Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 6 (2019), pp. 945-970

http://www.ripublication.com/gjpam.htm

Improvement of Jensen and Levinson Type Inequalities for Functions with Nondecreasing Increments

M. Maqsood Ali [*1], **Asif R. Khan** [†1], **Inam Ullah Khan** [‡1,2], **and Sumayyah Saadi** [§1]

[1]*Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan.*

[2]*Pakistan Shipowners' Govt. College, North Nazimabad, Karachi, Pakistan.*

Abstract

In this article we give improvements of Jensen inequality, Jensen-Boas inequality, Jensen-Steffensen inequality, Jensen-Mercer inequality and Neizgoda's inequality for functions with nondecreasing increments of convex type followed by related improvements of Levinson inequality. In this way many established results will become special cases of our results.

Keywords: Functions with nondecreasing increments, Levinson's inequality, Jensen-Steffensen inequality

2010 Mathematics Subject Classification: 26D10, 26D99

[*]maqsoodali81@yahoo.com
[†]asifrk@uok.edu.pk
[‡]zrishk@gmail.com
[§]sumayyahsaadi@uok.edu.pk

1. CONVEX FUNCTIONS ON NORMED LINEAR SPACES AND HILBERT SPACES

We start with some results from [10] that will help us in our main constructions:

Proposition 1. *A linear functional, having its domain contained in a normed linear space, is continuous ⇔ it is bounded.*

Following is the well known "Riesz Representation Theorem":

Proposition 2. *There is an inner product representation for each linear functional Ψ which is bounded on a Hilbert space H and is written as:*

$$\Psi(\theta) = \langle \theta, \vartheta \rangle \tag{1}$$

here ϑ depending upon Ψ has unique value that is evaluated through Ψ. The norm of ϑ is

$$\|\vartheta\| = \|\Psi\| = \sup_{0 \neq \theta \in D(\Psi)} \frac{|\Psi(\theta)|}{\|\theta\|}.$$

In [13, p. 128] we see an interesting result that if V is a normed linear space and U be an open convex subset in V then a convex function Ψ on U generates a supporting hyperplane at every point say $a_0 \in U$. This implies the presence of a linear functional h which in conituous on V and is charectarized as

$$\Psi(\theta) \geq \Psi(a_0) + h(\theta - a_0) \;\forall\; \theta \in U. \tag{2}$$

Such functionals h are said to be the support of Ψ at a_0 and the subdifferential of Ψ at the point a_0 is established through the set $\partial\Psi(a_0)$ of all these functionals h.

Now if V is a Hilbert space then we have a unique representation of all such functionals h as $h(\theta) = \langle \theta, \vartheta \rangle$ for $\theta \in V$ such that $\|\Psi\| = \|\vartheta\|$ by using Riesz representation theorem.

In this case inequality (2) becomes

$$\Psi(\theta) \geq \Psi(a_0) + \langle \theta - a_0, \vartheta \rangle \quad \text{for all} \quad \theta \in U \tag{3}$$

all these vectors ϑ are usually termed as subgradients and the set of all vectors ϑ constitute the subdifferential $\partial\Psi(a_0)$.

Using these facts we state our first main result:

Theorem 1. *If Ψ is a convex function on an open convex subset U of a Hilbert space H, then for all $\theta,\ a_0 \in U$*

$$\Psi(\theta) - \Psi(a_0) - \langle \theta - a_0, \vartheta \rangle \geq \big| |\Psi(\theta) - \Psi(a_0)| - \|\Psi\| \, \|\theta - a_0\| \big|,$$

where $\vartheta \in \partial\Psi(a_0)$.

Proof. Using (3) we have that for all $\theta, a_0 \in U$

$$\begin{aligned} \Psi(\theta) - \Psi(a_0) - \langle \theta - a_0, \vartheta \rangle &= |\Psi(\theta) - \Psi(a_0) - \langle \theta - a_0, \vartheta \rangle| \\ &\geq \big| |\Psi(\theta) - \Psi(a_0)| - |\langle \theta - a_0, \vartheta \rangle| \big| \end{aligned}$$

Using Cauchy-Schwartz inequality $|\langle \theta, y \rangle\| \leq \|\theta\|\|y\|$, we have

$$\Psi(\theta) - \Psi(a_0) - \langle \theta - a_0, \vartheta \rangle \geq \big| |\Psi(\theta) - \Psi(a_0)| - \|\theta - a_0\|\|\vartheta\| \big|$$

Since $\|\vartheta\| = \|\Psi\|$ by Riesz representation theorem, so we are done. □

Let U be an open subset of a Banach space E. Then the real valued function Ψ, defined on U, have one-sided directional derivatives (left(right) Gâteaux derivatives) at $a_0 \in U$ relative to (or may say in the direction of) v. These Gâteaux derivatives are defined through limits as:

$$\nabla_{+(-)}\Psi\left(a_0; v\right) := \lim_{t \to 0+(-)} \frac{\Psi\left(a_0 + tv\right) - \Psi\left(a_0\right)}{t}$$

It is worth mentioning that both of the one-sided directional derivatives $\nabla_{-}\Psi\left(a_0; v\right)$ and $\nabla_{+}\Psi\left(a_0; v\right)$ are subadditive as well as positively homogenous and their existence and equivalence gives us the directional derivative of Ψ relative to v at a_0, written as $\Psi'(a_0; v)$, which is nothing but the common value of $\nabla_{-}\Psi\left(a_0; v\right)$ and $\nabla_{+}\Psi\left(a_0; v\right)$. We infer that $\nabla_{-}\Psi\left(a_0; v\right)$ and $\nabla_{+}\Psi\left(a_0; v\right)$ are linear when they exist and obey the expression

$$\nabla_{+}\Psi\left(a_0; v\right) = -\nabla_{-}\Psi\left(a_0; -v\right)$$

To be more specific, the directional derivatives are actually the partial derivatives when taken in connection to canonical basis of $\mathbb{R}^n$. For the sake of simplicity we will denote right directional derivative by Ψ'_{+} and left directional derivative by Ψ'_{-} and for directional derivative by Ψ' if exist.

Following is another important result from [14, p. 12–13]

Proposition 3. *Let Ψ be convex function on an open set U of a real normed linear space $\mathbb{R}^n$. Then for all $\theta, \theta_0 \in U$ the inequality*

$$\Psi(\theta) - \Psi(\theta_0) \geq \langle \Psi'_{+}(\theta_0), \theta - \theta_0 \rangle$$

holds, where $\langle \cdot, \cdot \rangle$ represent usual inner product on $\mathbb{R}^n$.

Now let us recall an important result from literature by Levinson [11].

Proposition 4. *For a real valued function* Ψ *with nonnegative third derivative and defined on* $[0, 2a] \in \mathbb{R}$*, the inequality*

$$\frac{\sum_{i=1}^{n} q_i \Psi(\theta_i)}{Q_n} - \Psi\left(\frac{\sum_{i=1}^{n} q_i \theta_i}{Q_n}\right) \leq \frac{\sum_{i=1}^{n} q_i \Psi(\eta_i)}{Q_n} - \Psi\left(\frac{\sum_{i=1}^{n} q_i \eta_i}{Q_n}\right) \tag{4}$$

holds for $0 < \theta_i < a$, $\eta_i = 2a - \theta_i$, *for* $i \in \{1, \ldots, n\}$ $\quad q_i > 0$ *and* $Q_n = \sum_{i=1}^{n} q_i$.

2. FUNCTIONS WITH NONDECREASING INCREMENTS

In [2], H. D. Brunk introduced an important class of functions namely functions with nondecreasing increments. In order to properly define this class here we have some assumptions.

Let $\mathbb{R}^k$ is a $k-$dimensional vector lattice and the partial ordering for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ is given as $\mathbf{x} = (x_1, \ldots, x_k) \leq \mathbf{y} = (y_1, \ldots, y_k)$ if and only if $x_i \leq y_i$, where x_i, y_i are real for each $i \in \{1, \ldots, k\}$. Furthermore a set $\{\mathbf{z} \in \mathbb{R}^k : \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\}$ is called an *interval* $[\mathbf{x}, \mathbf{y}]$ and $\mathbf{0}$ shows the $k-$tuple $(0, \ldots, 0)$. Also for any $r, s \in \mathbb{R}$

$$r\mathbf{x} + s\mathbf{y} = (rx_1 + sy_1, \ldots, rx_k + sy_k).$$

Throughout this article, functions with nondecreasing increments will be abbreviated as F.W.N.D.I. We also suppose that $\mathbf{U}$ is an interval in $\mathbb{R}^k$ and for real weights $q_1, \ldots, q_n$ we define

$$Q_i = \sum_{j=1}^{i} q_j, \; i \in \{1, \ldots, n\} \quad \text{and clearly} \quad Q_n = \sum_{j=1}^{n} q_j.$$

In [2], we see the following definition of a F.W.N.D.I.:

Definition 1. A real-valued function Ψ defined on $\mathbf{U}$ is said to have nondecreasing increments if

$$\Psi(\alpha + \eta) - \Psi(\alpha) \leq \Psi(\beta + \eta) - \Psi(\beta) \tag{5}$$

whenever $\alpha \in \mathbf{U},\; \beta + \eta \in \mathbf{U},\; \mathbf{0} \leq \eta \in \mathbb{R}^k$, $\alpha \leq \beta$.

Brunk observed that inequality (5) does not imply continuity even for $k = 1$. It is very interesting to notice that for positively oriented lines these functions are convex. These are lines having nonnegative direction cosines and expressed as $\mathbf{X}(t) = \alpha t + \beta$ where $\mathbf{0} \leq \alpha$ and $\alpha, \beta \in \mathbf{U}$. Furthermore, these functions are said to be *Wright-convex* for $k = 1$ and from [14, p. 7] we know that the class of convex functions is properly contained in class of Wright-convex functions while class of Wright-convex itself is a proper subclass of $J-$convex functions. For some more properties and results regarding F.W.N.D.I. [2], [5], [6], [7] and [15] can be seen.

2.1 Some Inequalities for Functions with Nondecreasing Increments

Lets start with Jensen-Steffensen type inequality for F.W.N.D.I. from [15].

