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On the Projective Algebra of Kropina space

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Abstract

In this paper we studied the projective algebra and Lie algebra of the projective group $P(M,F)$. The projective algebra of Kropina space is characterized as a certain Lie subalgebra of the projective algebra $p(M,\alpha)$. Further, we proved that the Kropina space of vanishing S- curvaruter admits a non- α- affine projective vector field is Berwald space.
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1. INTRODUCTION

The algebraic structure of the projrctive group and the projective algebra in Finsler spaces is thus far from their Riemannian counterparts. There are complex hierarches in the projective group and algebra, i.e., some Lie sub algebras of the projective algebra of Finsler spaces can be distinguished by preserving several non- Riemannian quantities. The usual Rici tensor K_{jl} is not generally symmetric in the indices j and l. This fact may refuse the closeness property of the projective factor, i.e., $P_{i|j} = P_{j|i}$, where "$|i$" denotes the horizontal derivative. A projective vector field is said to be C- projective if its projective factor p has closness property. In the recent works [3] projective algebra of Randers metrics are introduced and the Lie algebras of C- projective vector fields and the well known non- Riemannian curvature and H- curvature is invariants of the algebras of C- projective vector fields are also introduced.

Given any Finsler metric F on an n- dimensional manifold M, consider the following three non- Riemannian quantities $\Xi = \Xi dx^i$, and $H = H_{ij}$ and $\sum = \sum_{ij} dx^i \bigotimes dx^j$ on the pull back tangent bundle π^*TM.

$$\begin{aligned}\Xi &= S_{.i|m}y^m - S_{|i},\\ H_{ij} &= S_{.i.j} = \tfrac{1}{2}S_{.i.j|m}y^m,\\ \textstyle\sum_{ij} &= \tfrac{1}{n+1}(S_{.i|j} - S_{.j|i}),\end{aligned}$$

where S denotes the S- curvature and "." and "|" denotes the vertical and horizontal covariant derivatives respectively, with respect to the Berwald connection. The quantity Ξ was first introduced by Shen in [7]. In fact, the above quantities do not depend to the choice of connection for performing horizontal derivatives and can be direct for the Finsler metric itself.

2. PRELIMINARIES OF CONFORMAL VECTOR FIELDS:

A metric on an n- dimensional manifold M is a function $F : TM \to [0, \infty)$ which has the following properties

- F is C^∞ on $TM_0 : TM\backslash\{0\}$
- F is Positivity 1- homogeneous on the fibers of tangent bundle TM .
- for each $y \in T_pM$ the following quadratic form g_y on T_pM is positive definite.

$$g_y(u, v) = g_{ij}(x, y)u^iv^j = \tfrac{1}{2}[F^2]_{y^iy^j}(x, y)u^iv^j.$$

Here $x = (x^i)$ denotes the coordinates of $p \in M$. The geodesics are characterized by $\frac{d^2c^i}{dt^2} + 2G^i(\dot{c}(t)) = 0$, where $G^i = \frac{1}{2}g^{il}\left\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\right\}$ are called geodesic coefficients of F.

The pair (M, F) is then called a Finsler space and we define a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1- form by $\beta = b_i(x)y^i$. A globally defined vector field G is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $G = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2G^i(x, y), \lambda \geq 0$. In (**??**), the function c is called the conformal factor.

Assume the following conventions :

$$G^i_j = \frac{\partial G^i}{\partial y^i}, \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}, \quad G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}.$$

Notice that the local functions G^i_{jk} give rise to a torison - free connection in π^*TM called the Berwald connection which is practical in this paper , see [5]. The local functions G^i_j define a non linear connection HTM spanned by horizontal frame $\left\{\frac{\delta}{\delta x^i}\right\}$, where $\frac{\delta}{\delta x^j} - G^i_j \frac{\partial}{\partial y^i}$. The nonlinear connection HTM splits TTM as $TTM = ker\pi_* \bigoplus HTM$, see [5]. A Finsler metric is called a Berwald metric if $G^i_{jk}(x, y)$ are functions of x only at every point $x \in M$, equivalently F is a Berwald metric if and only if $G^i_{jkl} = 0$.

