Bessel Function with Linear Differential Operator

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Abstract

In this paper, a class of analytic functions $f$ defined on the open unit disc satisfying

$$
\Re \left[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right) \right]^2 + \beta > \left| \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right|^2
$$

is studied, among other results, inclusion relations and applications involving a certain class of integral operator are also considered.

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1 INTRODUCTION

Let $A$ denote the class of all analytic function of the form

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
$$

in the open unit disc $U = \{ z : z \in \mathbb{C} : |z| < 1 \}$. Let $S$ be the subclass of $A$ consisting

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of univalent functions in $U$. We say that the function $f$ is convex when $f(U)$ is a convex set. Also, we say that a function $f$ is starlike with respect to the origin when $f(U)$ is a starlike set with respect to 0. By $K$ or $S^*$ we denote the subclasses of $A$ consisting of all functions which are convex or starlike respectively, while by $S^*(\delta)$ we denote the class of starlike functions of order $\delta$, $\delta \in [0,1)$.

In 1991, Goodman [7] introduced the class $UCV$ of uniformly convex functions. A function $f \in CV$ is the class $UCV$ if for every circular arc $\xi \subset U$, with center in $U$, the arc $f(\xi)$ is convex. A more useful characterization of class $UCV$ was given by Ma and Minda [11] (see also [17]) as:

$$f \in UCV \iff f \in A \text{ and } \Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in U).$$

In 1999, Kanas and Wisniowska [8] (see also [9]) introduced the class of $k$-uniformly convex functions, $k \geq 0$, denoted by $k-UCV$ and the class $k-ST$ related to $k-UCV$ by Alexandar type relation, i.e.,

$$f \in k-UCV \iff zf' \in k-ST \iff f \in A \text{ and } \Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in U).$$

In [8] and [9] respectively, their geometric definitions and connections with the conic domains were also considered. For a fixed $k \geq 0$, the class $k-UCV$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $U$ with any disk centered at $\zeta, |\zeta| \leq k$, onto a convex domain. The notion of $k$-uniform convexity is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center $\zeta$ is the origin and the class $k-UCV$ reduces to the class $kV$.

Moreover for $k = 1$ it coincides with the class of uniformly convex functions $UCV$ introduced by Goodman [7] and studied extensively by Rønning [17] and independently by Ma and Minda [11]. The class $k-UCV$ started much earlier in papers [5, 6] with some additional conditions but without the geometric interpretation.

We say that a function $f \in A$ is in the $S^*_{k,\gamma}, k \geq 0, \gamma \in \mathbb{C}\backslash\{0\}$, if and only if

$$\Re \left[1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1\right)\right] > k \frac{1}{\gamma} \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in U).$$
A lot of authors investigated the properties of the class $S_{k,\gamma}$ and their generalizations in several directions, e.g., see [1, 2, 6, 9, 15, 17, 19]. An analytic function $f$ is said to be subordinate to an analytic function $g$ (written as $f \prec g$) if and only if there exists an analytic function $\omega$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for } z \in U$$

such that

$$f(z) = g(\omega(z)) \text{ for } z \in U.$$ 

In particular, if $g$ is univalent in $U$, we have the following equivalence

$$f \prec g \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The convolution or Hadamard product of two functions of class $A$ is denoted and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where $f$ has the form (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \text{ for } z \in U.$$

Let us consider the following second-order linear homogeneous differential equation (see for details [3] and [4]):

$$z^2 \omega''(z) + b z \omega'(z) + \left[ d z^2 - v^2 + (1-b) v \right] \omega(z) = 0 \quad (v,b,d \in \mathbb{C}). \quad (1.2)$$

The function $\omega_{v,b,d}(z)$, which is called the generalized Bessel function of the first kind of order $v$, it is defined as a particular solution of (1.2). The function $\omega_{v,b,d}(z)$ has the familiar representation as

$$\omega_{v,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n! \Gamma\left(v + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \quad (1.3)$$

Here $\Gamma$ stands for the Euler gamma function. The series (1.3) permits the study of Bessel, modified Bessel, and spherical Bessel function altogether. It is worth mentioning that, in particular:

i) For $b = d = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order $u$.
defined by
\[ J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v+n+1)} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \] (1.4)

ii) For \( b = 1 \) and \( d = -1 \) in (1.3), we obtain the modified Bessel function of the first kind of order \( v \) defined by
\[ I_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v+n+1)} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \] (1.5)

iii) For \( b = 2 \) and \( d = 1 \) in (1.3), the function \( \omega_{\nu,b,d}(z) \) reduces to \( \sqrt{\frac{2}{\sqrt{\pi}}} S_v(z) \), where \( S_v \) is the spherical Bessel function of the first kind of order \( v \), defined by
\[ S_v(z) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v+n+3/2)} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \] (1.6)

