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On the Applications of Laplace - Aboodh Transforms in Engineering field and Comparison with Sumudu Transform

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Abstract

The current article introduced Double Aboodh transform, Triple Aboodh transform and their properties. The relation between Double Laplace and Double Aboodh transform, Double Sumudu and Double Aboodh transform, the comparison between Laplace transform, Aboodh transform and Sumudu transform given in this paper. Using Double Aboodh transform and Triple Aboodh transform solution of partial differential equations are given in this article. The results related to Aboodh transform are proved and Applications of Aboodh transform is shown to solve engineering problems.

Keywords: Aboodh Transform, Double Aboodh transform, Triple Aboodh transform, Laplace-Aboodh Duality, Kamal transform**.**

Mathematics Subject Classification: 35A22, 35G15, 44A35

1. INTRODUCTION

In an application of engineering, the partial differentiation plays a crucial role. To get to the bottom of these differential equations Integral transforms are used. The solutions of initial and boundary value problems are given by several Integral transforms methods [1-2]. The concept of Laplace transform is widely used in engineering. In Engineering Mechanics, Electrical Engineering the linear and partial differential equations and there solutions were solved by Laplace transform. Recently Double Laplace transform [3] is used to solve partial differential equations and integral equations. Karaballi, Eltayeb, Kalicman work on Sumudu transform developed by Watugula [4]. Kalicman [6-7] compare Sumudu and Laplace transform and solve problems in engineering. A. Babakhani work on double Laplace-Carson transforms and proved theorems for solving partial differential equations. Like of

Laplace –Carson transform Aboodh introduce Laplace- Aboodh transform and used it to solve ordinary and partial differential equations [14-15].

The main objective of this article is to compare Laplace, Aboodh and Sumudu transform. Also to introduce concept of Double and Triple Aboodh transform with their properties. In the first part the definition and duality with other transforms is given and in second part theorems with proof, applications of Aboodh transform are shown. This paper will provide a good platform for those who want to solve partial differential equations, engineering problems using Integral transforms. The Laplace-Aboodh transform is modification in Laplace transform and Aboodh gave it name Aboodh transform, so we will take the same name Aboodh transform throughout in this paper.

The Aboodh transform defined for $t \geq 0$the function $f(t), t > 0$ is,

Let f(t) be an exponential order function in the set A as

$$A = \{f: |f(t)| < Me^{|t|\alpha_j}, t \in (-1)^j \times [0, \infty), j = 1,2; M, \alpha_1, \alpha_2 > 0\}$$

Any function in the set A, the constant M is finite number and α_1, α_2 are finite or may be infinite numbers. Then the Aboodh transform is,

$$A[f(t)] = \frac{1}{p}\int_0^\infty e^{-pt} f(t)dt \ = G(p), \alpha_1 \leq p \leq \alpha_2 \tag{1}$$

And Inverse Aboodh transform is,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} pG(p)dp \ ; \gamma \geq 0 \tag{2}$$

Theorem 1.:- Let $f(t) \in A$ and the $A[f(t)] = G(p)$ is Aboodh transform such that $pG(p)$ is a meromorphic function, with singularities having $Re(p) < \gamma$ and there exists a circular region Γ_R with radius R and positive constants $M$and n with $pG(p) < \frac{M}{R^n}$ then the function $f(t)$ is given by,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} pG(p)dp = \sum residues \ of \ (e^{pt} pG(p)) \tag{3}$$

Proof of this theorem will be obtained using Aboodh-Laplace duality [16] and theorems on Laplace transform [2].

The properties of Aboodh transform are given below,

Let $A[f(t)] = G(p)$

1. $A\{af_1(t) + bf_2(t)\} = aA[f_1(t)] + bA[f_2(t)]$
2. $A[e^{at} f(t)] = \left(\frac{p-a}{p}\right) G(p - a)$
3. $A[tf(t)] = \left(-\frac{d}{dp} - \frac{1}{p}\right) G(p)$

4. $A\left[\int_0^t f(t)dt\right] = \frac{1}{p^2}G(p)$

5. $Let\ f_1(t) = \begin{cases} f(t-a), & t > a \\ 0 & t < a \end{cases}\ then\ A[f_1(t)] = e^{-ap}G(p)$

6. $A[f'(t)] = pG(p) - \frac{f(0)}{p}$

7. $A[f''(t)] = p^2G(p) - \frac{f'(0)}{p} - f(0)$

2. DOUBLE ABOODH TRANSFORM

Double Aboodh transform is defined for the continuous and an exponential order functions.

Let $f(x,y)$ be function of two variables defined in the first quadrant of the XOY plane. Consider the set

$C = \{f: |f(x,y)| < Me^{|x|\alpha_j + |y|\beta_j}, x, y \in (-1)^j \times [0,\infty), j = 1,2; M, \alpha_j, \beta_j > 0\}$ be the set of continuous and exponential order functions, Let f(x, y) be any function in the set C, the constant M is finite number and α_j, β_j are finite or may be infinite numbers. Then the Double Aboodh transform is,

$$A_2[f(x,y)] = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)dxdy \ = G(p,q), \alpha_1 < p < \alpha_2, \beta_1 < q < \beta_2 \quad (4)$$

And Inverse Double Aboodh transform is,

$$f(x,y) = \frac{1}{(-4\pi^2)}\int_{\gamma_1 - i\infty}^{\gamma_2 + i\infty}\int_{\gamma_3 - i\infty}^{\gamma_4 + i\infty} e^{px+qy}\ \ pqG(p,q)dp\,dq\ ;\gamma_j \geq 0 \quad (5)$$

Provided the integral exists and it is absolutely convergent in the given interval.

On the similar way one can defined Triple Aboodh transform as,

$$A_3[f(x,y,z)] = \frac{1}{pqs}\int_0^\infty \int_0^\infty e^{-(px+qy+sz)} f(x,y,z)dxdydz \ = G(p,q,s),$$

$\alpha_1 < p < \alpha_2, \beta_1 < q < \beta_2, \eta_1 < p < \eta_2$

And Inverse Double Aboodh transform is,

$$f(x,y,z) = \frac{1}{(-8\pi^3 i)}\int_{\gamma_1 - i\infty}^{\gamma_2 + i\infty}\int_{\gamma_3 - i\infty}^{\gamma_4 + i\infty}\int_{\gamma_5 - i\infty}^{\gamma_6 - i\infty} e^{px+qy}\ \ pqsG(p,q,s)dp\,dqds\ ;\gamma_j \geq 0$$

Provided the integral exists and it is absolutely convergent in the given interval.