Proposition 5. *Let a function of bounded variations* $B : [a, b] \to \mathbb{R}$*, such that*

$$B(a) \leq B(x) \leq B(b), \quad B(b) > B(a), \tag{6}$$

and a continuous nondecreasing map $\boldsymbol{\Upsilon}$ *from a real interval* $[a, b]$ *to* $\mathbf{U}$*. Then for continuous F.W.N.D.I.,* $\Psi : \mathbf{U} \to \mathbb{R}$*, we have*

$$\Psi\left(\frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b \Psi\left(\boldsymbol{\Upsilon}(t)\right)\, dB(t)}{\int_a^b dB(t)}, \tag{7}$$

where $\int_a^b \boldsymbol{\Upsilon}\, dB$ *is the vector* $(\int_a^b \Upsilon_1 d\, B, \ldots, \int_a^b \Upsilon_k\, dB)$*.*

Remark 1. If $\dfrac{\int_a^b \boldsymbol{\Upsilon}(t) dB(t)}{\int_a^b dB(t)} \in \mathbf{U}$ and we have either $B(x) \leq B(a)$ or $B(x) \geq B(b)$, then the reverse inequality in (5) holds.

The following proposition is the Jensen-Boas type inequality for functions with nondecreasing increments and it is proved in [6].

Proposition 6. *Let* $\boldsymbol{\Upsilon} : [a, b] \to \mathbf{U}$ *be a continuous and monotonic (either nonincreasing or nondecreasing) map in each of the* l *intervals* (b_{i-1}, b_i)*. Let* $B : [a, b] \to \mathbb{R}$ *is continuous or of bounded variation satisfying*

$$B(a) \leq B(a_1) \leq B(b_1) \leq B(a_2) \leq \cdots \leq B(b_{l-1}) \leq B(a_l) \leq B(b) \tag{8}$$

for all $a_i \in (b_{i-1}, b_i)$ $(b_0 = a,\ b_l = b)$*, and* $B(b) > B(a)$*. If* φ *is continuous function having nondecreasing increments in each of the* l *intervals* (b_{i-1}, b_i)*, then we have the following inequality*

$$\Psi\left(\frac{\int_a^b \boldsymbol{\Upsilon}(t) dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t)) dB(t)}{\int_a^b dB(t)}. \tag{9}$$

Remark 2. If $\dfrac{\int_a^b \boldsymbol{\Upsilon}(t) dB(t)}{\int_a^b dB(t)} \in \mathbf{U}$ and $\forall x \in a_i \ni (b_{i-1}, b_i)(b_0 = a, b_l = b)$ and we have either $B(x) \geq B(b)$ or $B(x) \leq B(a)$, then the reverse inequality in (9) holds.

Following is the Neizgoda's integral inequality for F.W.N.D.I. from [4].

Proposition 7. *Let* (Υ, Σ, μ) *be a measure space with positive finite measure* μ *and consider the weight function* $\omega : \Upsilon \to [0, +\infty)$*. Let* $\Theta : [a, b] \to \mathbf{U}$ *and* $\Psi : \Upsilon \times [a, b] \to \mathbf{U}$ *be two nondecreasing continuous mappings such that for each* $s \in \Upsilon$

$$\int_x^b \Psi(s,t)\, dB(t) \quad \leq \quad \int_x^b \Theta(s,t) dB(t), \quad \textit{for each} \quad x \in (a,b), \tag{10}$$

$$\textit{and} \quad \int_a^b \Psi(s,t)\, dB(t) \quad = \quad \int_a^b \Theta(s,t)\, dB(t), \tag{11}$$

where $B : [a, b] \to \mathbb{R}$ *is of bounded variation. Moreover, let* $I = \bigcup_{i=1}^{k}(a_i, b_i)$*,* $I^c = [a, b] \backslash I = \bigcup_{i=1}^{k+1}[b_{i-1}, a_i]$ *and* $|I^c| = \sum_{i=1}^{k+1}(a_i - b_{i-1})$ *where* $a = b_0 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k \leq a_{k+1} = b$ *is a partition of the interval* $[a, b]$*. Then for every continuous function* $\varphi : \mathbf{U} \to \mathbb{R}$ *with nondecreasing increments the inequality*

$$\varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t) dB(t) - \frac{1}{\int_\Upsilon \omega(s) d\mu(s)} \int_I \int_\Upsilon \omega(s)\Psi(s,t) d\mu(s) dB(t)\right)\right) \leq$$
$$\frac{1}{|I^c|}\left(\int_a^b \varphi(\Theta(t)) dB(t) - \frac{1}{\int_\Upsilon \omega(s) d\mu(s)} \int_I \int_\Upsilon \varphi(\Psi(s,t)) d\mu(s) dB(t)\right) \tag{12}$$

holds for each $s \in \Upsilon$.

Remark 3. It is interesting to notice that (12) still holds if the corresponding assumptions of Proposition 7 is replaced by the following:
Let $\Theta : [a, b] \to \mathbf{U}$ and $\Psi : \Upsilon \times [a, b] \to \mathbf{U}$ be two nonincreasing continuous mappings such that for each $s \in \Upsilon$

$$\int_a^x \Psi(s,t)\, dB(t) \quad \leq \quad \int_a^x \Theta(s,t)\, dB(t), \quad \text{for each} \quad x \in (a,b),$$

$$\text{and} \quad \int_a^b \Psi(s,t)\, dB(t) \quad = \quad \int_a^b \Theta(s,t)\, dB(t).$$

Now we state a result namely generalized Levinson inequality from [7].

Proposition 8. *Let* $B : [a, b] \to \mathbb{R}$ *be a function of bounded variation such that* (8) *holds and let* $\boldsymbol{\Upsilon}$ *be a continuous and nondecreasing map from* $[a, b] \subset \mathbb{R}$ *to an interval* $\mathbf{I} = [\mathbf{0}, \mathbf{d}] \subset \mathbb{R}^k$*,* $\mathbf{d} > \mathbf{0}$*. If* Ψ *is a continuous F.W.N.D.I. of order* 3 *on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$*, then*

$$\frac{\int_a^b \Psi\left(\boldsymbol{\Upsilon}(t)\right)\, dB(t)}{\int_a^b dB(t)} - \Psi\left(\frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}\right)$$
$$\leq \frac{\int_a^b \Psi\left(2\mathbf{d} - \boldsymbol{\Upsilon}(t)\right)\, dB(t)}{\int_a^b dB(t)} - \Psi\left(\frac{\int_a^b (2\mathbf{d} - \boldsymbol{\Upsilon}(t))\, dB(t)}{\int_a^b dB(t)}\right).$$

Proposition 9. *Let $B : [a, b] \to \mathbb{R}$ be a function of bounded variation such that* (6) *holds, and let Ψ be a continuous F.W.N.D.I. of order 3 on $[\mathbf{c}, \mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c}$. If $\mathbf{\Upsilon}(t) : [a, b] \to [\mathbf{c}, \mathbf{d} - \mathbf{a}]$ is a continuous and nondecreasing map, then*

$$\frac{\int_a^b \Psi\left(\mathbf{\Upsilon}(t)\right) \, dB(t)}{\int_a^b dB(t)} - \Psi\left(\frac{\int_a^b \mathbf{\Upsilon}(t) \, dB(t)}{\int_a^b dB(t)}\right) \leq \frac{\int_a^b \Psi\left(\mathbf{a} + \mathbf{\Upsilon}(t)\right) \, dB(t)}{\int_a^b dB(t)} - \Psi\left(\frac{\int_a^b (\mathbf{a} + \mathbf{\Upsilon}(t)) \, dB(t)}{\int_a^b dB(t)}\right).$$

holds.

We now prove another important result related to F.W.N.D.I.

3. INTEGRAL JENSEN-MERCER INEQUALITY FOR FUNCTIONS WITH NONDECREASING INCREMENTS

The integral version of Jensen-Mercer inequality for F.W.N.D.I. under the condition of Jensen-Boas inequality (8) is given below:

Theorem 2. *Let $B : [a, b] \to \mathbb{R}$ be a function of bounded variation satisfying*

$$B(a) \leq B(a_1) \leq B(b_1) \leq B(a_2) \leq \cdots \leq B(b_{l-1}) \leq B(a_l) \leq B(b)$$

for all $a_i \in (b_{i-1}, b_i)$ ($b_0 = a$, $b_l = b$), and $B(b) > B(a)$ and let $\mathbf{\Upsilon}$ be a continuous nondecreasing map from the real interval $[a, b]$ to the interval $\mathbf{U}$. If $\Psi : \mathbf{I} \to \mathbb{R}$ is a continuous F.W.N.D.I., then

$$\Psi\left(\boldsymbol{L} + \boldsymbol{M} - \frac{\int_a^b \mathbf{\Upsilon}(t) \, dB(t)}{\int_a^b dB(t)}\right) \leq \Psi(\boldsymbol{L}) + \Psi(\boldsymbol{M}) - \frac{\int_a^b \Psi\left(\mathbf{\Upsilon}(t)\right) \, dB(t)}{\int_a^b dB(t)} \tag{13}$$

where $\int_a^b \mathbf{\Upsilon} \, dB$ is the vector $(\int_a^b \Upsilon_1 d\, B, \ldots, \int_a^b \Upsilon_k \, dB)$, and $\boldsymbol{L} = (L_1, \ldots, L_k)$ and $\boldsymbol{M} = (M_1, \ldots, M_k)$ are two k-tuples related to $\boldsymbol{U}$ such that $\boldsymbol{L} \leq \Upsilon(t) \leq \boldsymbol{M}$ for all $t \in [a, b]$.
Also if $\dfrac{\int_a^b \mathbf{\Upsilon}(t) dB(t)}{\int_a^b dB(t)} \in \mathbf{U}$ and $\forall x \in a_i \ni (b_{i-1}, b_i)(b_0 = a, b_l = b)$ and we have either $B(x) \geq B(b)$ or $B(x) \leq B(a)$, then the inequality (13) *remains valid.*