For a Finsler metric F on an n- dimensional manifold M the Busemann - Hausdorff volume form $dV_F = \sigma_F(x)dx^1...dx^n$ is defined by

$$\sigma F(x) = \frac{Vol(B^n(1))}{Vol\left\{(y^i)\in R^n|F(y^i\frac{\partial}{\partial x^i})_x<1|\right\}}.$$

Assume $g = det(g_{ij}(x, y))$ and define $r(x, y) = ln\frac{\sqrt{g}}{\sigma F(x)}$. $\tau = \tau(x, y)$ is a scalar function on TM_0, which is called the distortion [5]. For a vector $y \in T_xM$, let $c(t)$, $-\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. The function $S(y) = \frac{d}{dt}\left[\tau(\dot{c}(t))\right]_{|t=0}$ is called the S- curvature with respect to Busemann - Hausdorff volume form. A Finsler space is said to be of isotropic S- curvature if there is a function $\sigma = \sigma(x)$ defined on M such that $S = (n + 1)\sigma(x)F$. It is called a Finsler space of constant S- curvature once σ is a constant. Every Berwald space is of vanishing S- curvature [5]. The E- curvature of the Finsler space (M, F) is defined by $E_y = E_{ij}(y)dx^i \bigotimes dx^j$, where $E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j}$, (M, F) is called a weakly - Berwald space if $E = 0$. It is easy to see that we have $E_{ij} = \frac{1}{2}G^r_{irj}$.

Definition 2.1. *The collection of all projective vector fields on a Finsler space (M, F) is finite dimensional Lie algebra with respect to the usual Lie bracket, called the projective algebra $p(M, F)$, and is the Lie algebra of the projective group $P(M, F)$.*

3. PROJECTIVE VECTOR FIELDS ON KROPINA SPACE

In this section, we characterized the some certain Lie sub algebra of the projective algebra of vanishing S- curvature which admits a non α- affine projective vector field is Berwald space of Kropina metric.

Recall that ; Let (M, α) be a Riemannian space and $\beta = b_i(x)y^i$ be a 1- form defined on M such that $\|\beta\|_x = sup\beta(y)/\alpha(y) < 1$. The Finsler metric $F = \frac{\alpha^2}{\beta}$ is called a Kropina space on a manifold M. Denote the geodesic spray coefficients of α and F by the notions G^i_α and G^i respectively and the Levi- Civita connection of α by ∇. Define $\nabla_j b_i$ by $(\nabla_j b_i)\theta^j = db_i - b_j\theta^j_i$, where $\theta^i = dx^i$ and $\theta^j_i = \tilde{\gamma}^j_{ik}dx^k$ denote the Levi- Civita connection forms and ∇ denotes its associated covariant derivation of α. Let us put

$$r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), \quad , s_{ij} = \frac{1}{2}(\nabla_j b_i - \nabla_i b_j)$$
$$s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \quad e_{ij} = r_{ij} + b_i s_j + b_j s_i.$$

Then G^i are given by

$$G^i = G^i_\alpha - \frac{1}{b^2}\left(\frac{r_{00}}{F} + s_0\right) y^i - \frac{F}{2} s^i_0,$$

where $r_{00} = r_{ij}y^iy^j$, $s_0 = s_iy^i$, $s^i_0 = s^i_jy^j$ and G^i_α denote the geodesic coefficients of α.

Theorem 3.1. *Let $(M, F = \frac{\alpha^2}{\beta})$ be a Kropina space and V be a vector field on M. Then V is F- projective iff V is α- projective and $L_{\hat{v}}\{\alpha S^i_0\} = 0$.*

Proof: Suppose that V is F- projective . Since, it preserves the Douglas tensor, i.e., $L_{\hat{v}} D^i_{jkl} = 0$. The sprays G^i of F and $\hat{G}^i = G^i_\alpha + T^i$ are projectively related and thus they have the same Douglas tensor, hence

$$D^i_{jkl} = \hat{D}^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left\{T^i - \frac{1}{n+1} T^m_m y^i\right\},$$

where $T^i = \frac{-\alpha}{2s} S^i_0 + \frac{1}{2(b^2 - 2s^2)}\left\{\frac{\alpha}{s} s_0 + r_{00}\right\} b^i$.
By direct calculation shows that $T^m_m = 0$. From this and take $T^i = \alpha S^i_0$. we have

$$L_{\hat{v}} D^i_{jkl} = L_{\hat{v}} T^i_{.j.k.l} = L_{\hat{v}}\{\alpha s^i_0\}_{.j.k.l} = 0.$$