Now, consider the function \( u_{\nu,b,d}(z) : \mathbb{C} \to \mathbb{C} \), defined by the transformation
\[ u_{\nu,b,d}(z) = 2^v \left( v + \frac{b+1}{2} \right)^{-\nu} z^{-\nu} \omega_{\nu,b,d}(\sqrt{z}). \] (1.7)

By using the well-known Pochhammer symbol (or the shifted factorial) \((\lambda)_\mu\) defined, for \( \lambda, \mu \in \mathbb{C} \) and in terms of the Euler \( \Gamma \) function, by
\[ (\lambda)_\mu := \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \]
and \((\lambda)_0 = 1\), we obtain for the function \( u_{\nu,b,d}(z) \) the following representation
\[ u_{\nu,b,d}(z) = \sum_{n \geq k} \frac{(-d)^n}{n!} \left( v + \frac{b+1}{2} \right)_n z^n, \]
where \( k = v + \frac{b+1}{2} \neq 0, -1, -2, \ldots \). This function is analytic on \( \mathbb{C} \) and satisfies the
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second order linear differential equation

\[ 4z^2 u''(z) + 2(2\nu + b + 1)uz'(z) + dz u(z) = 0. \]

Now, we introduce the function \( \varphi_{\nu,b,d}(z) \) defined in terms of generalized Bessel function \( \omega_{\nu,b,d}(z) \), defined by

\[ \varphi_{\nu,b,d}(z) = zu_{\nu,b,d}(z) \]

\[ = 2^\nu \Gamma(\nu + \frac{b + 1}{2}) z^{\frac{b+1}{2}} \omega_{\nu,b,d}(\sqrt{z}) \]

\[ = z + \sum_{n=1}^{\infty} \frac{(-d)^n z^{n+1}}{4^n n!(k)_n}, \text{ where } k = \left( \nu + \frac{b + 1}{2} \right) \]

\[ = g(k,d,z) \]

Motivated by Selvaraj and Karthikeyan [14], we define the following

\[ D^m_\lambda(k,d)f(z) : U \rightarrow U \] by

\[ D_\lambda(k,d)f(z) = f(z) \ast g(k,d,z) \quad (1.8) \]

\[ D^1_\lambda(k,d)f(z) = (1-\lambda)(f(z) \ast g(k,d,z))' + \lambda z (f(z) \ast g(k,d,z))' \quad (1.9) \]

\[ \vdots \]

\[ D^m_\lambda(k,d)f(z) = D^1_\lambda(D^{m-1}_\lambda(k,d)f(z)) \quad (1.10) \]

If \( f \in \mathcal{A} \), then from (1.9) and (1.10) we may easily deduce that

\[ D^m_\lambda(k,d)f(z) = z + \sum_{n=2}^{\infty} \frac{(1+(n-1)\lambda)^m (-d)^{n+1} k^n z^n}{4^{n-1} (n-1)! (k)^{n-1}} \quad (1.11) \]

where \( m \in N_0 = N \cup \{0\} \) and \( \lambda \geq 0 \).

It can be easily verified from definition of (1.11) that

\[ \lambda z \left( D^m_\lambda(k,d)f(z) \right)' = D^{m+1}_\lambda(k,d)f(z) - (1-\lambda) D^m_\lambda(k,d)f(z) \]

(1.12)
In the special cases of the $D^n_k(k,d)f(z)$, we obtain the following operators related to the Bessel function:

i) Choosing $m = 0$ in (1.11) we get the Deniz operator

$$B_k^c = z + \sum_{n=2}^{\infty} \frac{(-d)^{n-1} a_n z^n}{4^{n-1}(n-1)!(k)^{n-1}}.$$ 

ii) Choosing $m = 0$ and $b = d = 1$ in (1.11) we obtain the operator $J_t : A \rightarrow A$ related with Bessel function, defined by

$$J_t f(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} a_n z^n}{4^{n-1}(n-1)!(v+1)^{n-1}}.$$ 