Aboodh-Laplace Duality:

Theorem 2:- If $A_2[f(x,y)] = G(p,q)$ double Aboodh transform and $L_2[f(x,y)] =$

$\overline{F}(p,q)$ be double Laplace transform then

$$G(p,q) = \frac{1}{pq}\overline{F}(p,q) \text{ and } \overline{F}(p,q) = pqG(p,q) \tag{6}$$

Proof:- Consider $G(p,q) = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)dxdy = \frac{1}{pq}\overline{F}(p,q)$

Aboodh-Sumudu Duality:

Theorem 3:- If $A_2[f(x,y)] = G(p,q)$ double Aboodh transform and $S_2[f(x,y)] = SF(p,q)$ double Sumudu transform then

$$SF(p,q) = \frac{1}{p^2q^2}\ G\left(\frac{1}{p},\frac{1}{q}\right) \text{ and } G(p,q) = \frac{1}{p^2q^2}\ SF\left(\frac{1}{p},\frac{1}{q}\right) \tag{7}$$

Proof:- Consider $G(p,q) = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)dxdy$

$$= \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-(\frac{x}{1/p}+\frac{y}{1/q})} f(x,y)dxdy = \frac{1}{p^2q^2}\ SF\left(\frac{1}{p},\frac{1}{q}\right)$$

Theorem 4:- Let $f(x,y) \in C$ and it is absolutely integrable function having first order partial derivative and mixed second order partial derivative $\frac{\partial f}{\partial x \partial y}(x,y) \in C$ such that $0 < x, y < \infty$ and if Double Aboodh transform $A_2[f(x,y)] = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)dxdy\ = G(p,q), \alpha_1 < p < \alpha_2, \beta_1 < q < \beta_2$ is absolutely convergent then

$$f(x,y) = \frac{1}{(-4\pi^2)}\int_{\gamma_1 - i\infty}^{\gamma_2 + i\infty}\int_{\gamma_3 - i\infty}^{\gamma_4 + i\infty} e^{px+qy}\ \ pqG(p,q)dp\,dq\ \ ;\gamma_j \geq 0$$

This result directly follows from the results of Double Laplace transform and Aboodh- Laplace duality.

Theorem 5:- Let $A_2[f(x,y)] = G(p,q)$, $A_3[f(x,y,z)] = G(p,q,s)$ then

$$A_2[af_1(x,y) + bf_2(x,y)] = aG_1(p,q) + bG_2(p,q).$$

$$A_3[af_1(x,y,z) + bf_2(x,y,z)] = aG_1(p,q,s) + bG_2(p,q,s)$$

Proof: Consider, $A_2[af_1(x,y) + bf_2(x,y)]$

$$= \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-px-qy}[af_1(x,y) + bf_2(x,y)]dxdy$$

$$= \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-px-qy} af_1(x,y)dxdy + \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-px-qy} bf_2(x,y)dxdy$$

$$= aG_1(p,q) + bG_2(p,q) \tag{8}$$

Theorem 6:- Let $A[f(x)] = G(p)$ and the Heaviside's unit step function

$H(x-y) = \begin{cases} 1 & ; if\ x \geq y \\ 0 & ; if\ x < y \end{cases}$ then

1. $A_2[f(x)H(x-y)] = \frac{1}{q^2}\left[G(p) - \frac{p+q}{p}G(p+q)\right]$

2. $A_2[f(x)H(y-x)] = \frac{p+q}{pq^2}[G(p+q)]$

3. $A_2[f(x)H(x+y)] = \frac{p-q}{pq^2}[G(p+q)]$

4. $A_2[H(x-y)] = \frac{1}{qp^2(p+q)}$

Proof: 1. Consider, $A_2[f(x)H(x-y)] = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-px-qy} f(x)H(x-y)dxdy$

$$= \frac{1}{pq}\int_0^\infty \int_y^\infty e^{-px-qy} f(x)dxdy$$

By changing the order of integration,

$$A_2[f(x)H(x-y)] = \frac{1}{pq}\int_0^\infty \int_0^x e^{-px-qy} f(x)dxdy$$

$$A_2[f(x)H(x-y)] = \frac{1}{pq^2}\int_0^\infty [1 - e^{-qx}]e^{-px} f(x)dx$$

$$A_2[f(x)H(x-y)] = \frac{1}{q^2}\left[G(p) - \frac{p+q}{p}G(p+q)\right]$$

2. Consider, $A_2[f(x)H(y-x)] = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-px-qy} f(x)H(y-x)dxdy$

$$= \frac{1}{pq}\int_0^\infty \int_x^\infty e^{-px-qy} f(x)dxdy$$

$$A_2[f(x)H(x-y)] = \frac{1}{pq^2}\int_0^\infty [e^{-qx}]e^{-px} f(x)dx$$

$$A_2[f(x)H(x-y)] = \frac{p+q}{pq^2}G(p+q)$$

3. Consider, $A_2[f(x)H(x+y)] = \frac{1}{pq}\int_0^\infty \int_0^\infty e^{-px-qy} f(x)H(x+y)dxdy$

$$= \frac{1}{pq}\int_0^\infty \int_{-y}^\infty e^{-px-qy} f(x)dxdy$$

By changing the order of integration,

$$A_2[f(x)H(x-y)] = \frac{1}{pq}\int_0^\infty \int_{-x}^\infty e^{-px-qy} f(x)dxdy$$

$A_2[f(x)H(x-y)] = \frac{1}{pq^2}\int_0^\infty [e^{qx}]e^{-px}f(x)dx$

$A_2[f(x)H(x-y)] = \frac{p-q}{pq^2}G(p-q)$

4. Consider, $A_2[H(x-y)] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}H(x-y)dxdy$

$$= \frac{1}{pq}\int_0^\infty\int_y^\infty e^{-px-qy}dxdy$$

$$A_2[f(x)H(x-y)] = \frac{1}{p^2q}\frac{1}{(p+q)}$$

Theorem 7:- Let $A_2[f(x,y)] = G(p,q)$ and $A_3[f(x,y,z)] = G(p,q,s)$ the Heaviside's unit step function for two variable and three variables are defined as,

$$H(x-a,y-b) = \begin{cases} 1 & ;if\ x > a, y > b \\ 0 & ;if\ x < a\ \&/or\ \ y < b \end{cases}$$

$$H(x-a,y-b,z-c) = \begin{cases} 1 & ;if\ x > a, y > b, z > c \\ 0 & ;otherwise \end{cases}$$

then for any real constants $a, b, c > 0$,

$$A_2[f(x-a,y-b)H(x-a,y-b)] = e^{-ap-bq}G(p,q). \tag{9}$$

$$A_3[f(x-a,y-b,z-c)H(x-a,y-b,z-c)] = e^{-ap-bq-cs}G(p,q,s) \tag{10}$$

Proof: Consider, $A_2[f(x-a,y-b)H(x-a,y-b)]$

$$= \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}f(x-a,y-b)H(x-a,y-b)dxdy$$

$$= \frac{1}{pq}\int_a^\infty\int_b^\infty e^{-px-qy}f(x-a,y-b)dxdy$$

By using the substitution $x-a=u, y-b=v$ then

$$A_2[f(x-a,y-b)H(x-a,y-b)]$$

$$= \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-p(u+a)-q(v+b)}f(u,v)dudv$$

$$= e^{-ap-bq}\ G(p,q)$$

Similar calculation can be done for second proof.