Proof. Under the conditions (8) for a continuous nondecreasing map $\mathbf{\Upsilon}$ from the real interval $[a, b]$ to the interval $\mathbf{U}$, we have

$$\Psi\left(\mathbf{L}+\mathbf{M}-\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right)=\Psi\left[\frac{1}{\int_a^b dB(t)}\int_a^b (\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t))\,dB(t)\right]$$

now if $\boldsymbol{\Upsilon}$ is a continuous nondecreasing map from the real interval $[a,b]$ to the interval $\mathbf{U}$, then $\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t)$ is also a continuous nondecreasing map for $\forall t\in[a,b]$. Therefore by using Proposition 6 we have

$$\Psi\left(\mathbf{L}+\mathbf{M}-\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right)\le\frac{1}{\int_a^b dB(t)}\int_a^b \Psi(\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t))\,dB(t)$$

now by using Lemma 1 of [1] we get

$$\Psi\left(\mathbf{L}+\mathbf{M}-\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right)\le\Psi(\mathbf{L})+\Psi(\mathbf{M})-\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))\,dB(t)}{\int_a^b dB(t)}$$

now if $\frac{\int_a^b \boldsymbol{\Upsilon}(t)dB(t)}{\int_a^b dB(t)}\in\mathbf{U}$ and $\forall x\in a_i\ni(b_{i-1},b_i)(b_0=a,b_l=b)$ and we have either $B(x)\ge B(b)$ or $B(x)\le B(a)$, then by using Lemma 1 of [1] and Remark 2 we have

$$\begin{aligned}\Psi\left(\mathbf{L}+\mathbf{M}-\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right)&\le\Psi(\mathbf{L})+\Psi(\mathbf{M})-\Psi\left(\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right)\\&\le\Psi(\mathbf{L})+\Psi(\mathbf{M})-\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))\,dB(t)}{\int_a^b dB(t)}\end{aligned}$$

which completes the proof. □

We now state the integral Jensen-Mercer inequality under the condition of Jensen-Steffensen inequality (16) is given below:

Corollary 1. *Let $B:[a,b]\to\mathbb{R}$ be a function of bounded variation satisfying*

$$B(a)\le B(x)\le B(b),\quad B(b)>B(a)$$

and let $\boldsymbol{\Upsilon}$ be a continuous nondecreasing map from the real interval $[a,b]$ to the interval $\mathbf{U}$. If $\Psi:\mathbf{I}\to\mathbb{R}$ is a continuous F.W.N.D.I., then

$$\Psi\left(\boldsymbol{L}+\boldsymbol{M}-\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right)\le\Psi(\boldsymbol{L})+f(\boldsymbol{M})-\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))\,dB(t)}{\int_a^b dB(t)}\tag{14}$$

where $\int_a^b \boldsymbol{\Upsilon}\,dB$ is the vector $(\int_a^b \Upsilon_1 d\,B,\ldots,\int_a^b \Upsilon_k\,dB)$, and $\boldsymbol{L}=(L_1,\ldots,L_k)$ and $\boldsymbol{M}=(M_1,\ldots,M_k)$ are two k-tuples related to $\boldsymbol{U}$ such that $\boldsymbol{L}\le\Upsilon(t)\le\boldsymbol{M}$ for all

$t \in [a, b]$.
Also if $\dfrac{\int_a^b \boldsymbol{\Upsilon}(t)dB(t)}{\int_a^b dB(t)} \in \mathbf{U}$ *and we have either* $B(x) \geq B(b)$ *or* $B(x) \leq B(a)$*, then the inequality* (14) *still remains valid.*

Proof. It is nothing but a straight forward case of Theorem 2 for $l = 1$. □

Following is the discrete version of the above inequality:

Theorem 3. *Let* $\Psi : \mathbf{U} \to \mathbb{R}$ *be a continuous F.W.N.D.I. and let* $\mathbf{x}^{(i)} \in \mathbf{U}, i \in \{1, \ldots, n\}$ *satisfy the condition*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad \textit{or} \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

if ***q*** *is a real n-tuple such that* $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad$ *for* $\quad i \in \{1, \ldots, n\}$, *then we have*

$$\Psi\left(\boldsymbol{L} + \boldsymbol{M} - \frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}\right) \leq \Psi(\boldsymbol{L}) + \Psi(\boldsymbol{M}) - \frac{\sum_{i=1}^n q_i \Psi(\mathbf{x}^{(i)})}{Q_n} \tag{15}$$

where $\mathbf{L}, \mathbf{M}$ *are two* $k-tuples$ *in* $\mathbf{U}$ *such that* $\boldsymbol{L} \leq \mathbf{x}^{(i)} \leq \boldsymbol{M}$ *for all* $i \in \{1, \ldots, n\}$. *Also if* $\dfrac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n} \in \mathbf{U}$ *and there exist* $i \in \{1, \ldots, n\}$ *such that for any* $m \in \{2, \ldots, n\}$ $Q_i \leq 0$ *and* $i < m$ *and* $Q_n - Q_{i-1} \leq 0$ *for* $i > m$ *then the inequality* (15) *still remains valid.*

Remark 4. (i) Theorem 1 of [1] becomes special case of our Theorem 3 if we first take **q** as a nonnegative n-tuple with $Q_n > 0$ and then as a real tuple such that $q_1 > 0, q_i \leq 0, i \in \{2, \ldots, n\}, Q_n > 0$.
(ii) Taking $k = 1$ and **q** as a positive n-tuple with $Q_n = 1$ then Theorem 1.2 of [12] and all its related results become special cases of our Theorem 3.

In the rest of this article we assume that $\boldsymbol{\Upsilon}$ is a continuous nondecreasing map from the real interval $[a, b]$ to the interval $\mathbf{I} \subset \mathbb{R}^k$ of the form $\boldsymbol{\Upsilon}(t) = \mathbf{a}t + \mathbf{b}$ where $\mathbf{0} \leq \mathbf{a}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Hence a F.W.N.D.I. Ψ along positively oriented lines with positive direction cosines is convex over this domain $\mathbf{U}$ and we will call this function a F.W.N.D.I. of convex type.

Before going onto our main results we need the following important lemma with respect to the Proposition 3.

Lemma 1. *Let* $\Psi : \mathbf{U} \to \mathbb{R}$ *be a F.W.N.D.I. of convex type then* $\forall\ \mathbf{x}, \mathbf{x_0} \in \mathbf{U}$ *we have*

$$\Psi(\mathbf{x}) - \Psi(\mathbf{x_0}) - \langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle \geq \big| |\Psi(\mathbf{x}) - \Psi(\mathbf{x_0})| - \|\Psi'_+(\mathbf{x_0})\| \|\mathbf{x} - \mathbf{x_0}\| \big|$$

where $\langle \cdot, \cdot \rangle$ *represent usual inner product on* $\mathbb{R}^n$ *and* $\Psi'_+(\mathbf{x_0})$ *represents the right-directional derivative of* Ψ *at* $\mathbf{x_0}$.

Proof. Since Ψ is continuous F.W.N.D.I. of convex type, therefore $\forall\ \mathbf{x}, \mathbf{x_0} \in \mathbf{U}$ from Theorem 3 we have

$$\Psi(\mathbf{x}) - \Psi(\mathbf{x_0}) \geq \langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle$$

which can be written as

$$\begin{aligned} &\Psi(\mathbf{x}) - \Psi(\mathbf{x_0}) - \langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle = |\Psi(\mathbf{x}) - \Psi(\mathbf{x_0}) - \langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle| \\ &\geq \big| |\Psi(\mathbf{x}) - \Psi(\mathbf{x_0})| - |\langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle| \big| \end{aligned}$$

Now applying Cauchy-Schwartz Inequality $|\langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle| \leq \|\Psi'_+(\mathbf{x_0})\| \|\mathbf{x} - \mathbf{x_0}\|$, we conclude

$$\Psi(\mathbf{x}) - \Psi(\mathbf{x_0}) - \langle \Psi'_+(\mathbf{x_0}), \mathbf{x} - \mathbf{x_0} \rangle \geq \big| |\Psi(\mathbf{x}) - \Psi(\mathbf{x_0})| - \|\Psi'_+(\mathbf{x_0})\| \|\mathbf{x} - \mathbf{x_0}\| \big|.$$

□

4. IMPROVEMENTS OF THE JENSEN-BOAS INEQUALITY

Theorem 4. *Let all the assumptions of Proposition* 6 *be valid for the F.W.N.D.I. of convex type. Then we have*

$$\begin{aligned} &\frac{\int_a^b f(\boldsymbol{\Upsilon}(t)) dB(t)}{\int_a^b dB(t)} - \Psi\left(\frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)} \right) \\ &\qquad \geq \left| \frac{\int_a^b |\Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})| dB(t)}{\int_a^b dB(t)} - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \frac{\int_a^b \|\boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|, \end{aligned}$$

where

$$\bar{\boldsymbol{\Upsilon}} = \frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}.$$

Proof. Setting $\mathbf{x} = \boldsymbol{\Upsilon}(t)$ and $\mathbf{x_0} = \bar{\boldsymbol{\Upsilon}}$ for all $t \in [a, b]$ in Lemma 1 we get

$$\Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}}) - \langle \Psi'_+(\bar{\boldsymbol{\Upsilon}}), \boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}} \rangle \geq \big| |\Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})| - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \|\boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| \big|$$

By integrating with respect to function of bounded variation B over interval $[a,b]$ we get

$$\begin{aligned}\int_a^b f(\boldsymbol{\Upsilon}(t))dB(t) - \Psi(\bar{\boldsymbol{\Upsilon}})\int_a^b dB(t) \\ \geq \int_a^b \big||\Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})| - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\|\boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\|\big|dB(t) \\ \geq \left|\int_a^b |\Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})|dB(t) - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\int_a^b \|\boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\|dB(t)\right|\end{aligned}$$

and after dividing by $\int_a^b dB(t) > 0$ we get what we wanted. □

Remark 5. Under the assumptions of Remark 2 the inequality in Theorem 4 get reversed.

The following result gives the improvement of the Jensen-Steffensen inequality.

