), we have

$$
\frac{z\left(D^m_\lambda(k,d)F(z)\right)'}{D^m_\lambda(k,d)F(z)} \in \text{co}[F(U)] \subseteq K_\alpha(z).
$$

Since $K_\alpha(z)$ is a convex univalent and $p(z) \prec K_\alpha(z)$.

Hence $F = \psi \ast f \in Q^n_\beta(k,d,\alpha,\beta,\gamma)$.

REFERENCES


Bessel Function with Linear Differential Operator