Theorem 8:- Let $A_2[f_1(x,y)] = G_1(p,q)$ and $A_2[f_2(x,y)] = G_2(p,q)$ are absolutely convergent then

$$A_2[f_1(x,y) ** f_2(x,y)] = pqG_1(p,q)G_2(p,q). \tag{11}$$

Where $f_1(x,y) ** f_2(x,y) = \int_0^x\int_0^y f_1(x-a,y-b)f_2(a,b)dadb$ is called convolution product of $f_1(x,y)$ and $f_2(x,y)$.

Proof: Consider, $A_2[f_1(x,y) ** f_2(x,y)]$

$= \frac{1}{pq}\int_0^\infty\int_0^\infty\int_0^x\int_0^y e^{-px-qy}[f_1(x-a,y-b)f_2(a,b)dadb]dxdy$

By changing the order of integration in the above integral,

$= \frac{1}{pq}\int_0^\infty\int_0^\infty f_2(a,b)\int_a^\infty\int_b^\infty e^{-px-qy}[f_1(x-a,y-b)dxdy]dadb$

$= \frac{1}{pq}\int_0^\infty\int_0^\infty f_2(a,b)\int_0^\infty\int_0^\infty e^{-px-qy}[f_1(x-a,y-b)H(x-a,y-b)dxdy]dadb$

$= F_1(p,q)F_2(p,q)$ Using equation (9) get the result.

Same way one can do it for Triple Aboodh transform as,

Let $A_3[f_1(x,y,z)] = G_1(p,q,s)$ and $A_3[f_2(x,y,z)] = G_2(p,q,s)$ are absolutely convergent then $A_2[f_1(x,y,z) *** f_2(x,y,z)] = pqsG_1(p,q,s)G_2(p,q,s)$.

Where $f_1(x,y,z) *** f_2(x,y,z) = \int_0^x\int_0^y\int_0^z f_1(x-a,y-b,z-c)f_2(a,b,c)dadbdc$ is called convolution product of $f_1(x,y,z)$ and $f_2(x,y,z)$.

Theorem 9:- Let $A[f(x)] = G(p)$ and $A[f(y)] = G(q)$ and $f(x+y) \in C$ then

$$A_2[f(x+y)] = \frac{1}{pq(p-q)}[qG(q) - pG(p)]$$

Proof: Consider, $A_2[f(x+y)] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}f(x+y)dxdy$

Put $x+y=u$ and changing the order of integration,

$$A_2[f(x+y)] = \frac{1}{pq}\int_0^\infty\int_0^u e^{-x(p-q)}e^{-qu}f(u)dxdu$$

$$A_2[f(x+y)] = \frac{1}{pq(p-q)}[qG(q) - pG(p)]$$

Examples:

a) If $f(x,y) = 1$ for $x,y > 0$ then

$A_2[1] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-(px+qy)}dxdy = \frac{1}{p^2q^2}$ (12)

b) If $f(x,y) = e^{(ax+by)}$ for all x,y then

$A_2\left[e^{(ax+by)}\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-(px+qy)}e^{(ax+by)}dxdy = \frac{1}{pq(p-a)(q-b)}$ (13)

c) If $f(x,y) = cos(ax+by)$ then

$A_2[cos(ax+by)] = \frac{pq-ab}{pq(p^2+a^2)(q^2+b^2)}$ (14)

d) $A_2[sin(ax+by)] = \frac{pb+qa}{pq(p^2+a^2)(q^2+b^2)}$ (15)

e) $A_2[sinh(ax+by)] = \frac{pb+qa}{pq(p^2-a^2)(q^2-b^2)}$ (16)

f) $A_2[cosh(ax+by)] = \frac{pq-ab}{pq(p^2-a^2)(q^2-b^2)}$ (17)

g) $A_2[(xy)^n] = \begin{cases} \frac{(n!)^2}{(pq)^{n+2}} & , n \text{ is integer} \\ \frac{(\gamma(n+1))^2}{(pq)^{n+2}} & , n \text{ is not an integer} \end{cases}$ (18)

Theorem 10:- Let $f(x,y)$ then for any pair of real constants $a, b > 0$

$$A_2\left[e^{(-ax-by)}f(x,y)\right] = \frac{(p-a)(q-b)}{pq}G(p-a, q-b) \quad (19)$$

Proof:- Consider $A_2\left[e^{(-ax-by)}f(x,y)\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-(px+qy)}e^{(-ax-by)}f(x,y)dxdy$

$= \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-x(p+a)}e^{-y(q+b)}f(x,y)dxdy = \frac{(p-a)(q-b)}{pq}G(p-a, q-b)$

Theorem 11:- Let $A[f(x)] = G(p)$ and $A[g(y)] = G(q)$ then $A_2[f(x)] = \frac{1}{pq^2}G(p)$ and

$$A_2[g(y)] = \frac{1}{qp^2}G(q).$$

Proof: Consider, $A_2[f(x)] = \frac{1}{pq}\int_0^\infty e^{-qx}\int_0^\infty e^{-px}f(x)dxdy = \frac{1}{pq^2}G(p)$

Similar way for the proof of $A_2[g(y)] = \frac{1}{qp^2}G(q)$

Theorem 12:- Let $A[f(x)] = G(p)$ and $A[g(y)] = G(q)$ then

$$A_2[f(ax)g(by)] = \frac{1}{a^2b^2}G(p/a)G(q/b).$$

Proof: Consider, $A_2[f(ax)g(by)] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}f(ax)g(by)dxdy$

$$= \frac{1}{pq}\int_0^\infty e^{-px}f(ax)dx\int_0^\infty e^{-qy}g(by)dy$$

Using substitution $ax = u, by = v$ in the integration get the required proof

$$A_2[f(ax)g(by)] = \frac{1}{a^2b^2}G(p/a)G(q/b)$$

Theorem 13:- Let f(x, y) be periodic function in C then

$$A_2[f(x,y)] = \frac{1}{pq(1-e^{-pT_1-qT_2})}\int_0^{T_1}\int_0^{T_2} e^{-px-qy} f(x,y)dxdy. \qquad (20)$$

$$A_3[f(x,y,z)] = \frac{1}{pqs(1-e^{-pT_1-qT_2-sT_3})}\int_0^{T_1}\int_0^{T_2}\int_0^{T_3} e^{-px-qy} f(x,y)dxdy$$

Proof will obvious from Duality of Double Aboodh-Laplace transform.