Corollary 2. *Let all the assumptions of Proposition* 5 *be valid the F.W.N.D.I. of convex type. Then*

$$\begin{aligned}\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \Psi\left(\frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}\right) \\ \geq \left|\frac{\int_a^b |\Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})|dB(t)}{\int_a^b dB(t)} - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\frac{\int_a^b \|\boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\|dB(t)}{\int_a^b dB(t)}\right|\end{aligned}$$

where

$$\bar{\boldsymbol{\Upsilon}} = \frac{\int_a^b \boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}.$$

Proof. The Theorem 4 generates the desired result by using (8) for $l = 1$. □

Remark 6. (i)Under the assumptions of Remark 1 the inequality in Corollary 2 get reversed.
(ii) If we take B as an increasing and bounded function with $B(a) \neq B(b)$ then the integral inequality of Theorem 2.1 of [3] becomes special case of our result for $k = 1$.

The analogous discrete inequality of the above result is given as

Theorem 5. *Let* $\Psi : \mathbf{U} \to \mathbb{R}$ *be a continuous F.W.N.D.I. of convex type, let* $\boldsymbol{q}$ *be a real n-tuple such that* $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1,\ldots,n\}$ *and let* $\mathbf{x}^{(i)} \in \mathbf{U}, i \in \{1,\ldots,n\}$ *be such that*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad or \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

then

$$\begin{aligned}&\frac{\sum_{i=1}^{n} q_i \Psi\left(\mathbf{x}^{(i)}\right)}{Q_n} - \Psi\left(\frac{\sum_{i=1}^{n} q_i \mathbf{x}^{(i)}}{Q_n}\right) \\ &\geq \left| \frac{\sum_{i=1}^{n} q_i \left|\Psi\left(\mathbf{x}^{(i)}\right) - \Psi\left(\bar{\mathbf{x}}\right)\right|}{Q_n} - \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^{n} q_i \|\mathbf{x}^{(i)} - \bar{\mathbf{x}}\|}{Q_n} \right|\end{aligned}$$

where

$$\bar{\mathbf{x}} = \frac{\sum_{i=1}^{n} q_i \mathbf{x}^{(i)}}{Q_n}.$$

Remark 7. (i)If $\frac{\sum_{i=1}^{n} q_i \mathbf{x}^{(i)}}{Q_n} \in \mathbf{U}$ and there exist $i \in \{1, \ldots, n\}$ such that for any $m \in \{2, \ldots, n\}$ $Q_i \leq 0$ for $i < m$ and $Q_n - Q_{i-1} \leq 0$ for $i > m$ then the inequality in Theorem 5 is reversed.
(ii) If we take positive q_i for $i \in \{1, \ldots, n\}$ then the discrete inequality of Theorem 2.1 of [3] becomes a special case of our result for $k = 1$.

5. IMPROVEMENTS OF THE JENSEN-MERCER INEQUALITY

Theorem 6. *Let all the assumptions of Theorem* 2 *be valid for the F.W.N.D.I. of convex type. Then we have*

$$\begin{aligned}&\Psi(\mathbf{L}) + \Psi(\mathbf{M}) - \frac{\int_a^b f(\boldsymbol{\Upsilon}(t)) dB(t)}{\int_a^b dB(t)} - \Psi\left(\mathbf{L} + \mathbf{M} - \frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}\right) \\ &\geq \left| \frac{\int_a^b |\Psi(\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})| dB(t)}{\int_a^b dB(t)} - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \frac{\int_a^b \|\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|\end{aligned}$$

where $\mathbf{L}, \mathbf{M}$ *are two* $k-tuples$ *in* $\mathbf{U}$ *such that* $\boldsymbol{L} \leq \boldsymbol{\Upsilon}(t) \leq \boldsymbol{M}$ *for all* $t \in [a, b]$ *with* $\bar{\boldsymbol{\Upsilon}} = \mathbf{L} + \mathbf{M} - \frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}$.

Proof. For $t \in [a, b]$, set $\mathbf{x} = \mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t)$ and $\mathbf{x_0} = \bar{\boldsymbol{\Upsilon}}$ in Lemma 1 we have

$$\begin{aligned}&\Psi(\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}}) - \langle \Psi'_+(\bar{\boldsymbol{\Upsilon}}), \mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}} \rangle \\ &\geq \left| |\Psi(\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})| - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \|\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| \right|\end{aligned}$$

by integrating with respect to function of bounded variation B over interval $[a,b]$ we get

$$\begin{aligned}
&\int_a^b \Psi(\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t))dB(t) - \Psi(\bar{\boldsymbol{\Upsilon}})\int_a^b dB(t) \\
&\geq \int_a^b \Big||\Psi(\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})| - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\|\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t)-\bar{\boldsymbol{\Upsilon}}\|\Big|dB(t) \\
&\geq \left|\int_a^b |\Psi(\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t)) - \Psi(\bar{\boldsymbol{\Upsilon}})|dB(t) - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\int_a^b \|\mathbf{L}+\mathbf{M}-\boldsymbol{\Upsilon}(t)-\bar{\boldsymbol{\Upsilon}}\|dB(t)\right|.
\end{aligned}$$

and which leads to the desired result by applying Lemma 1 of [1] and dividing by $\int_a^b dB(t) > 0$. $\square$

The analogous discrete inequality for the above result is given as:

Theorem 7. *Let* $f : \mathbf{U} \to \mathbb{R}$ *be a continuous F.W.N.D.I. of convex type and let* $\mathbf{x}^{(i)} \in \mathbf{U}, i \in \{1,\ldots,n\}$ *satisfy the condition*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad or \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

if ***q*** *be a real n-tuple such that* $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1,\ldots,n\}$, *then we have*

$$\begin{aligned}
&\Psi(\mathbf{L}) + \Psi(\mathbf{M}) - \frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{x}^{(i)}\right)}{Q_n} - \Psi\left(\mathbf{L}+\mathbf{M}-\frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}\right) \geq \\
&\left|\frac{\sum_{i=1}^n q_i \left|\Psi\left(\mathbf{L}+\mathbf{M}-\mathbf{x}^{(i)}\right) - \Psi\left(\bar{\mathbf{x}}\right)\right|}{Q_n} - \left\|\Psi'_+\left(\bar{\mathbf{x}}\right)\right\| \frac{\sum_{i=1}^n q_i \|\mathbf{L}+\mathbf{M}-\mathbf{x}^{(i)}-\bar{\mathbf{x}}\|}{Q_n}\right|,
\end{aligned}$$

where $\bar{\mathbf{x}} = \mathbf{L}+\mathbf{M}-\frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}$ *and* ***L*** $= (l_1,\ldots,l_k)$ *and* ***M*** $= (m_1,\ldots,m_k)$ *are two* k*-tuples related to* ***U*** *such that* ***L*** $\leq \mathbf{x}^{(i)} \leq$ ***M*** *for all* $i \in \{1,\ldots,n\}$.

6. IMPROVEMENTS OF THE NEIZGODA'S INEQUALITY

Theorem 8. *Let all the assumptions of Proposition 7 hold for F.W.N.D.I. of convex type, then we have*

$$\begin{aligned}
&\frac{1}{|I^c|}\left(\int_a^b \varphi(\Theta(t))dB(t) - \frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \omega(s)\varphi(\Psi(s,t))d\mu(s)dB(t)\right)\\
&\quad - \varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \omega(s)\Psi(s,t)d\mu(s)dB(t)\right)\right)\\
&\qquad \geq \left| \frac{\int_\Upsilon \omega(s)|\varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right)\varphi(\bar{\boldsymbol{\Upsilon}})|d\mu(s)}{\int_\Upsilon \omega(s)d\mu(s)}\right.\\
&\qquad \left. - \|\varphi'_+(\bar{\boldsymbol{\Upsilon}})\| \frac{\int_\Upsilon \omega(s)\|\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right) - \bar{\boldsymbol{\Upsilon}}\|d\mu(s)}{\int_\Upsilon \omega(s)d\mu(s)}\right|
\end{aligned}$$

where

$$\bar{\boldsymbol{\Upsilon}} = \left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \omega(s)\Psi(s,t)d\mu(s)dB(t)\right)\right).$$

Proof. Setting $\mathbf{x} = \frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)$ and $\mathbf{x_0} = \bar{\boldsymbol{\Upsilon}}$ for all $s \in \boldsymbol{\Upsilon}$ in Lemma 1 we get

$$\begin{aligned}
&\Psi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right) - \Psi(\bar{\boldsymbol{\Upsilon}}) -\\
&\langle \Psi'_+(\bar{\boldsymbol{\Upsilon}}), \frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right) - \bar{\boldsymbol{\Upsilon}}\rangle \geq \Big|\|\Psi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t)\right.\right.\\
&\left.\left. - \int_I \Psi(s,t)dB(t)\right)\right) - \Psi(\bar{\boldsymbol{\Upsilon}})| - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\|\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right) - \bar{\boldsymbol{\Upsilon}}\|\Big|
\end{aligned}$$

multiplying by $\omega(s)$ and integrating with respect $\mu(s)$ over interval Υ we get

$$\begin{aligned}
&\int_{\Upsilon} \omega(s)\varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right)d\mu(s) - \varphi(\bar{\boldsymbol{\Upsilon}})\int_{\Upsilon}\omega(s)d\mu(s)\\
&\quad\geq \int_{\Upsilon}\omega(s)\big||\Psi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right) - \Psi(\bar{\boldsymbol{\Upsilon}})|\\
&\quad - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\|\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right) - \bar{\boldsymbol{\Upsilon}}\|\big|d\mu(s)\\
&\geq \left|\int_{\Upsilon}\omega(s)|\Psi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right) - \Psi(\bar{\boldsymbol{\Upsilon}})|d\mu(s)\right.\\
&\left. - \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\|\int_{\Upsilon}\omega(s)\|\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right) - \bar{\boldsymbol{\Upsilon}}\|d\mu(s)\right|
\end{aligned}$$

and after dividing by $\int_{\Upsilon}\omega(s)d\mu(s) > 0$ and using Lemma 1 of [8] we get our desired result. □

Remark 8. The above inequality still holds for the following assumptions:
Let $\Theta : [a,b] \to \mathbf{U}$ and $\Psi : \Upsilon \times [a,b] \to \mathbf{U}$ be two nonincreasing continuous mappings such that for each $s \in \Upsilon$

$$\begin{aligned}
\int_a^x \Psi(s,t)\,dB(t) &\leq \int_a^x \Theta(t)\,dB(t), \quad \text{for each} \quad x \in (a,b),\\
\text{and} \quad \int_a^b \Psi(s,t)\,dB(t) &= \int_a^b \Theta(t)\,dB(t),
\end{aligned}$$