iii) Choosing $m = 0$, $b = 1$ and $d = -1$ in (1.11) we obtain the operator $J_r : A \rightarrow A$ related with modified Bessel function, defined by

$$J_r f(z) = z + \sum_{n=2}^{\infty} \frac{(1)^{n-1} a_n z^n}{4^{n-1}(n-1)!(v+1)^{n-1}}.$$ 

iv) Choosing $m = 0$, $b = 2$ and $d = 1$ in (1.11) we obtain the operator $S_v : A \rightarrow A$ related with spherical Bessel function, defined by

$$S_v f(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} a_n z^n}{4^{n-1}(n-1)!(v+3/2)^{n-1}}.$$ 

**Definition 1.1** A function $f(z) \in A$ is in the class $S^\ast(\alpha, \beta, \gamma)$, $\gamma \in \mathbb{C}\setminus\{0\}$ if and only if

$$\text{Re} \left[ e^{i \alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right) \right]^2 + \beta > \left| \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right|^2.$$ 

(1.13)
Remark 1  The above differential inequality can be equivalently written as

\[ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \leq e^{-i\alpha} (h(z) \cos \alpha + i \sin \alpha) = K_\alpha(z), \]

where \( h(z) = 1 - \frac{\beta}{\cos^2 \alpha} + \frac{2}{\pi} \log \frac{1 + \sqrt{(z + \theta)}}{1 - \sqrt{(z + \theta)}} \), \( \theta = \left( \frac{e^\mu - 1}{e^\mu + 1} \right)^2 \) and \( \mu = \sqrt{\beta \pi} / 2 \cos \alpha \).

Definition 1.2  A function \( f(z) \in A \) is in the class \( Q^m_\alpha(k, d, \alpha, \beta, \gamma) \) if and only if

\[
\text{Re} \left[ e^{ia} \left( 1 + \frac{1}{\gamma} (J(\lambda, m, k, d, z) - 1) \right) \right] > \left( \frac{1}{\gamma} (J(\lambda, m, k, d, z) - 1) \right)^2,
\]

where

\[ J(\lambda, m, k, d, z) = \frac{z(D^{m}_\lambda(k, d)f(z))'}{D^{m}_\lambda(k, d)f(z)}. \]  \hspace{1cm} (1.14)

Since \( K_\alpha(z) = e^{ia} (h(z) \cos \alpha + i \sin \alpha) \) is a convex univalent function, we can write Definition 1.1 in subordination form

\[ f \in Q^m_\alpha(k, d, \alpha, \beta, \gamma) \Leftrightarrow f \in A \text{ and } J(\lambda, m, k, d, z) < K_\alpha(z) \quad (z \in U). \]

Special Cases

For \( \lambda = 0 \) and \( f \in Q^m_\alpha(k, d, \alpha, \beta, \gamma) \) we have \( D^{m}_\gamma(k, d)f(z) \in S^r(\alpha, \beta, \gamma). \)

2  PRELIMINARY RESULTS

Lemma 1  [13] Let \( h \) be a convex univalent function in \( U \) with \( \text{Re}(\lambda h(z) + \mu) > 0 \), where \( \mu \in \mathbb{C}, \lambda \in \mathbb{C} \setminus \{0\}, \), \( z \in U \). If \( p \) is analytic in \( U \) with \( p(0) = h(0) \), then

\[ p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} < h(z). \]
Lemma 2 [12] Let \( d \) be a complex number with \( \text{Re} \, d > 0 \). Suppose that \( \Psi : \mathbb{C}^2 \to \mathbb{C} \) is continuous and satisfies the conditions \( \text{Re} \, \Psi(ix, y) \leq 0 \), when \( x \) is real and
\[
y \leq (-|d - ix|^2)/(2\text{Re} \, d).
\]
If \( p \) is analytic in \( U \) with \( p(0) = d \) and \( \text{Re} \left[ \Psi \left( p(z), zp'(z) \right) \right] > 0 \) for \( z \in U \), then \( \text{Re} \, p(z) > 0 \) in \( U \).

Lemma 3 [18] Let \( f \) and \( g \) be in the class \( kV \) and \( S^* \) respectively. Then, for every function \( F \) analytic in \( U \), we have
\[
\frac{f(z) \ast g(z)F(z)}{f(z) \ast g(z)} \in \overline{\text{co}}[F(U)], \; z \in U,
\]
where \( \overline{\text{co}}[F(U)] \) denotes the closed convex hull of the set \( F(U) \).