Therefore, we have the functions $H^i(x, y), (i = 1, 2...n)$ quadratic in y such that

$$L_{\hat{v}}\left\{\alpha s^i_0\right\} = H^i. \tag{3.1}$$

Now, let us put $t_{ij} = L_{\hat{v}} a_{ij}$.
Here, observe that

$$L_{\hat{v}}\left\{\alpha s^i_0\right\} = \frac{t_{00}}{2\alpha} s^i_0 + \alpha L_{\hat{v}} s^i_0. \tag{3.2}$$

Using the (3.2), the equation (3.1) can be re-written as follows:

$$t_{00} s^i_0 + 2\alpha^2 L_{\hat{v}} s^i_0 = \alpha H^i. \tag{3.3}$$

Here, we see that $\alpha^2 = a_{ij}(x)y^iy^j$, $t_{00}s^i_0 = (t_{ij}(x)s^i_k(x))\, y^iy^jy^k$ and $L_{\hat{v}} s^i_0 = (L_v s^i_k)(x)y^k$ are polynomials in $y^1, y^2, ...y^n$.
Hence, the left hand side of (3.3) is a polynomial in $y^1, y^2, ...y^n$ for every i, while the right hand side is not. It follows immediately that $H^i = 0$ for every index i and (3.1) reads as $L_{\hat{v}}\{\alpha s^i_0\} = 0$.
Conversely, we know that the geodesic co-efficients of F are of the following form:

$$
\begin{aligned}
G^i &= G^i_\alpha - \tfrac{F}{2}s^i_0 - \tfrac{1}{2b^2F}(Fs_0+r_{00})(2y^i - Fb^i) \\
&= G^i_\alpha - \tfrac{1}{b^2F}(Fs_0+r_{00})y^i - \tfrac{F}{2}s^i_0 \\
&= G^i_\alpha - \frac{1}{b^2}\left(\frac{r_{00}}{F}+s_0\right)y^i - \frac{F}{2}s^i_0.
\end{aligned}
\tag{3.4}
$$

Since $L_{\hat{v}}\left\{\alpha s^i_0\right\} = 0$ and $L_{\hat{v}}G^i = py^i$, from this we have

$$L_{\hat{v}}G^i = L_{\hat{v}}\left\{G^i_\alpha - \tfrac{1}{b^2}\left(\tfrac{r_{00}}{F}+s_0\right)y^i\right\} = py^i,$$

and finally we obtain

$$L_{\hat{v}}G^i_\alpha = \left\{p - L_{\hat{v}}\left(\tfrac{1}{b^2}\left(\tfrac{r_{00}}{F}+s_0\right)\right)\right\}y^i.$$

It shows that V is a α-projective vector field. Conversly, suppose that V is a α- projective $\left(i.e., L_{\hat{v}}G^i_\alpha = w_0 y^i forsome 1 - forms w_0 = w_k(x)y^k on M\right)$ and $L_{\hat{v}}\left\{\alpha s^i_0\right\} = 0$.
From the equation (3.4) it follows

$$
\begin{aligned}
L_{\hat{v}}G^i &= L_{\hat{v}}\left\{G^i_\alpha - \tfrac{1}{b^2}(\tfrac{r_{00}}{F}+s_0)y^i - \tfrac{F}{2}s^i_0\right\} \\
&= L_{\hat{v}}G^i_\alpha - L_{\hat{v}}\tfrac{1}{b^2}\left(\tfrac{r_{00}}{F}+s_0\right)y^i. \\
&= \left\{w_0 - L_{\hat{v}}\left(\tfrac{1}{b^2}(\tfrac{r_{00}}{F}+s_0)\right)\right\}y^i
\end{aligned}
$$

which proves that V is F- projective vector field.

Lemma 3.1. *Let* $(M, F = \frac{\alpha^2}{\beta})$ *be an* n*- dimensional Kropina space. If* $s^i_j \neq 0$ *then* v *is* F*- projective vector field iff it is a* α*- homothety and* $L_{\hat{v}}d\beta = \mu d\beta$ *where* $L_{\hat{v}}a_{ij} = t_{ij} = 2\mu a_{ij}$.