The analogous discrete inequality of the above theorem is given as:

Theorem 9. *Suppose that $\Psi : \mathbf{U} \to \mathbb{R}$ be a continuous F.W.N.D.I. of convex type. Let there be m elements in $\acute{\mathbf{a}} = \left(\boldsymbol{a}^{(1)},\ldots,\boldsymbol{a}^{(m)}\right)$ such that $\boldsymbol{a}^{(j)} \in \boldsymbol{U}$ is a k-tuple for $j \in \{1,\ldots,m\}$. Also there are $n \times m$ elements represented as $\mathbf{x}^{(ij)} \in \boldsymbol{U}$ s.t. every $\mathbf{x}^{(ij)}$ is a k-tuple for $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m\}$ and all these $\mathbf{x}^{(ij)}$ can be arranged in a $n \times m$ matrix $X = (\mathbf{x}^{(ij)})$. Let $\boldsymbol{q}$ be a real n-tuple such that $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1,\ldots,n\}$ and If $\acute{\mathbf{a}}$ majorizes each row of Υ, that is,*

$$\mathbf{x}^{(i.)} = \left(\mathbf{x}^{(i1)},\ldots,\mathbf{x}^{(im)}\right) \prec \left(\boldsymbol{a}^{(1)},\ldots,\boldsymbol{a}^{(m)}\right) = \acute{\mathbf{a}} \, \textit{for each}\ i \in \{1,\ldots,n\},$$

then we have the inequality

$$\sum_{j=1}^{m} \Psi\left(\mathbf{a}^{(j)}\right) - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m-1} q_i \Psi\left(\mathbf{x}^{(ij)}\right)}{Q_n} - \Psi\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m-1} q_i \mathbf{x}^{(ij)}}{Q_n}\right) \geq$$
$$\left| \frac{\sum_{i=1}^{n} q_i \left|\Psi\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)}\right) - \Psi\left(\bar{\mathbf{x}}\right)\right|}{Q_n} - \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^{n} q_i \| \sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)} - \bar{\mathbf{x}}\|}{Q_n} \right|$$

where

$$\bar{\mathbf{x}} = \sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m-1} q_i \mathbf{x}^{(ij)}}{Q_n}.$$

7. IMPROVEMENTS OF THE LEVINSON INEQUALITY FOR FUNCTIONS WITH NONDECREASING INCREMENTS

Here we first state the improvement of the Levinson inequality by using Jesen-Boas Inequality.

Theorem 10. *Let $B : [a, b] \to \mathbb{R}$ be a function of bounded variation such that (8) holds, and let $\mathbf{\Upsilon}$ be a continuous and nondecreasing map from $[a, b] \subset \mathbb{R}$ to an interval $\mathbf{I} = [\mathbf{0}, \mathbf{d}] \subset \mathbb{R}^k$, $\mathbf{d} > \mathbf{0}$. If Ψ is a continuous F.W.N.D.I. of convex type of order 3 on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, then*

$$\frac{\int_a^b f(2\mathbf{d} - \mathbf{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \Psi\left(2\mathbf{d} - \bar{\mathbf{\Upsilon}}\right) - \frac{\int_a^b f(\mathbf{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} + \Psi\left(\bar{\mathbf{\Upsilon}}\right)$$
$$\geq \left| \frac{\int_a^b |\Psi(2\mathbf{d} - \mathbf{\Upsilon}(t)) - \Psi(\mathbf{\Upsilon}(t)) - \Psi(2\mathbf{d} - \bar{\mathbf{\Upsilon}}) + \Psi(\bar{\mathbf{\Upsilon}})| dB(t)}{\int_a^b dB(t)} - \|\Psi'_+(2\mathbf{d} - \bar{\mathbf{\Upsilon}})\| + \|\Psi'_+(\bar{\mathbf{\Upsilon}})\| \frac{\int_a^b \|\mathbf{\Upsilon}(t) - \bar{\mathbf{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|.$$

where

$$\bar{\mathbf{\Upsilon}} = \frac{\int_a^b \mathbf{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}.$$

Proof. If Ψ is a F.W.N.D.I. of order 3 on $\mathbf{J}$, then

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}\Delta_{\mathbf{s}}\Psi(\mathbf{x}) \geq 0 \quad (\mathbf{x}, \mathbf{x} + \mathbf{h} + \mathbf{t} + \mathbf{s} \in \mathbf{J},\ \mathbf{0} \leq \mathbf{h}, \mathbf{t}, \mathbf{s} \in \mathbb{R}^k),$$

i.e.,

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}(\Psi(\mathbf{x}+\mathbf{s})-\Psi(\mathbf{x}))\geq 0. \tag{16}$$

If $\mathbf{x}\in\mathbf{I}$ and $\mathbf{s}=2\mathbf{d}-2\mathbf{x}$, we have

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}(\Psi(2\mathbf{d}-\mathbf{x})-\Psi(\mathbf{x}))\geq 0$$

i.e. the function $\mathbf{x}\mapsto\Psi(2\mathbf{d}-\mathbf{x})-\Psi(\mathbf{x})$ is a F.W.N.D.I. of order 2, i.e., it is a F.W.N.D.I. Now, by replacing $\Psi(\mathbf{x})$ by $\Psi(2\mathbf{d}-\mathbf{x})-\Psi(\mathbf{x})$ in Theorem 4, we obtain Theorem 10. □

Theorem 11. *Let $B:[a,b]\to\mathbb{R}$ be a function of bounded variation such that (8) holds, and let Ψ be a continuous F.W.N.D.I. of convex type of order 3 on $[\mathbf{r},\mathbf{d}]\subset\mathbb{R}^k$. Let $\mathbf{0}<\mathbf{p}<\mathbf{d}-\mathbf{r}$. If $\boldsymbol{\Upsilon}(t):[a,b]\to[\mathbf{r},\mathbf{d}-\mathbf{p}]$ is a continuous and nondecreasing map, then*

$$\begin{aligned}\frac{\int_a^b f(\mathbf{p}+\boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)}-\Psi\left(\mathbf{p}+\bar{\boldsymbol{\Upsilon}}\right)-\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)}+\Psi\left(\bar{\boldsymbol{\Upsilon}}\right)\\ \geq\left|\frac{\int_a^b|\Psi(\mathbf{p}+\boldsymbol{\Upsilon}(t))-\Psi(\boldsymbol{\Upsilon}(t))-\Psi(\mathbf{p}+\bar{\boldsymbol{\Upsilon}})+\boldsymbol{\Psi}(\bar{\boldsymbol{\Upsilon}})|\mathbf{dB}(\mathbf{t})}{\int_a^b dB(t)}\right.\\ \left.-\|\Psi'_+(\mathbf{p}+\bar{\boldsymbol{\Upsilon}})\|+\|\boldsymbol{\Psi}'_+(\bar{\boldsymbol{\Upsilon}})\|\frac{\int_{\mathbf{a}}^{\mathbf{b}}\|\boldsymbol{\Upsilon}(\mathbf{t})-\bar{\boldsymbol{\Upsilon}}\|\mathbf{dB}(\mathbf{t})}{\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{dB}(\mathbf{t})}\right|,\end{aligned}$$

where

$$\bar{\boldsymbol{\Upsilon}}=\frac{\int_a^b\boldsymbol{\Upsilon}(t)\,dB(t)}{\int_a^b dB(t)}.$$

Proof. Using (16) for $\mathbf{s}=\mathbf{c}=\text{constant}\in\mathbb{R}^k$, we have $\Psi(\mathbf{p}+\mathbf{x})-\Psi(\mathbf{x})$ is a F.W.N.D.I., now applying Theorem 4 to get the Theorem 11. □

We now give the improvement of the Levinson inequality in accordance to Jensen-Steffensen inequality.

Corollary 3. *Let $B:[a,b]\to\mathbb{R}$ be a function of bounded variation such that (6) holds, and let $\boldsymbol{\Upsilon}$ be a continuous and nondecreasing map from $[a,b]\subset\mathbb{R}$ to an interval $\mathbf{I}=[\mathbf{0},\mathbf{d}]\subset\mathbb{R}^k$, $\mathbf{d}>\mathbf{0}$. If Ψ is a continuous F.W.N.D.I. of convex type of order 3 on*

$\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, *then we have*

$$\frac{\int_a^b f(2\mathbf{d} - \mathbf{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \Psi\left(2\mathbf{d} - \bar{\mathbf{\Upsilon}}\right) - \frac{\int_a^b \Psi(\mathbf{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} + \Psi\left(\bar{\mathbf{\Upsilon}}\right)$$
$$\geq \left| \frac{\int_a^b |\Psi(2\mathbf{d} - \mathbf{\Upsilon}(t)) - \Psi(\mathbf{\Upsilon}(t)) - \Psi(2\mathbf{d} - \bar{\mathbf{\Upsilon}}) + \Psi(\bar{\mathbf{\Upsilon}})| dB(t)}{\int_a^b dB(t)} \right.$$
$$\left. - \|\Psi'_+(2\mathbf{d} - \bar{\mathbf{\Upsilon}})\| + \|\Psi'_+(\bar{\mathbf{\Upsilon}})\| \frac{\int_a^b \|\mathbf{\Upsilon}(t) - \bar{\mathbf{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|$$

where

$$\bar{\mathbf{\Upsilon}} = \frac{\int_a^b \mathbf{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}.$$

Proof. It can be directly obtained from Theorem 10 by taking $l = 1$ in (8). □

The analogues discrete result for the above corollary is given as

Theorem 12. *Let* $\boldsymbol{q}$ *be a real n-tuple such that* $0 \leq Q_i \leq Q_n$, *with* $Q_n > 0$ *for* $i \in \{1, \ldots, n\}$,$\mathbf{x}^{(i)} \in [\mathbf{0}, \mathbf{d}]^n \subset \mathbf{U}$ *with* $(\mathbf{d} > \mathbf{0})$ *be such that*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad \textbf{or} \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

then for a continuous F.W.N.D.I. of convex type Ψ *of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, *we have*

$$\frac{\sum_{i=1}^n q_i \Psi\left(2\mathbf{d} - \mathbf{x}^{(i)}\right)}{Q_n} - \Psi\left(2\mathbf{d} - \bar{\mathbf{x}}\right) - \frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{x}^{(i)}\right)}{Q_n} + \Psi\left(\bar{\mathbf{x}}\right) \geq$$
$$\left| \frac{\sum_{i=1}^n q_i \left|\Psi\left(2\mathbf{d} - \mathbf{x}^{(i)}\right) - \Psi\left(\mathbf{x}^{(i)}\right) - \Psi\left(2\mathbf{d} - \bar{\mathbf{x}}\right) + \Psi\left(\bar{\mathbf{x}}\right)\right|}{Q_n} \right.$$
$$\left. - \|\Psi'_+\left(2\mathbf{d} - \bar{\mathbf{x}}\right)\| + \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^n q_i \|\mathbf{x}^{(i)} - \bar{\mathbf{x}}\|}{Q_n} \right|,$$

where

$$\bar{\mathbf{x}} = \frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}$$

Remark 9. For $k = 1$, the Theorem 4.1 of [9] becomes a special case of our Theorem 12.