Lemma 4 [16] Let the function \( \phi(z) \) given by
\[
\phi(z) = \sum_{n=1}^{\infty} B_n z^n
\]
be convex in \( U \). Suppose also that the function \( h(z) \) given by
\[
h(z) = \sum_{n=1}^{\infty} h_n z^n
\]
is holomorphic in \( U \). If \( h(z) \prec \phi(z), \; z \in U \), then \( |h_n| \leq B_1 \), \( n \in \mathbb{N} = \{1,2,3,\ldots\} \).

3 COEFFICIENT INEQUALITIES

A function \( f(z) \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Let \( \sum \) denote the class of bi-univalent functions defined in the unit disk \( U \).

Definition 3.1 Let \( h : U \to \mathbb{C} \) be a convex univalent function such that \( h(0) = 1 \) and \( h(\overline{z}) = \overline{h(z)} \), for \( z \in U \) and \( \text{Re}(h(z)) > 0 \). A function \( f \in \sum \) is said to be in the class...
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\[ S^{\gamma, \alpha}_{\omega}(\alpha, \gamma, \omega \in \mathbb{C} \setminus \{0\}) \text{ if the following conditions are satisfied:} \]

\[ f \in \sum, \quad e^{i\omega \left( 1 + \frac{1}{\gamma} \left( \frac{z f'(z)}{f(z)} - 1 \right) \right)} \sim h(z) \cos \alpha + i \sin \alpha, \quad z \in U \] (3.1)

and

\[ e^{i\omega \left( 1 + \frac{1}{\gamma} \left( \frac{z g'(\omega)}{g(\omega)} - 1 \right) \right)} \sim h(\omega) \cos \alpha + i \sin \alpha, \quad \omega \in U, \] (3.2)

where

\[ g(\omega) = f^{-1}(\omega) = \omega - a_2 \omega^2 + (2a_2^2 - a_3) \omega^3 - (5a_2^3 - 5a_2 a_3 + a_4) \omega^4 + \cdots, \quad \alpha \in \left( -\frac{\pi}{2}, -\frac{\pi}{2} \right). \]

**Definition 3.2** A function \( f \in \sum \) given by (1.1) is said to be in the class \( S^{\gamma, m}(\alpha, \gamma, k, d, h) \) if the following conditions are satisfied:

\[ e^{i\omega \left( 1 + \frac{1}{\gamma} \left( \frac{z D^m_{\alpha}(k,d) f(z)}{D^m_{\alpha}(k,d) f(z)} - 1 \right) \right)} \sim h(z) \cos \alpha + i \sin \alpha, \quad z \in U \] (3.3)

and

\[ e^{i\omega \left( 1 + \frac{1}{\gamma} \left( \frac{z D^m_{\alpha}(k,d) g(\omega)}{D^m_{\alpha}(k,d) g(\omega)} - 1 \right) \right)} \sim h(\omega) \cos \alpha + i \sin \alpha, \quad \omega \in U. \] (3.4)

where \( g(\omega) = f^{-1}(\omega), \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, z, \omega \in U \) on specializing the parameter \( \lambda \).

**Theorem 1** Let \( f \) given by (1.1) be in the class \( \sum S^{\gamma, m}(\alpha, \gamma), \) then

\[ |a_2| \leq \sqrt{\frac{\gamma}{\prod_{i} \parallel B_i \parallel}} \cos \alpha \]

and

\[ |a_1| \leq 4 \parallel \gamma \parallel \prod_{i} \parallel B_i \parallel \cos \alpha. \]
Theorem 2. Let \( f \) given by (1.1) be in the class \( S^\gamma_{\alpha,m}(\alpha, \gamma, k, d, h) \), then

\[
|a_2| \leq \frac{2 |\gamma| B_1 |\cos \alpha|}{(1+2\gamma)^{m} d^2} \left( \frac{1}{8(k)^2} - \frac{(1+\gamma)^{2m} d^2}{8(k)^2} \right)
\]

and

\[
|a_3| \leq \frac{2 |\gamma|^2 B_1 |\cos \alpha|}{(1+2\gamma)^{m} d^2} \left( \frac{1}{8(k)^2} - \frac{(1+\gamma)^{2m} d^2}{8(k)^2} \right).
\]