Theorem 14:- If $A_2[f(x,y)] = G(p,q)$and $A_3[f(x,y,z)] = G(p,q,s)$ then

1. $A_2[xf(x,y)] = -\left(\frac{\partial}{\partial p} + \frac{1}{p}\right) G(p,q)$,

$$A_2[yf(x,y)] = -\left(\frac{\partial}{\partial q} + \frac{1}{q}\right) G(p,q) \qquad (21)$$

2. $A_3[xf(x,y,z)] = -\left(\frac{\partial}{\partial p} + \frac{1}{p}\right) G(p,q,s)$,

$$A_3[yf(x,y,z)] = -\left(\frac{\partial}{\partial q} + \frac{1}{q}\right) G(p,q,s),$$

$$A_3[zf(x,y,z)] = -\left(\frac{\partial}{\partial s} + \frac{1}{s}\right) G(p,q,s) \qquad (22)$$

3. $A_2[xy\, f(x,y)] = \left(\frac{\partial^2}{\partial p \partial q} + \frac{1}{p}\frac{\partial}{\partial q} + \frac{1}{q}\frac{\partial}{\partial p} + \frac{1}{pq}\right) G(p,q)$ (23)

4. $A_2[x^2\, f(x,y)] = \left[\frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p}\right] G(p,q)$ (24)

5. $A_2[y^2\, f(x,y)] = \left[\frac{\partial^2}{\partial q^2} + \frac{1}{q}\frac{\partial}{\partial q}\right] G(p,q)$ (25)

Proof: 1. Consider, $\frac{\partial}{\partial p} G(p,q) = \int_0^\infty\int_0^\infty \frac{\partial}{\partial p}\left(\frac{1}{pq} e^{-px-qy} f(x,y)\right) dxdy$

$$= \int_0^\infty\int_0^\infty \left(\frac{-1}{pq} e^{-px-qy} x - \frac{1}{p^2 q} e^{-px-qy}\right) f(x,y)dxdy$$

$$\therefore A_2[xf(x,y)] = -\left(\frac{\partial}{\partial p} + \frac{1}{p}\right) G(p,q)$$

and $\frac{\partial G}{\partial q}(p,q) = \int_0^\infty\int_0^\infty \frac{\partial}{\partial q}\left(\frac{1}{pq} e^{-px-qy} f(x,y)\right) dxdy$

$$= \int_0^\infty\int_0^\infty \left(\frac{-1}{pq} e^{-px-qy} y - \frac{1}{q^2 p} e^{-px-qy}\right) f(x,y)dxdy$$

$$\therefore A_2[yf(x,y)] = -\left(\frac{\partial}{\partial q} + \frac{1}{q}\right) G(p,q)$$

The proof of second is similar to 1.

3. Consider, $A_2[xyf(x,y)] = \left(\frac{\partial}{\partial p}+\frac{1}{p}\right)\left(\frac{\partial}{\partial q}+\frac{1}{q}\right)G(p,q)$

$$= \left(\frac{\partial^2}{\partial p \partial q}+\frac{1}{p}\frac{\partial}{\partial q}+\frac{1}{q}\frac{\partial}{\partial p}+\frac{1}{pq}\right)G(p,q)$$

4. Consider, $A_2[x^2 f(x,y)] = \left(\frac{\partial}{\partial p}+\frac{1}{p}\right)\left(\frac{\partial}{\partial p}+\frac{1}{p}\right)G(p,q)$

$$= \left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}\right)G(p,q)$$

5. Consider, $A_2[y^2\, f(x,y)] = \left(\frac{\partial}{\partial q}+\frac{1}{q}\right)\left(\frac{\partial}{\partial q}+\frac{1}{q}\right)G(p,q)$

$$= \left(\frac{\partial^2}{\partial q^2}+\frac{1}{q}\frac{\partial}{\partial q}\right)G(p,q)$$

Theorem 15:- Let $f(x,y), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2} \in C$ and $A_2[f(x,y)] = G(p,q), A[f(x)] = G(p)$ then

1. $A_2\left[\frac{\partial f}{\partial x}(x,y)\right] = pA_2[f(x,y)] - \frac{1}{p}A[f(0,y)]$ (26)

2. $A_2\left[\frac{\partial f}{\partial y}(x,y)\right] = qA_2[f(x,y)] - \frac{1}{q}A[f(x,0)]$ (27)

3. $A_2\left[\frac{\partial^2 f}{\partial x^2}(x,y)\right] = p^2 A_2[f(x,y)] - A[f(0,y)] - \frac{1}{p}A[f_x(0,y)]$ (28)

4. $A_2\left[\frac{\partial^2 f}{\partial y^2}(x,y)\right] = q^2 A_2[f(x,y)] - A[f(x,0)] - \frac{1}{q}A[f_y(x,0)]$ (29)

5. $A_2\left[\frac{\partial^2 f}{\partial x \partial y}(x,y)\right] = pqA_2[f(x,y)] - \frac{p}{q}A[f(x,0)] - \frac{q}{p}A[f(0,y)] - \frac{1}{pq}f(0,0)$ (30)

Proof: 1. Consider, $A_2\left[\frac{\partial f}{\partial x}(x,y)\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}\frac{\partial f}{\partial x}dxdy$

$$= \frac{1}{pq}\int_0^\infty e^{-qy}\left(\int_0^\infty e^{-px}\frac{\partial f}{\partial x}dx\right)dy$$

$$= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-px-qy}\, f(x,y)dxdy - \frac{1}{pq}\int_0^\infty e^{-qy}\, f(0,y)dy$$

$$= pA_2[f(x,y)] - \frac{1}{p}A[f(0,y)]$$

2. Consider, $A_2\left[\frac{\partial f}{\partial y}(x,y)\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}\frac{\partial f}{\partial y}dxdy$

$$= \frac{1}{pq}\int_0^\infty e^{-px}\left(\int_0^\infty e^{-qy}\frac{\partial f}{\partial y}dy\right)dx$$

$$= \frac{1}{p}\int_0^\infty\int_0^\infty e^{-px-qy} f(x,y)dxdy - \frac{1}{pq}\int_0^\infty e^{-px} f(x,0)dx$$

$$= qA_2[f(x,y)] - \frac{1}{q}A[f(0,y)]$$

3. Consider, $A_2\left[\frac{\partial^2 f}{\partial x^2}(x,y)\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}\frac{\partial^2 f}{\partial x^2}dxdy$

$$= \frac{1}{pq}\int_0^\infty e^{-qy}\left(\int_0^\infty e^{-px}\frac{\partial^2 f}{\partial x^2}dx\right)dy$$