Corollary 4. *Let $B : [a, b] \to \mathbb{R}$ be a function of bounded variation such that (6) holds, and let Ψ be a continuous F.W.N.D.I. of order 3 on $[\mathbf{r}, \mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{p} < \mathbf{d} - \mathbf{r}$. If $\boldsymbol{\Upsilon}(t) : [a, b] \to [\mathbf{r}, \mathbf{d} - \mathbf{p}]$ is a continuous and nondecreasing map, then*

$$\begin{aligned}
&\frac{\int_a^b \Psi(\mathbf{p} + \boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \Psi\left(\mathbf{p} + \bar{\boldsymbol{\Upsilon}}\right) - \frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} + \Psi\left(\bar{\boldsymbol{\Upsilon}}\right) \\
&\quad \geq \left| \frac{\int_a^b |\Psi(\mathbf{p} + \boldsymbol{\Upsilon}(t)) - \Psi(\boldsymbol{\Upsilon}(t)) - \Psi(\mathbf{p} + \bar{\boldsymbol{\Upsilon}}) + \Psi(\bar{\boldsymbol{\Upsilon}})| dB(t)}{\int_a^b dB(t)} \right. \\
&\qquad \left. - \|\Psi'_+(\mathbf{p} + \bar{\boldsymbol{\Upsilon}})\| + \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \frac{\int_a^b \|\boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|,
\end{aligned}$$

where

$$\bar{\boldsymbol{\Upsilon}} = \frac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b \, dB(t)}.$$

Proof. It is a straight forward result from Theorem 11 with $l = 1$ in (8). $\square$

The analogues discrete result for the above Corollary is given as

Theorem 13. *Let **w** be a nonnegative n-tuple such that **q** be a real n-tuple such that $0 \leq Q_i \leq Q_n$, with $Q_n > 0$ for $i \in \{1, \ldots, n\}$, $\mathbf{x}^{(i)} \in [\mathbf{r}, \mathbf{d} - \mathbf{p}]^n \subset \mathbf{U}$ with $\mathbf{0} < \mathbf{p} < \mathbf{d} - \mathbf{r}$ be such that*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad \textbf{or} \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

then for a continuous F.W.N.D.I. Ψ of convex type of order 3 on $\mathbf{J} = [\mathbf{r}, \mathbf{d}]$, we have

$$\begin{aligned}
&\frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{p} + \mathbf{x}^{(i)}\right)}{Q_n} - \Psi\left(\mathbf{p} + \bar{\mathbf{x}}\right) - \frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{x}^{(i)}\right)}{Q_n} + \Psi\left(\bar{\mathbf{x}}\right) \\
&\quad \geq \left| \frac{\sum_{i=1}^n q_i \left|\Psi\left(\mathbf{p} + \mathbf{x}^{(i)}\right) - \Psi\left(\mathbf{x}^{(i)}\right) - \Psi\left(\mathbf{p} + \bar{\mathbf{x}}\right) + \Psi\left(\bar{\mathbf{x}}\right)\right|}{Q_n} \right. \\
&\qquad \left. - \|\Psi'_+\left(\mathbf{p} + \bar{\mathbf{x}}\right)\| + \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^n q_i \|\mathbf{x}^{(i)} - \bar{\mathbf{x}}\|}{Q_n} \right|
\end{aligned}$$

where

$$\bar{\mathbf{x}} = \frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}$$

We now give the improvement of the Levinson inequality in accordance to Jensen-Mercer inequality.

Theorem 14. *Let $B : [a,b] \to \mathbb{R}$ be a function of bounded variation such that* (6) *holds, and let* $\boldsymbol{\Upsilon}$ *be a continuous and nondecreasing map from* $[a,b] \subset \mathbb{R}$ *to an interval* $\mathbf{I} = [\mathbf{0}, \mathbf{d}] \subset \mathbb{R}^k$, $\mathbf{d} > \mathbf{0}$. *If* Ψ *is a continuous F.W.N.D.I. of convex type having order* 3 *on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, *then*

$$\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \frac{\int_a^b f(2\mathbf{d} - \boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \Psi\left(2\mathbf{d} - \bar{\boldsymbol{\Upsilon}}\right) + \Psi\left(\bar{\boldsymbol{\Upsilon}}\right)$$
$$\geq \left| \frac{\int_a^b |\Psi(\mathbf{L} + \mathbf{M} - 2\mathbf{d} + \boldsymbol{\Upsilon}(t)) - \Psi(\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t)) - \Psi(2\mathbf{d} - \bar{\boldsymbol{\Upsilon}}) + \Psi(\bar{\boldsymbol{\Upsilon}})| dB(t)}{\int_a^b dB(t)} \right.$$
$$\left. + - \|\Psi'_+(2\mathbf{d} - \bar{\boldsymbol{\Upsilon}})\| \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \frac{\int_a^b \|\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|.$$

Where $\bar{\boldsymbol{\Upsilon}} = \mathbf{L} + \mathbf{M} - \dfrac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}$ *and* $\mathbf{L}, \mathbf{M}$ *are two* $k - tuples$ *in* $\mathbf{U}$*such that* $\boldsymbol{L} \leq \boldsymbol{\Upsilon}(t) \leq \boldsymbol{M}$ *for all* $t \in [a,b]$.

Proof. Adopting the Theorem 10 and replacing $\Psi(\mathbf{x})$ by $\Psi(2\mathbf{d} - \mathbf{x}) - \Psi(\mathbf{x})$ in Theorem 6 we get the desired result. □

The analogues discrete inequality of the above result is given as

Theorem 15. *Let* $\boldsymbol{q}$ *be a real n-tuple such that* $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1, \ldots, n\}$ *and let for* $i \in \{1, \ldots, n\}, \mathbf{x}^{(i)} \in [\mathbf{0}, \mathbf{d}]^n \subset \mathbf{U}$ *with* $(\mathbf{d} > \mathbf{0})$ *be such that*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad or \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

then for a continuous F.W.N.D.I., Ψ *of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, *we have*

$$\frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{x}^{(i)}\right)}{Q_n} - \Psi\left(2\mathbf{d} - \bar{\mathbf{x}}\right) - \frac{\sum_{i=1}^n q_i \Psi\left(2\mathbf{d} - \mathbf{x}^{(i)}\right)}{Q_n} + \Psi\left(\bar{\mathbf{x}}\right)$$
$$\geq \left| \frac{\sum_{i=1}^n q_i \left| \Psi\left(\mathbf{L} + \mathbf{M} - 2\mathbf{d} + \mathbf{x}^{(i)}\right) - \Psi\left(\mathbf{L} + \mathbf{M} - \mathbf{x}^{(i)}\right) - \Psi\left(2\mathbf{d} - \bar{\mathbf{x}}\right) + \Psi\left(\bar{\mathbf{x}}\right) \right|}{Q_n} \right.$$
$$\left. + - \|\Psi'_+\left(2\mathbf{d} - \bar{\mathbf{x}}\right)\| \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^n q_i \|\mathbf{L} - \mathbf{M} - \mathbf{x}^{(i)} - \bar{\mathbf{x}}\|}{Q_n} \right|,$$

where $\bar{\mathbf{x}} = \mathbf{L} + \mathbf{M} - \frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}$*and* $\boldsymbol{L} = (l_1, \ldots, l_k)$ *and* $\boldsymbol{M} = (m_1, \ldots, m_k)$ *are two* k*-tuples related to* $\boldsymbol{U}$ *such that* $\boldsymbol{L} \leq \mathbf{x}^{(i)} \leq \boldsymbol{M}$ *for all* $i \in \{1, \ldots, n\}$.

Theorem 16. *Let $B : [a, b] \to \mathbb{R}$ be a function of bounded variation such that (6) holds, and let Ψ be a continuous F.W.N.D.I. of order 3 on $[\mathbf{r}, \mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{p} < \mathbf{d} - \mathbf{r}$. If $\boldsymbol{\Upsilon}(t) : [a, b] \to [\mathbf{r}, \mathbf{d} - \mathbf{p}]$ is a continuous and nondecreasing map, then*

$$\frac{\int_a^b \Psi(\boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \frac{\int_a^b \Psi(\mathbf{p} + \boldsymbol{\Upsilon}(t))dB(t)}{\int_a^b dB(t)} - \Psi\left(\mathbf{p} + \bar{\boldsymbol{\Upsilon}}\right) + \Psi\left(\bar{\boldsymbol{\Upsilon}}\right)$$
$$\geq \left| \frac{\int_a^b |\Psi(\mathbf{L} + \mathbf{M} - \mathbf{p} - \boldsymbol{\Upsilon}(t)) - \Psi(\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t)) - \Psi(\mathbf{p} + \bar{\boldsymbol{\Upsilon}}) + \Psi(\bar{\boldsymbol{\Upsilon}})| dB(t)}{\int_a^b dB(t)} \right.$$
$$\left. + - \|\Psi'_+(\mathbf{p} + \bar{\boldsymbol{\Upsilon}})\| \|\Psi'_+(\bar{\boldsymbol{\Upsilon}})\| \frac{\int_a^b \|\mathbf{L} + \mathbf{M} - \boldsymbol{\Upsilon}(t) - \bar{\boldsymbol{\Upsilon}}\| dB(t)}{\int_a^b dB(t)} \right|,$$

where $\bar{\boldsymbol{\Upsilon}} = \mathbf{L} + \mathbf{M} - \dfrac{\int_a^b \boldsymbol{\Upsilon}(t)\, dB(t)}{\int_a^b dB(t)}$ *and* $\mathbf{L}, \mathbf{M}$ *are two* $k-tuples$ *in* $\mathbf{U}$ *such that* $\boldsymbol{L} \leq \boldsymbol{\Upsilon}(t) \leq \boldsymbol{M}$ *for all* $t \in [a, b]$.