Proof. It follows from (3.3) and (3.4) that

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( z \left( D^m_{\alpha,k} (k,d) f(z) \right)' \right) - 1 \right) = p(z) \cos \alpha + i \sin \alpha, \quad z \in U
\]

and

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \omega \left( D^m_{\alpha,k} (k,d) g(\omega) \right)' \right) - 1 \right) = q(\omega) \cos \alpha + i \sin \alpha, \quad \omega \in U,
\]

where \( p(z) \sim h(z) \) and \( q(\omega) \sim h(\omega) \) have the forms

\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots
\]

and

\[
q(\omega) = 1 + q_1 \omega + q_2 \omega^2 + \cdots
\]

respectively. It follows from (3.5) and (3.6) that

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{z \left( D^m_{\alpha,k} (k,d) f(z) \right)'}{\gamma} \right) - 1 \right) = p(z) \cos \alpha
\]

(3.7)

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{\omega \left( D^m_{\alpha,k} (k,d) g(\omega) \right)'}{\gamma} \right) - 1 \right) = q(\omega) \cos \alpha
\]

(3.8)

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{(2)(1+2\gamma)^{m} d^2}{4(2)! (k)^2} a_3 - \left( \frac{(1+\gamma)^{m} (-d)}{4(1)! (k)^2} \right) a_2^2 \right) \right) = p_2 \cos \alpha
\]

(3.9)

\[
e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{(1+\lambda)^m (-d)}{4(1)! (k)^2} a_2 \right) \right) = q_1 \cos \alpha
\]
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and

\[ \frac{e^{ia}}{\gamma} \left( \frac{(4+2\lambda)^m d^2}{4\gamma^2(2)!k_2} a_2^2 - \frac{(2+2\lambda)^m d^2}{4\gamma^2(2)!k_2} a_3 \right) = q_2 \cos \alpha. \quad (3.10) \]

From (3.7) and (3.9), we obtain

\[ p_1 = -q_1. \]

Adding (3.8) and (3.10), we get

\[ \frac{e^{ia}}{\gamma} \left( \frac{(1+2\lambda)^m d^2}{8(k_2) - \frac{(1+\lambda)^2m d^2}{8(k_2)} a_1 \right) = (p_2 + q_2) \cos \alpha. \quad (3.11) \]

Since \( p, q \in h(U) \), applying Lemma 4, we have

\[ |p_m| = \left| \frac{p^{(m)}(0)}{m!} \right| \leq |B_1|, \quad m \in N \quad (3.12) \]

\[ |q_m| = \left| \frac{q^{(m)}(0)}{m!} \right| \leq |B_1|, \quad m \in N. \quad (3.13) \]

Applying (3.12), (3.13) and Lemma 4 for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we get

\[ |a_2| \leq \frac{2 |B_1| \cos \alpha}{(1+2\lambda)^m d^2} \frac{8(k_2)}{8(k_2)^2} \quad (3.14) \]

Subtracting (3.10) from (3.8), we get

\[ \frac{e^{ia}}{\gamma} \left( \frac{(1+2\lambda)^m d^2}{8(k_2) - \frac{(1+2\lambda)^m d^2}{8(k_2)} a_3 \right) = (p_2 - q_2) \cos \alpha \quad (3.15) \]

or, equivalently

\[ a_3 = \left( \frac{\gamma}{e^{ia}} \right) \left( \frac{(p_2 - q_2) \cos \alpha}{8(k_2)^2} \right) + \left( \frac{\gamma}{e^{ia}} \right)^2 \left( \frac{(p_2 + q_2) \cos \alpha}{8(k_2)^2} \right) \]

\[ \left( \frac{(1+2\lambda)^m d^2}{8(k_2)^2} \right) \left( \frac{(1+\lambda)^2m d^2}{8(k_2)^2} \right) . \]
Applying (3.12), (3.13) and Lemma 4 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we get

\[
|a_1| \leq \frac{2 |\gamma|^2 |B_1| \cos \alpha}{(1 + 2\lambda)^m d^2} - \frac{(1 + \lambda)^{2m} d^2}{8(k)_2}.
\]

This complete the proof of Theorem 2.

**Corollary 1** Let \( f \in A \) be bi-convex function of order \( \beta \) then \( |a_2| \leq \sqrt{1 - \beta} \) and \( |a_1| \leq 1 - \beta \).