$$= p^2A_2[f(x,y)] - A[f(0,y)] - \frac{1}{p}A[f_x(0,y)]$$

4. Consider, $A_2\left[\frac{\partial^2 f}{\partial y^2}(x,y)\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}\frac{\partial^2 f}{\partial y^2}dxdy$

$$= \frac{1}{pq}\int_0^\infty e^{-px}\left(\int_0^\infty e^{-qy}\frac{\partial^2 f}{\partial y^2}dy\right)dx$$

$$= q^2A_2[f(x,y)] - A[f(x,0)] - \frac{1}{q}A[f_y(x,0)]$$

5. Consider, $K_2\left[\frac{\partial^2 f}{\partial x\partial y}(x,y)\right] = \frac{1}{pq}\int_0^\infty\int_0^\infty e^{-px-qy}\frac{\partial^2 f}{\partial x\partial y}dxdy$

$$= \frac{1}{pq}\int_0^\infty e^{-qy}\left(\int_0^\infty e^{-px}\frac{\partial^2 f}{\partial x\partial y}dx\right)dy$$

$$= pqA_2[f(x,y)] - \frac{p}{q}A[f(x,0)] - \frac{q}{p}A[f(0,y)] + \frac{1}{pq}f(0,0)$$

Comparison between Laplace, Aboodh and Sumudu transform:

Let us take the example given in [7] for the comparison of Laplace and Sumudu transform

Consider the steady-state temperature distribution function $u(x,y)$ in a long square bar with one face held at constant temperature u_0 and other face held at zero temperature is given by

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad (31)$$

With the boundary conditions, $u(0,y) = 0, u(x,0) = 0, u(\pi,y) = 0, u(x,\pi) = u_0$

The solution using Laplace transform one can see [7]. Here we discuss the solution using Aboodh and Sumudu transform.

1) Using Aboodh transform

Taking Double Aboodh transform of both sides of equation (31) one will get,

$$p^2A_2[u(x,y)] - A[u(0,y)] - \frac{1}{p}A[u_x(0,y)] + q^2A_2[u(x,y)] - A[u(x,0)] - \frac{1}{q}A\left[u_y(x,0)\right] = 0$$

Let $A[u_x(0,y)] = AH(q), A\left[u_y(x,0)\right] = AG(p)$ and $A_2[u(x,y)] = AU(p,q)$ one will get,

$$AU(p,q) = \frac{AH(q)}{p(p^2+q^2)} + \frac{AG(p)}{q(p^2+q^2)}$$

Taking Inverse Aboodh transform with respect to 'p'

$$AU(x,q) = \frac{AH(q)}{q}sin(qx) + \frac{1}{q}\int_0^x AG(v)cos\, q(x-v)dv$$

Now using third condition $u(\pi,y) = 0$ and simplifying the terms with using trigonometry and $\int_0^\pi f(x)dx = \int_0^x f(x)dx + \int_x^\pi f(x)dx$, one will get

$$U(x,q) = \frac{1}{qsin(q\pi)}\int_0^x cosqv\, sinq(\pi - x)AG(v)dv - \frac{1}{qsin(q\pi)}\int_x^\pi sin(qx)cosq(\pi - v)AG(v)dv$$

To use result given in equation (3) one will get

$$qU(x,q) = \frac{1}{sin(q\pi)}\int_0^x sin(qv)sinq(\pi - x)AG(v)dv - \frac{1}{sin(q\pi)}\int_x^\pi sin(qx)sinq(\pi - v)AG(v)dv$$

We get poles at $q = \pm n$ and residues are,

$$r_1 = -\frac{e^{ny}}{\pi}\int_0^\pi AG(v)sinnxcosnvdv,\ r_2 = \frac{e^{-ny}}{\pi}\int_0^{\pi A} G(v)sinnxcosnvdv$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{-2}{\pi}sinhny\left[\int_0^\pi G(v)cosnvdv\right]sinnx$$

Now using last condition and concept of Fourier series one will get,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4u_0}{n\pi}\frac{sinhny}{sinhn\pi}sinnx, \quad n = odd$$

2) Using Sumudu transform

Taking Double Sumudu transform of both sides of equation (31) one will get,

$$\frac{1}{p^2}S_2[u(x,y)] - \frac{1}{p^2}S[u(0,y)] - \frac{1}{p}S[u_x(0,y)] + \frac{1}{q^2}S_2[u(x,y)] - \frac{1}{q^2}S[u(x,0)] - \frac{1}{q}S\left[u_y(x,0)\right] = 0$$

Let $S[u_x(0,y)] = SH(q), S\left[u_y(x,0)\right] = SG(p)$ and $S_2[u(x,y)] = SU(p,q)$ one will get,

$$SU(p,q) = \frac{pq^2}{p^2+q^2}SH(q) + \frac{p^2q}{p^2+q^2}SG(p)$$

$$= \frac{p}{\left(1+\frac{p^2}{q^2}\right)}SH(q) + \frac{p^2}{q\left(1+\frac{p^2}{q^2}\right)}SG(p)$$

Taking Inverse Sumudu transform with respect to 'p'

$$SU(x,q) = H_1(q)sin\left(\frac{x}{q}\right) + \int_0^x G_1(v)sin\left(\frac{x-v}{q}\right)dv$$

where $qSH(q) = H_1(q), pSG(p) = G_1(p)$

Now using third condition $u(\pi,y) = 0$ and simplifying the terms and using trigonometry and $\int_0^\pi f(x)dx = \int_0^x f(x)dx + \int_x^\pi f(x)dx$, one will get

$$SU(x,q) = \frac{-1}{sin\left(\frac{\pi}{q}\right)}\int_0^x sin\left(\frac{v}{q}\right)sin\left(\frac{\pi-x}{q}\right)G_1(v)dv - \frac{1}{sin\left(\frac{\pi}{q}\right)}\int_x^\pi sin\left(\frac{x}{q}\right)sin\left(\frac{\pi-v}{q}\right)G_1(v)dv$$

To use result given in [7] replace q by $\frac{1}{q}$ and multiply by $\frac{1}{q}$, one will get

$$\frac{1}{q}SU\left(x,\frac{1}{q}\right) = \frac{-1}{qsin(q\pi)}\int_0^x sin(qv)sinq(\pi-x)G(v)dv - \frac{1}{qsin(q\pi)}\int_x^\pi sin(qx)sinq(\pi-v)G(v)dv$$

We get poles at $q = 0, q = \pm n$

For $q = 0$ we get residue 0 and at $q = \pm n$

we get $r_1 = \frac{e^{ny}}{n\pi}\int_0^\pi G(v)sinnxsinnvdv, r_2 = \frac{e^{-ny}}{n\pi}\int_0^\pi G(v)sinnxsinnvdv$

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{n\pi}sinhny\left[\int_0^{\pi}G(v)sinnvdv\right]sinnx$$

Now using last condition and concept of Fourier series one will get,

$$u(x,y)=\sum_{n=1}^{\infty}\frac{4u_0}{n\pi}\frac{sinhny}{sinhn\pi}sinnx, n=odd$$

Here we can see that the solution exists by Sumudu transform and the solution of Laplace- Aboodh transform is same as that of Sumudu transform and Laplace transform [7].