Proof. Adopting Theorem 11 and replacing $\Psi(\mathbf{x})$ by $\Psi(\mathbf{p} + \mathbf{x}) - \boldsymbol{\Psi}(\mathbf{x})$ in Theorem 6 we get the desired result. □

The analogous discrete inequality of the above result is given as

Theorem 17. *Let $\boldsymbol{q}$ be a real n-tuple such that $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1, \dots, n\}$ and let for $i \in \{1, \dots, n\}, \mathbf{x}^{(i)} \in [\mathbf{r}, \mathbf{d} - \mathbf{p}]^n \subset \mathbf{U}$ with $\mathbf{0} < \mathbf{p} < \mathbf{d} - \mathbf{r}$ be such that*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad or \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$$

then for a continuous F.W.N.D.I. Ψ of order three on $\mathbf{J} = [\mathbf{r}, \mathbf{d}]$, we have

$$\frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{x}^{(i)}\right)}{Q_n} - \Psi\left(\mathbf{p} + \bar{\mathbf{x}}\right) - \frac{\sum_{i=1}^n q_i \Psi\left(\mathbf{p} + \mathbf{x}^{(i)}\right)}{Q_n} + \Psi\left(\bar{\mathbf{x}}\right) \geq$$
$$\left| \frac{\sum_{i=1}^n q_i \left|\Psi\left(\mathbf{L} + \mathbf{M} - \mathbf{p} - \mathbf{x}^{(i)}\right) - \Psi\left(\mathbf{L} - \mathbf{M} - \mathbf{x}^{(i)}\right) - \Psi\left(\mathbf{p} + \bar{\mathbf{x}}\right) + \Psi\left(\bar{\mathbf{x}}\right)\right|}{Q_n} \right.$$
$$\left. + - \|\Psi'_+\left(\mathbf{p} + \bar{\mathbf{x}}\right)\| \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^n q_i \|\mathbf{L} - \mathbf{M} - \mathbf{x}^{(i)} - \bar{\mathbf{x}}\|}{Q_n} \right|$$

where $\bar{\mathbf{x}} = \mathbf{L} + \mathbf{M} - \frac{\sum_{i=1}^n q_i \mathbf{x}^{(i)}}{Q_n}$ *and* $\boldsymbol{L} = (l_1, \dots, l_k)$ *and* $\boldsymbol{M} = (m_1, \dots, m_k)$ *are two k-tuples related to* $\boldsymbol{U}$*such that* $\boldsymbol{L} \leq \mathbf{x}^{(i)} \leq \boldsymbol{M}$ *for all* $i \in \{1, \dots, n\}$.

We now state improvement of the Levinson inequality by using Neizgoda's inequality of Theorem 7.

Theorem 18. *Let (Υ, Σ, μ) be a measure space with positive finite measure μ and consider the weight function $\omega : \Upsilon \to [0, +\infty)$. Let Θ and Ψ be two nondecreasing continuous mappings from $[a, b] \subset \mathbb{R}$ to an interval $\mathbf{I} = [\mathbf{0}, \mathbf{d}] \subset \mathbb{R}^k$, $\mathbf{d} > \mathbf{0}$ such that for each $s \in \Upsilon$*

$$\begin{aligned} \int_x^b \Psi(s,t)\, dB(t) &\le \int_x^b \Theta(t) dB(t), \quad \textit{for each} \quad x \in (a,b), \\ \textit{and} \quad \int_a^b \Psi(s,t)\, dB(t) &= \int_a^b \Theta(t)\, dB(t), \end{aligned}$$

where $B : [a, b] \to \mathbb{R}$ is of bounded variation. Moreover, let $I = \bigcup_{i=1}^{k}(a_i, b_i)$, $I^c = [a, b] \backslash I = \bigcup_{i=1}^{k+1}[b_{i-1}, a_i]$ and $|I^c| = \sum_{i=1}^{k+1}(a_i - b_{i-1})$ where $a = b_0 \le a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k \le a_{k+1} = b$ is a partition of the interval $[a, b]$. Then for every continuous function φ with nondecreasing increments of convex type having order 3 on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, the following inequality holds.

$$\begin{aligned} &\frac{1}{|I^c|}\left(\frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \varphi(\Psi(s,t))d\mu(s)dB(t)\right) \\ &- \frac{1}{|I^c|}\left(\frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \varphi(2\mathbf{d} - \Psi(s,t))d\mu(s)dB(t)\right) - \varphi\left(2\mathbf{d} - \bar{\mathbf{\Upsilon}}\right) + \varphi\left(\bar{\mathbf{\Upsilon}}\right) \\ &\ge \Bigg| \frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\Bigg[\int_\Upsilon \omega(s)|\varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I (2\mathbf{d} - \Psi(s,t))dB(t)\right)\right) \\ &- \varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right) - \varphi\left(2\mathbf{d} - \bar{\mathbf{\Upsilon}}\right) + \varphi(\bar{\mathbf{\Upsilon}})|d\mu(s)\Bigg] + \|\varphi'_+(2\mathbf{d} - \bar{\mathbf{\Upsilon}})\| \\ &- \|\varphi'_+(\bar{\mathbf{\Upsilon}})\| \frac{\int_\Upsilon \omega(s)\|\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I (2\mathbf{d} - \Psi(s,t))dB(t)\right)\right) - \bar{\mathbf{\Upsilon}}\|d\mu(s)}{\int_\Upsilon \omega(s)d\mu(s)}\Bigg|. \end{aligned} \tag{17}$$

where

$$\bar{\mathbf{\Upsilon}} = \left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \omega(s)\Psi(s,t)d\mu(s)dB(t)\right)\right).$$

Proof. Following Theorem 10 and replacing $\varphi(\Psi)$ by $\varphi(2\mathbf{d} - \Psi) - \varphi(\Psi)$ in the Theorem 8 we get the desired result.

□

Remark 10. The above inequality still holds for the following assumptions:
Let $\Theta : [a,b] \to \mathbf{U}$ and $f : \Upsilon \times [a,b] \to \mathbf{U}$ be two nonincreasing continuous mappings such that for each $s \in \Upsilon$

$$\begin{aligned} \int_a^x \Psi(s,t)\,dB(t) &\le \int_a^x \Theta(t)\,dB(t), \quad \text{for each} \quad x \in (a,b), \\ \text{and} \quad \int_a^b \Psi(s,t)\,dB(t) &= \int_a^b \Theta(t)\,dB(t), \end{aligned}$$

The discrete analogous inequality for the above result is given as

Theorem 19. *Let there are m elements in $\acute{\mathbf{a}} = \left(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(m)}\right)$ in a way that every $\boldsymbol{a}^{(j)} \in \boldsymbol{I}$ is a k-tuple for $j \in \{1,\ldots,m\}$. Also there be $n \times m$ elements represented as $\mathbf{x}^{(ij)} \in [\mathbf{0},\mathbf{d}]^n \subset \mathbf{U}$ with $(\mathbf{d} > \mathbf{0})$ such that every $\mathbf{x}^{(ij)}$ is a k-tuple for $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m\}$ and all these $\mathbf{x}^{(ij)}$ can be arranged in a $n \times m$ matrix $X = (\mathbf{x}^{(ij)})$. Let $\boldsymbol{q}$ be a real n-tuple such that $0 \le Q_i \le Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1,\ldots,n\}$. If $\acute{\mathbf{a}}$ majorizes each row of Υ, that is,*

$$\mathbf{x}^{(i.)} = \left(\mathbf{x}^{(i1)}, \ldots, \mathbf{x}^{(im)}\right) \prec \left(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(m)}\right) = \acute{\mathbf{a}}\, \textit{for each}\ i \in \{1,\ldots,n\},$$

then for a continuous F.W.N.D.I. Ψ of order three on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, we have the inequality

$$\begin{aligned} &\frac{\sum_{i=1}^n \sum_{j=1}^{m-1} q_i \Psi\left(\mathbf{x}^{(ij)}\right)}{Q_n} + \Psi\left(2\mathbf{d} - \bar{\mathbf{x}}\right) - \frac{\sum_{i=1}^n \sum_{j=1}^{m-1} q_i \Psi\left(2\mathbf{d} - \mathbf{x}^{(ij)}\right)}{Q_n} + \Psi\left(\bar{\mathbf{x}}\right) \ge \\ &\left| \frac{\sum_{i=1}^n q_i \left| \Psi\left(\sum_{j=1}^m \mathbf{a}^{(j)} - \sum_{j=1}^{m-1}\left(2\mathbf{d} - \mathbf{x}^{(ij)}\right)\right) - \Psi\left(\sum_{j=1}^m \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)}\right) - \Psi\left(2\mathbf{d} - \bar{\mathbf{x}}\right) + \Psi\left(\bar{\mathbf{x}}\right) \right|}{Q_n} \right. \\ &\left. - \left\|\Psi'_+\left(2\mathbf{d} - \bar{\mathbf{x}}\right)\right\| + \left\|\Psi'_+\left(\bar{\mathbf{x}}\right)\right\| \frac{\sum_{i=1}^n q_i \left\| \sum_{j=1}^m \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)} - \bar{\mathbf{x}} \right\|}{Q_n} \right|, \end{aligned}$$

where

$$\bar{\mathbf{x}} = \sum_{j=1}^m \mathbf{a}^{(j)} - \frac{\sum_{i=1}^n \sum_{j=1}^{m-1} q_i \mathbf{x}^{(ij)}}{Q_n}.$$