**Corollary 2** Let \( f \in A \) satisfy the condition \( 1 + \frac{zf''}{f'}(z) < h(z) \) and \( 1 + \frac{zg''}{g'}(w) < h(w) \) then \( |a_2| \leq \frac{B_1 \sqrt{B_i}}{\sqrt{2 |B_i|^2 + 2B_1 - 2B_2}} \) and \( |a_3| \leq \frac{1}{2} (B_i + |B_2 - B_1|) \).

**Corollary 3** Let \( f \) be given by (1.1) and \( g = f^{-1} \). If \( f \) and \( g \) satisfies the condition

\[
(1 - \lambda) \frac{zf'}{f(z)} + \lambda \left( 1 + \frac{zf''}{f'(z)} \right) < h(z)
\]

and \( (1 - \lambda) \frac{wg'}{g(w)} + \lambda \left( 1 + \frac{wg''}{g'(w)} \right) < g(w) \),

then

\[
|a_2| \leq \frac{B_1 \sqrt{B_i}}{\sqrt{(1 + \lambda) |B_i|^2 + (1 + \lambda)(B_1 - B_2)|}} \quad (3.16)
\]

and

\[
|a_3| \leq \frac{B_1 + |B_2 - B_1|}{1 + \lambda} \quad (3.17)
\]
Remark 2 Put $\lambda = 1$ in corollary 3, we get the result $|a_2| \leq \frac{B_1 \sqrt{B_1}}{2 |B_1^2 + 2B_1 - 2B_2|}$ and $|a_1| \leq \frac{1}{2} (B_1 + |B_2 - B_1|)$ of corollary 2.

4 CLOSURE PROPERTY

Theorem 3 Let

$$F(\lambda, m, k, d, f(z)) = D^m_\lambda (k, d) f(z).$$

Then $f \in Q^w_\lambda (k, d, \alpha, \beta, \gamma)$ if and only if $F(\lambda, m, k, d, f(z)) \in S^\prime (\alpha, \beta, \gamma)$.

Proof. Let $F(\lambda, m, k, d, f(z)) \in S^\prime (\alpha, \beta, \gamma)$, then

$$\operatorname{Re} \left[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{zF^\prime(\lambda, m, k, d, f(z))(z)}{F(\lambda, m, k, d, f(z))} - 1 \right) \right) \right]^2 + \beta > \left( \frac{zF^\prime(\lambda, m, k, d, f(z))(z)}{F(\lambda, m, k, d, f(z))} - 1 \right)^2.$$  \hfill (4.2)

Thus (4.1) together with (4.2) implies

$$\operatorname{Re} \left[ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( J(\lambda, m, k, d, z) - 1 \right) \right) \right]^2 + \beta > \left( \frac{1}{\gamma} \left( J(\lambda, m, k, d, z) - 1 \right) \right)^2,$$

where $J(\lambda, m, k, d, z)$ is given by (1.14). Therefore $f(z) \in Q^w_\lambda (k, d, \alpha, \beta, \gamma)$.

Converse is immediate.

Theorem 4 For $m \geq 1$, $Q^{w+}_\lambda (k, d, \alpha, \beta, \gamma) \subseteq Q^w_\lambda (k, d, \alpha, \beta, \gamma)$.

Proof. Let $f \in Q^{w+}_\lambda (k, d, \alpha, \beta, \gamma)$ and

$$z \left( D^m_\lambda (k, d) f(z) \right)' \right) = p(z),$$

where $p$ is analytic in $U$ and $p(0) = 1$.  \hfill (4.3)
From (1.12) and (4.3) and after some simplification, we obtain

\[ \frac{D_{\lambda}^{m+1}(k,d)f(z)}{D_{\lambda}^m(k,d)f(z)} = \lambda p(z) + (1 - \lambda). \quad (4.4) \]

By logarithmic differentiation of (4.4), we have

\[ \frac{z \left( D_{\lambda}^{m+1}(k,d)f(z) \right)'}{D_{\lambda}^{m+1}(k,d)f(z)} = p(z) + \frac{\lambda z p'(z)}{(1-\lambda) + \lambda p(z)}. \quad (4.5) \]

Since \( f \in Q_{\lambda}^m(k,d,\alpha,\beta,\gamma) \), so

\[ p(z) + \frac{\lambda z p'(z)}{(1-\lambda) + \lambda p(z)} \prec K_{\alpha}(z). \]

Thus by using Lemma 1, \( p(z) \prec K_{\alpha}(z) \) and hence \( f \in Q_{\lambda}^m(k,d,\alpha,\beta,\gamma) \).