Applications to solve Engineering Problems:

Let us define $A[u(t)=AU(p), A_2[u(x,t)]=AU(p,q)$ be Aboodh transform and Double Aboodh transform

1. A particle moving along plane curve at any time t with coordinates (x,y) are given by

$$\frac{dy}{dt}+2x=sin2t,\ \frac{dx}{dt}-2y=cos2t, t>0 \text{ if at } t=0, x=1, y=0$$

Let us find the curve using Aboodh transform

$$pAY(p)-\frac{y(0)}{p}+2AX(p)=\frac{2}{p(p^2+4)} \text{ and}$$

$$pAX(p)-\frac{x(0)}{p}-2AY(p)=\frac{1}{(p^2+4)}$$

Simplifying and finding it for x and y one will get,

$$AX(p)=\frac{1}{p(p^2+4)}+\frac{1}{(p^2+4)} \text{ gives } x(t)=\frac{1}{2}sin2t+cos2t \text{ and}$$

$$AY(p)=\frac{2}{p(p^2+4)} \text{ gives } y(t)=sin2t$$

Laplace transform gives the same result.

2. Consider the example for **Electrical system.** The currents i_1 and i_2 in mesh are given by the differential equations

$$\frac{di_1}{dt}-wi_2=acoskt\ ,\ \frac{di_2}{dt}+wi_1=asinkt$$

Let us find the currents using Aboodh transform if at $t=0, i_1=0, i_2=0$

Using Aboodh transform and simplifying one will get,

$$Ai_1(p) = \frac{a(p^2+w)}{p(p^2+k^2)(p^2+w^2)} \text{ we get}$$

$i_1 = \frac{a}{w+k}(sinwt + sinkt)$ Similarly one will get

$$i_2 = \frac{a}{w+k}(coswt - coskt)$$

3. Let us find the voltage v in a line of length l, t seconds after the ends are suddenly grounded satisfies **Radio Equations** [18 p.983]

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$

Subject to the conditions $v(0,t) = v(l,t) = 0$, $v(x,0) = v_0 sin\left(\frac{\pi x}{l}\right), \left(\frac{\partial v}{\partial t}\right)_{t=0} = 0$

Appling Aboodh transform of both side and simplifying the terms, one will get

$$\frac{d^2[AV(x,q)]}{dx^2} - LCq^2[AV(x,q)] = -v_0 LCsin\left(\frac{\pi x}{l}\right)$$

Where $AV(x,q) = A[v(x,t)]$, After solving the equation we get,

$$AV(x,q) = C_1 e^{\sqrt{LC}qx} + C_2 e^{-\sqrt{LC}qx} + \frac{v_0 sin\left(\frac{\pi x}{l}\right)}{q^2 + \frac{\pi^2}{l^2 LC}}$$

Using given conditions one will get $C_1 = C_2 = 0$

$$AV(x,q) = \frac{v_0 sin\left(\frac{\pi x}{l}\right)}{q^2 + \frac{\pi^2}{l^2 LC}}$$

Taking Inverse Aboodh transform $v(x,t) = v_0 sin\left(\frac{\pi x}{l}\right) cos\left(\frac{\pi t}{l\sqrt{LC}}\right)$

4. Consider a **Mechanical system** with two degrees of freedom, satisfies the equations $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$, $2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$,

and initially $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanishes at $t = 0$

Appling Aboodh transform and simplifying the terms one will get

$AX(p) = \frac{8}{p^2(4p^2+9)}$, $AY(p) = \frac{12}{p^3(4p^2+9)}$

Gives $x(t) = \frac{8}{9}\left(1 - cos\frac{3}{2}t\right)$ and $y(t) = \frac{8}{9}\left(t - \frac{2}{3}sin\frac{3}{2}t\right)$

5. Consider the example related to **Civil engineering**, The deflection of a beam of length l, clamped horizontally at both ends and loaded at $x = l/4$ by a weight W, is given by

$$EI\frac{d^4y}{dx^4} = W\delta\left(x - \frac{l}{4}\right)$$

Taking Aboodh transform and simplifying the terms, one will get

$$AY(p) = \frac{c_1}{p^3} + \frac{c_2}{p^5} + \frac{W}{EI}\frac{1}{p^4}e^{-lp/4}$$

where $y'(0) = c_1, y'''(0) = c_2$ Taking Inverse Aboodh transform

$$y(t) = c_1 x + c_2\frac{x^3}{6} + \frac{W}{6EI}\left(x - \frac{l}{4}\right)^3 H\left(x - \frac{l}{4}\right)$$

Using given conditions, the final answer,

$$y(t) = \frac{7W}{128EI}l^2 x - \frac{W}{8EI}x^3 + \frac{W}{6EI}\left(x - \frac{l}{4}\right)^3 H\left(x - \frac{l}{4}\right)$$

6. Consider the wave equation

$$u_{tt} - u_{xx} = -3e^{2x+t}\ ,\ 0 < x < \infty, t > 0$$

With initial and boundary conditions,

$$u(0,t) = 2e^t\ ,\ \ t \geq 0$$

$$u(x,0) = e^{2x} + e^x = \left.\frac{\partial u}{\partial t}\right|_{t=0}\ ,\qquad 0 < x < \infty$$

$$u_x(0,t) = 3e^t,\ \ x > 0$$

Let us take Double Aboodh transform of both sides of given equation

$A_2[u_{tt}] - A_2[u_{xx}] = A_2[-3e^{2x+t}]$

$\therefore q^2 A_2[u(x,t)] - A[u(x,0)] - \frac{1}{q}A[u_t(x,0)] - p^2 A_2[u(x,t)] + A[u(0,t)] + \frac{1}{p}A[u_x(0,t)] = \frac{-3}{pq(p-2)(q-1)}$

After substituting the given conditions, given equation is reduces in to,

$U(p,q) = \frac{(q+1)(q-1)(p-2)+(p-1)(q^2-2p^2+p+2)}{pq(p-1)(p-2)(q-1)(q-p)(q+p)}$ Now using equation (3) one can get