Theorem 20. *Let (Υ, Σ, μ) be a measure space with positive finite measure μ and consider the weight function $\omega : \Upsilon \to [0, +\infty)$. Let Θ and Ψ be two nondecreasing continuous mappings from $[a,b] \subset \mathbb{R}$ to an interval $[\mathbf{r}, \mathbf{d} - \mathbf{p}] \subset \mathbb{R}^k$, with $\mathbf{0} < \mathbf{p} < \mathbf{d} - \mathbf{r}$ such that for each $s \in \Upsilon$*

$$\begin{aligned} \int_x^b \Psi(s,t)\,dB(t) &\le \int_x^b \Theta(t) dB(t), \quad \textit{for each} \quad x \in (a,b), \\ \textit{and} \quad \int_a^b \Psi(s,t)\,dB(t) &= \int_a^b \Theta(t)\,dB(t), \end{aligned}$$

where $B : [a,b] \to \mathbb{R}$ *is of bounded variation. Moreover, let* $I = \bigcup_{i=1}^{k}(a_i, b_i)$, $I^c = [a,b]\backslash I = \bigcup_{i=1}^{k+1}[b_{i-1}, a_i]$ *and* $|I^c| = \sum_{i=1}^{k+1}(a_i - b_{i-1})$ *where* $a = b_0 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k \leq a_{k+1} = b$ *is a partition of the interval* $[a,b]$*. Then for every continuous function* φ *with nondecreasing increments of convex type having order* 3 *on* $\mathbf{J} = [\mathbf{r}, \mathbf{d}] \subset \mathbb{R}^k$*, the following inequality holds.*

$$\begin{aligned}
&\frac{1}{|I^c|}\left(\frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \varphi(\Psi(s,t))d\mu(s)dB(t)\right) \\
&-\frac{1}{|I^c|}\left(\frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \varphi(\mathbf{p}+\Psi(s,t))d\mu(s)dB(t)\right) - \varphi\left(2\mathbf{p}+\bar{\mathbf{\Upsilon}}\right) + \varphi\left(\bar{\mathbf{\Upsilon}}\right) \\
&\geq \left|\frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\left[\int_\Upsilon \omega(s)|\varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I(\mathbf{p}+\Psi(s,t))dB(t)\right)\right)\right.\right. \\
&-\left.\varphi\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I \Psi(s,t)dB(t)\right)\right) - \varphi\left(\mathbf{p}+\bar{\mathbf{\Upsilon}}\right) + \varphi(\bar{\mathbf{\Upsilon}})|d\mu(s)\right] + \|\varphi'_+(\mathbf{p}-\bar{\mathbf{\Upsilon}})\| \\
&-\left.\|\varphi'_+(\bar{\mathbf{\Upsilon}})\|\frac{\int_\Upsilon \omega(s)\|\left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \int_I(\mathbf{p}+\Psi(s,t))dB(t)\right)\right) - \bar{\mathbf{\Upsilon}}\|d\mu(s)}{\int_\Upsilon \omega(s)d\mu(s)}\right|,
\end{aligned} \tag{18}$$

where

$$\bar{\mathbf{\Upsilon}} = \left(\frac{1}{|I^c|}\left(\int_a^b \Theta(t)dB(t) - \frac{1}{\int_\Upsilon \omega(s)d\mu(s)}\int_I\int_\Upsilon \omega(s)\Psi(s,t)d\mu(s)dB(t)\right)\right)$$

Proof. Following Theorem 11 and replacing $\varphi(\mathbf{f})$ by $\varphi(\mathbf{p}+\mathbf{f}) - \varphi(\mathbf{f})$ in Theorem 8 we get the desired result. $\square$

Remark 11. The above inequality still holds for the following assumptions:
Let $\Theta : [a,b] \to \mathbf{U}$ and $f : \Upsilon \times [a,b] \to \mathbf{U}$ be two nonincreasing continuous mappings such that for each $s \in \Upsilon$

$$\begin{aligned}
\int_a^x \Psi(s,t)\,dB(t) &\leq \int_a^x \Theta(t)\,dB(t), \quad \text{for each} \quad x \in (a,b), \\
\text{and} \quad \int_a^b \Psi(s,t)\,dB(t) &= \int_a^b \Theta(t)\,dB(t),
\end{aligned}$$

The analogous discrete inequality for the above result is given as:

Theorem 21. *Let there are* m *elements in* $\acute{\mathbf{a}} = \left(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(m)}\right)$ *in a way that every* $\boldsymbol{a}^{(j)} \in \boldsymbol{I}$ *is a* k*-tuple for* $j \in \{1,\ldots,m\}$*. Also there are* $n \times m$ *elements represented as* $\mathbf{x}^{(ij)} \in [\mathbf{r}, \mathbf{d}-\mathbf{p}]^n \subset \mathbf{U}$ *with* $\mathbf{0} < \mathbf{p} < \mathbf{d}-\mathbf{r}$ *such that every* $\mathbf{x}^{(ij)}$ *is a* k*-tuple for* $i \in \{1,\ldots,n\}$*,* $j \in \{1,\ldots,m\}$ *and all these* $\mathbf{x}^{(ij)}$ *can be arranged in a* $n \times m$

matrix $X = (\mathbf{x}^{(ij)})$. *Let* $\boldsymbol{q}$ *be a real n-tuple such that* $0 \leq Q_i \leq Q_n, \quad with \quad Q_n > 0 \quad for \quad i \in \{1, \ldots, n\}$. *If* $\acute{\mathbf{a}}$ *majorizes each row of* Υ*, that is,*

$$\mathbf{x}^{(i.)} = \left(\mathbf{x}^{(i1)}, \ldots, \mathbf{x}^{(im)}\right) \prec \left(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(m)}\right) = \acute{\mathbf{a}} \textit{ for each } i \in \{1, \ldots, n\},$$

then for a continuous F.W.N.D.I. Ψ *of order three on* $\mathbf{J} = \mathbf{J} = [\mathbf{r}, \mathbf{d}]$*, we have the inequality*

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{m-1} q_i \Psi\left(\mathbf{x}^{(ij)}\right)}{Q_n} + \Psi\left(\mathbf{p} + \bar{\mathbf{x}}\right) - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m-1} q_i \Psi\left(\mathbf{p} + \mathbf{x}^{(ij)}\right)}{Q_n} + \Psi\left(\bar{\mathbf{x}}\right) \geq$$
$$\left| \frac{\sum_{i=1}^{n} q_i \left| \Psi\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1}\left(\mathbf{p} + \mathbf{x}^{(ij)}\right)\right) - \Psi\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)}\right) - \Psi\left(\mathbf{p} + \bar{\mathbf{x}}\right) + \Psi\left(\bar{\mathbf{x}}\right) \right|}{Q_n} \right.$$
$$\left. - \|\Psi'_+\left(\mathbf{p} + \bar{\mathbf{x}}\right)\| + \|\Psi'_+\left(\bar{\mathbf{x}}\right)\| \frac{\sum_{i=1}^{n} q_i \| \sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)} - \bar{\mathbf{x}} \|}{Q_n} \right|,$$

where

$$\bar{\mathbf{x}} = \sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m-1} q_i \mathbf{x}^{(ij)}}{Q_n}.$$

ACKNOWLEDGMENTS

This research study is funded by Dean's Research Grant, Dean Sciences, University of Karachi, Karachi, Pakistan.

REFERENCES

[1] M. Klaričić Bakula, Anita Matković and J. Pečarić, On a variant of Jensen's inequality for functions with nondecreasing increments, *J. Korean Math. Soc.*, **45** (3) (2008), 821–834.

[2] H. D. BRUNK, Integral inequalities for functions with nondecreasing increments., *Pacific J. Math.* **14** (1964), 783–793.

[3] S. HUSSAIN, J. E. PEČARIĆ, An improvment of Jensen's Inquality With Some Apllications, *Asian Euorepian Journal of Mathematics.*, **2** (1) (2009), 85–94.

[4] M. Maqsood Ali and Asif R. Khan, Generalized Integral Niezgoda's Inequalities, *Journal of Inequalities and Special Functions*, **10**, (1) 2019, 60-76.

[5] Asif R. Khan, J. Pečarić, M. Praljak and S. Varošanec, General Linear Inequalities and Positivity / Higher order convexity. *Monographs in inequalities* **12**, Element, Zagreb, 2017., pp. 269. -

[6] Asif R. Khan and Sumayyah Saadi, Generalized Jensen-Mercer Inequality for Functions with Nondecreasing increments, *Abs. Appl. Anal.*, **2016** (2016), Article ID 5231476, 1-12.

[7] Asif R. Khan, Josip Pečarić and Sanja Varosanec, On some inequalities for functions with non-decreasing increments of higher order, *J. Inequal. Appl.* **2013**, 2013:8 1-14

[8] A.R Khan, Inam Ullah Khan, An Extension of Jensen-Mercer Inequality For Functions With Nondecreasing Increments,*Journal of Inequalities and Special Functions*,(In Press)

[9] M. Adil Khan, M. Anwar, J. Jakšetić and J. Pečarić, On some improvements of the Jensen inequality with some applications, *J. Inequal. Appl.*, **2009** (2009), Article ID 323615, 1–15.

[10] Erwin Kreszig, *Introductory functional analysis with applications*, John Wiley & Sons, New York 1978.

[11] Levinson, N, Generalization of an inequality of Ky Fan. *J. Math. Anal. Appl.* **8**, (1964) 133-134.

[12] A. McD. Mercer, A variant of Jensen's inequality, *J. Ineq. Pure and Appl. Math.*, **4**(4) (2003), Article 73.

[13] C. P. Niculescu and L. E. Persson, *Convex functions and their applications. A contemporary approach*, CMS Books in Mathematics, Vol. **23**, Springer-Verlage, New York 2006.

[14] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, *Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[15] J. E. Pečarić, On some inequalities for functions with nondecreasing increments, *J. Math. Anal. Appl.* **98** (1984), 188–197.