Let us consider the Bernardi integral operator \( F_\mu \) given by

\[ F_\mu f(z) = \frac{\mu+1}{z^\mu} \int_0^1 t^{\mu-1} f(t)dt. \quad (4.6) \]

For \( \mu \) with \( \text{Re} \, \mu > -1 \) the operator has the property \( F_\mu : \mathbb{A} \rightarrow \mathbb{A} \), (see, for instance, [10], p.11).

**Theorem 5** Let \( \mu > -1 \) and \( f \in Q_{\lambda}^m(k,d,\alpha,\beta,\gamma) \), then \( F_\mu f \in Q_{\lambda}^m(k,d,\alpha,\beta,\gamma) \).

**Proof.** Let \( f \in Q_{\lambda}^m(k,d,\alpha,\beta,\gamma) \) and

\[ \frac{z \left( D_{\lambda}^m(k,d)F_\mu f(z) \right)'}{D_{\lambda}^m(k,d)F_\mu f(z)} = p(z), \quad (4.7) \]

where the function \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). From (4.6), we have

\[ z(F_\mu f)'(z) + \mu F_\mu f(z) = (\mu+1)f(z) \]

and so

\[ z \left( D_{\lambda}^m(k,d)F_\mu f(z) \right)' = (\mu+1)(D_{\lambda}^m(k,d)f(z) - \mu D_{\lambda}^m(k,d)F_\mu f(z)). \quad (4.8) \]
Bessel Function with Linear Differential Operator

Then, by using equations (4.7) and (4.8), we obtain

\[
\frac{(\mu+1)(D^m_{\lambda}(k,d)f(z))}{D^m_{\lambda}(k,d)F_{\mu}(f)(z)} = p(z) + \mu \tag{4.9}
\]

By logarithmic differentiation of (4.9), we have

\[
\frac{z\left(D^m_{\lambda}(k,d)f(z)\right)'}{D^m_{\lambda}(k,d)f(z)} = p(z) + \frac{zp'}{p(z)+\mu}, \quad (z \in \mu).
\]

Hence by Lemma 1, we conclude that \( p(z) \prec K_{\lambda}(z) \) in \( U \) which implies that \( F_{\mu}f \in Q^m_{\lambda}(k,d,\alpha,\beta,\gamma) \).

**Theorem 6** Let \( f \in Q^m_{\lambda}(k,d,\alpha,\beta,\gamma) \) and \( \psi \in kV \). If \( 0 < \gamma \leq 1 \), then \( \psi \ast f \in Q^m_{\lambda}(k,d,\alpha,\beta,\gamma) \).

**Proof.** Let \( F = \psi \ast f \). If \( f \in Q^m_{\lambda}(k,d,\alpha,\beta,\gamma) \) then the condition (4.3) is satisfied with \( p \prec K_{\lambda}(z) \). Using the usual convolution properties and (4.3),

\[
p(z) = \frac{z\left(D^m_{\lambda}(k,d)F(z)\right)'}{D^m_{\lambda}(k,d)F(z)} = \frac{z\left(D^m_{\lambda}(k,d)(\psi \ast f)\right)'}{D^m_{\lambda}(k,d)(\psi \ast f)} = \frac{\psi \ast D^m_{\lambda}(k,d)f}{\psi \ast D^m_{\lambda}(k,d)f} = \frac{\psi \ast z\left(D^m_{\lambda}(k,d)f\right)'}{\psi \ast D^m_{\lambda}(k,d)f} = \frac{\psi \ast (P(z)F(z))}{\psi \ast F(z)}. \tag{4.10}
\]
By Theorem(3), the function \( F(z) = D_n^m(k,d)f(z) \in S'(\alpha, \beta, \gamma) \).

Hence, by Lemma (3), we have

\[
\frac{z \left( D_n^m(k,d)F(z) \right)'}{D_n^m(k,d)F(z)} \in \overline{\text{co}} [F(U)] \subseteq K_\alpha(z).
\]

Since \( K_\alpha(z) \) is a convex univalent and \( p(z) \prec K_\alpha(z) \).

Hence \( F = \psi \ast f \in Q_n^m(k,d,\alpha,\beta,\gamma) \).

REFERENCES