$\Rightarrow u(x,t) = e^{2x+t} - e^{x+t}$

7. Consider the equation

$$u_{xx} - u_{tt} - u_t - u = 0 \,, \; x, t > 0$$

With initial and boundary conditions,

$$u(0,t) = e^{-t} \,, \; u(x,0) = e^{x} \,, \; u_x(0,t) = e^{-t}, \; u_t(x,0) = 0$$

Let us take Aboodh transform of both sides of given equation

$$p^2 A_2[u(x,t)] - A[u(0,t)] - \frac{1}{p} A[u_x(0,t)] - q^2 A_2[u(x,t)] + A[u(x,0)] + \frac{1}{q} A[u_t(x,0)]$$

$$-qA_2[f(x,y)] + \frac{1}{q} A[f(x,0)] - A_2[u(x,t)] = 0$$

After substituting the given conditions, given equation is reduces in to,

$$U(p,q) = \frac{p^2 - q^2 - q - 1}{pq(q+1)(p-1)(p^2 - q^2 - q - 1)} \therefore u(x,t) = e^{x-t}$$

8. Consider the Fourier heat equation in a quarter plane

$$u_t = K u_{xx} \,, \; x \geq 0, \;\; t > 0$$

With initial and boundary conditions,

$$u(x,0) = 0 \,, \qquad u(x,t) \to 0, x \to \infty$$

$$u(0,t) = 2u_0 \,, \qquad u_x(0,t) = 0 \,,$$

Let us take Aboodh transform of both sides of given equation

$$A_2[u_t] = K A_2[u_{xx}]$$

After substituting the given conditions, given equation is reduces in to,

$$U(p,q) = \frac{2u_0 k}{q^2(q - kp^2)}$$

$$\therefore u(x,t) = A^{-1}\left(\frac{u_0}{q^2}\left[e^{\sqrt{\frac{q}{k}}x} + e^{-\sqrt{\frac{q}{k}}x}\right]\right)$$

$$\Longrightarrow u(x,t) = u_0 erf_c\left(\frac{x}{2\sqrt{kt}}\right)$$

This is because we want finite solution.

9. A thin membrane of great extent is released from rest in the position $u = f(x,y)$ [1] one can find the displacement as,

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right), \qquad x > 0, \;\; y > 0$$

Taking Double Aboodh transform of above equation and taking t as a constant,

$$\frac{d^2U(p,q)}{dt^2} = c^2(p^2U(p,q) + q^2U(p,q),$$

$$U(p,q) = Acos\left((c\sqrt{p^2+q^2}\,)t\right) + Bsin\left((c\sqrt{p^2+q^2}\,)t\right)$$

Using initial condition, at $t = 0$ $u = f(x,y)$ and $u_t(x,y) = 0$, one can get

$$A = G(p,q) = A_2[f(x,y)], \;\; B = 0$$

$$u(x,y) = \frac{-1}{4\pi}\int_0^\infty\int_0^\infty e^{px+qy} pqG(p,q)dpdq$$

TABLE 1: ABOODH TRANSFORM OF FUNCTIONS

Aboodh Transform of Functions		
Sr. No.	Function $f(t)$	Aboodh Transform $G(p)$
1	e^{at}	$\frac{1}{p(p-a)}$
2	$cosat$	$\frac{1}{p^2+a^2}$
3	$sinat$	$\frac{a}{p(p^2+a^2)}$
4	$coshat$	$\frac{1}{p^2-a^2}$
5	$sinhat$	$\frac{a}{p(p^2-a^2)}$
6	$t^n \quad ,n \geq 0$	$\frac{n!}{p^{n+1}}$
7	$t^a \quad ,a > -1$	$\frac{\gamma(a+1)}{p^{n+1}}$
8	$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{p^{3/2}}$
9	$2\sqrt{t}$	$\frac{\sqrt{\pi}}{p^{5/2}}$
10	$t^n e^{-at}$	$\frac{\gamma(n+1)}{p(p+a)^n}$

Aboodh Transform of Functions		
Sr. No.	Function $f(t)$	Aboodh Transform $G(p)$
11	$e^{at}cosbt$	$\frac{p-a}{p[(p-a)^2+b^2]}$
12	$e^{at}sinbt$	$\frac{b}{p[(p-a)^2+b^2]}$
13	$tcosat$	$\frac{1}{p}\frac{(p^2-a^2)}{(p^2+a^2)^2}$
14	$tsinat$	$\frac{2a}{(p^2+a^2)^2}$
15	$\frac{sinat}{t}$	$\frac{1}{p}tan^{-1}(a/p)$
16	$Si(t)=\int_0^t \frac{sinx}{x}dx$	$\frac{1}{p^2}\ cot^{-1}(p)$
17	$Ci(t)=-\int_t^\infty \frac{cosx}{x}dx$	$-\frac{1}{2p^2}\ log(p^2+1)$
18	$sinhat\ sinat$	$\frac{2a^2}{p^4+4a^4}$
19	$sinhat-sinat$	$\frac{2a^3}{p(p^4-a^4)}$
20	$coshat-cosat$	$\frac{2a^2}{p^4-a^4}$
21	$cosat-cosbt$	$\frac{(b^2-a^2)}{(p^2+a^2)(p^2+b^2)}$
22	$e^{at}-e^{bt}$	$\frac{1}{p}\frac{(a-b)}{(p-a)(p-b)}$
23	$ae^{at}-be^{bt}$	$\frac{(a-b)}{(p-a)(p-b)}$
24	$\frac{1}{t}(e^{bt}-e^{at})$	$\frac{1}{p}log\left(\frac{p-a}{p-b}\right)$
25	$t^{-\frac{1}{2}}e^{-a/t}$	$\frac{\sqrt{\pi}}{p^{3/2}}\ e^{-2\sqrt{ap}}$

Aboodh Transform of Functions		
Sr. No.	Function $f(t)$	Aboodh Transform $G(p)$
26	$t^{-\frac{3}{2}}e^{-a/t}$	$\frac{\sqrt{\pi}}{\sqrt{ap}}\ e^{-2\sqrt{ap}}$, $a>0$
27	$t^{-\frac{3}{2}}e^{-a^2/4t}$	$\frac{2\sqrt{\pi}}{ap}\ e^{-a\sqrt{p}}$, $a>0$
28	$t^{-\frac{1}{2}}e^{-a^2/4t}$	$\frac{\sqrt{\pi}}{p^{3/2}}\ e^{-a\sqrt{p}}$, $a\geq 0$
29	$e^{-a^2t^2/4}$	$\frac{\sqrt{\pi}}{pa}\ e^{-\left(\frac{p^2}{a^2}\right)}erf_c\left(\frac{p}{a}\right)$, $a>0$
30	$erf\left(\frac{t}{2a}\right)$	$\frac{1}{p^2}\ e^{\left(a^2p^2\right)}erf_c(ap)$, $a>0$
31	$erf\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{1}{p^2}\left(1-e^{-(a\sqrt{p})}\right)$, $a\geq 0$
32	$erf_c\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{1}{p^2}e^{-(a\sqrt{p})}$, $a\geq 0$
33	$t^{\frac{1}{2}}e^{-a^2/4t}$	$\frac{\sqrt{\pi}}{2}\left[\frac{1}{p^{5/2}}+\frac{a}{p^2}\right]e^{-(a\sqrt{p})}$, $a\geq 0$
34	$e^{a^2t}erf\left(a\sqrt{t}\right)$	$\frac{a}{p^{3/2}(p-a^2)}$
35	$e^{a^2t}erf_c\left(a\sqrt{t}\right)$	$\frac{1}{p^{3/2}(\sqrt{p}-a^2)}$
36	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}erf\left(a\sqrt{t}\right)$	$\frac{1}{\sqrt{p}(p-a^2)}$
37	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}erf_c\left(a\sqrt{t}\right)$	$\frac{1}{p(\sqrt{p}-a)}$
38	$\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}e^{-a^2/4t}-aerf_c\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{1}{p^{5/2}}\ e^{-a\sqrt{p}}$, $a\geq 0$
39	$\frac{1}{\sqrt{t+a}}$	$\frac{\sqrt{\pi}}{p^{3/2}}e^{ap}er\,f_c\left(\sqrt{ap}\right)$, $a>0$
40	$t^{-\frac{1}{2}}e^{-2a\sqrt{t}}$, $a\geq 0$	$\frac{\sqrt{\pi}}{p^{3/2}}e^{a/p}er\,f_c\left(\frac{a}{\sqrt{p}}\right)$, $a>0$

Aboodh Transform of Functions		
Sr. No.	Function $f(t)$	Aboodh Transform $G(p)$
41	$\frac{1}{\sqrt{t}}\cos(2a\sqrt{t})$	$\frac{\sqrt{\pi}}{p^{3/2}}\ e^{-a/p}$
42	$\frac{1}{\sqrt{t}}\sin(2a\sqrt{t})$	$\frac{\sqrt{\pi}}{p^{5/2}}\ e^{-a/p}$
43	$cosh(2a\sqrt{t})$	$\frac{\sqrt{\pi a}}{p^{3/2}}\ e^{a/p}$
44	$sinh(2a\sqrt{t})$	$\frac{\sqrt{\pi a}}{p^{5/2}}\ e^{a/p}$
45	$\frac{sin(2a\sqrt{t}}{t}$	$\frac{\pi}{p}erf(a/\sqrt{p})$
46	$\delta(t-a)$	$\frac{1}{p}e^{-ap} \quad ,a \geq 0$
47	$H(t-a)$	$\frac{1}{p^2}e^{-ap} \quad ,a \geq 0$
48	$\delta'(t-a)$	$e^{-ap} \quad ,a \geq 0$
49	$\delta^n(t-a)$	$p^{n-1}e^{-ap} \quad ,a \geq 0$
50	$Ei(-t) = \int_t^\infty \frac{e^{-x}}{x}dx$	$\frac{1}{p^2}log(1+p)$

3. CONCLUSION

In this article Double Aboodh transform and Triple Aboodh transform are introduced and their properties are proved. They are used to partial differential equations with initial and boundary conditions.

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REFERENCES

[1] L. Debnath and D. Bhatta, 2015, Integral Transforms and Their Applications, 2nd ed., Taylor and Francis Group. Boca Raton: Chapman and Hall/ CRC Press.

[2] Schiff, J. L.,1999, The Laplace Transforms, New York: Springer.

[3] L. Debnath, 2016, “The Double Laplace Transform and Their Properties with Applications to Functional, Integral and partial differential Equations,” Int. J. Appl. Comput. Math., vol. 2, pp. 223–241.

[4] A. Kalicman, H. E. Gadain, 2010, “On the Applications of Laplace and Sumudu Transforms,” J Franklin Inst, vol. 347, pp. 848–862.

[5] A. Kalicman, H Eltyab and A, Atan, 2011, “A note on the Comparison between Laplace and Sumudu transforms,” Bulletin of the Iranian Mathematical Society, Vol. 37, pp 131-141.

[6] G. K, Watugula, 1993, “Sumudu Transform: A new Integral Transform to solve differential equations and control Engineering Problems,” Int J Math Educ Sci Technol, vol. 24, pp. 35–43.

[7] F. B. M. Belgacem, A. A. Karaballi and S. Kalla, 2003 “Analytical investigations of the Sumudu transform and applications to integral production equations,” Math Probl Eng, vol. 3, pp. 103–118.

[8] F. B. M. Belgacem, A. A. Karaballi, 2006 “Sumudu transform fundamental properties investigations and applications,” Journal of Applied Mathematics and Stochastic Analysis, vol. 1, pp. 1–23.

[9] Hassan Eltayeb and Adem Kalicman, 2010 “On some applications of a new integral transform,” Int. journal of math. Analysis, Vol. 4, pp.123-132.

[10] Hassan Eltayeb and Adem Kiliçman, 2010” On Double Sumudu Transform and Double Laplace Transform,” Malaysian Journal of Mathematical Sciences, Vol. 4, pp.17-30.

[11] K. S. Aboodh, 2013 “The new integral transform ‘Aboodh Transform’,” Global joural of pure and applied mathematics, Vol. 9, pp. 35-43.

[12] M M. A Mahgoub, 2016, “The new integral transform ‘Mahgoub Transform’,” Advances in theoretical and applied mathematics, Vol. 11, pp. 391-398.

[13] M M. A Mahgoub and A A Alshikh, 2017 “An application of new transform ‘Mahgoub Transform’ to partial differential equations,” Mathematical theory and Modeling, Vol. 07, pp. 7-9.

[14] S. B. Kiwane and Sandip M. Sonawane, 2018, “Mahgoub Transform Fundamental Properties and Applications,” International Journal of Computer and Mathematical Sciences, Vol. 7, pp.500-511.

[15] A. Kamal . H. Sedeeg, 2016, “The New Integral Transform ‘Kamal

Transform',” Advances in Theoretical and Applied Mathematics, Vol. 11, pp.451-458.

[16] N. E .H .Taha, R .I. Nuruddeen and A .K. H. Sedeeg, 2017, “Dualities between Aboodh & Mahgoub integral transforms and Some famous integral transforms,” British journal of applied science & technology, Vol. 20, pp.1-8.

[17] Sandip M. Sonawane and S. B. Kiwane, 2019, “Double Kamal Transform Properties and Applications,” Journal of Applied Science and Computations, Vol. 6, pp.1727-1739.

[18] N.P. Bali and Manish Goyal, 2009, “A Textbook of Engineering Mathematics,” Laxmi Publications, 7th Edition.