Proof: Suppose that $s^i_j \neq 0$. By theorem (3.3) V is F- projective vector field iff it is α-projective and $L_{\hat{v}}\left\{\alpha s^i_0\right\} = 0$. Let us suppose $t_{ij} = L_{\hat{v}}a_{ij}$ and since $L_{\hat{v}}\left\{\alpha s^i_0\right\} = 0$. Therefore, equation (3.3) becomes

$$t_{00}s^i_0 + 2\alpha^2 L_{\hat{v}}s^i_0 = 0. \tag{3.5}$$

It follows that α^2 divides $t_{00}s^i_0$ for every index i. This equivalent to that $s^i_0 = 0$ (or) α^2 divedes t_{00} which means that V is conformal vector field on (M, α). Since $s^i_j \neq 0$. Since V is α- projective, it follows that V is an α- homothety and there is a constant μ such that $L_{\hat{v}}a_{ij} = 2\mu a_{ij}$.
Now, from (3.5) we obtain $L_v s^i_j = -\mu s^i_j$.
Here, observe that

$$\begin{aligned} L_v s_{ij} &= L_v \left\{a_{ik} s_j^k\right\} \\ &= \left(L_v a_{ik}\right) s_j^k + a_{ik} L_v s_j^k \\ &= 2\mu a_{ik} s_j^k - \mu a_{ik} s_j^k - \mu a_{ik} s_j^k \\ &= \mu s_{ij} \quad . \end{aligned}$$

It shows that $L_{\hat{v}} d\beta = \mu d\beta$.

Theorem 3.2. *Let $(M, F = \frac{\alpha^2}{\beta})$ be a Kropina space of vanishing s-curvature and dimension $n \geq 3$. If (M, F) admits a non α-affine projective vector field V then (M, F) is a Berwald space.*

Proof: Let us assume that F be a Kropina space of isometric s- curvature $s = (n+1)\sigma(x)F$ and V be an non - affine projective vector field [1]. From this results, shows that

$$\frac{r_{00}}{b^2 F} = 2\sigma(x)\left(\frac{\beta}{\alpha^2} - \beta^2\right).$$

Suppose that, there is a function ψ such that $\psi(x, y)$ is linear with respect to y such that $L_{\hat{v}} G^i = 4y^y, \quad \frac{r_{00}}{b^2 F} = 2\sigma(x)\left(\frac{\beta}{\alpha^2} - \beta^2\right)$.
Suppose that, there is a function ψ such that $\psi(x, y)$ is linear with respect to y such that $L_{\hat{v}} G^i = \psi y^i$.
Now, by applying theorem (3.3) we have

$$L_{\hat{v}} G^i = V_{\hat{v}} \tilde{G}^i + L_{\hat{v}}\left(\sigma(\frac{\beta}{\alpha^2} - \beta) y^i\right) - L_{\hat{v}} s_0 y^i = \psi y^i.$$

put $t_{ij} = L_{\hat{v}} a_{ij}$. It is well known that $L_{\hat{v}} y^i = 0$, it follows that $t_{00} = L_{\hat{v}} \alpha^2$ and

$$L_{\hat{v}} \tilde{G}^i = \frac{\beta}{\alpha^2} L_{\hat{v}} \sigma y^i - \beta L_{\hat{v}} \sigma y^i + \frac{t_{00}}{2\alpha} c y^i - \sigma y^i L - \hat{v}\beta - L_{\hat{v}} s_0 y^i = \psi y^i.$$

Recall that the natural coordinates system $(x^i, y^i), \phi^{-1}(U)$ and $x \in U$, we can regard each terms of the above equation as a polynomial in $y^1, y^2 ... y^n$.
Multiplying the two sides of the last equation by α. Then we obtain the following identity.

$$Rat^i + \alpha Irrat^i = 0, \quad i = 1, 2 ... n,$$

where Rat^i and $Irrat^i$ are polynomials given by

$$Rat^i = \alpha^2 L_{\hat{v}} \sigma y^i + \frac{1}{2} \mu a_{ij} \sigma y^i$$

$$Irrat^i = L_{\hat{v}}\tilde{G}^i - \left(\frac{\beta}{\alpha^2}L_{\hat{v}}\sigma + \sigma L_{\hat{v}}\beta + L_{\hat{v}}s_0 + \psi\right) y^i.$$

Now, assume that $s = 0$. By lemma (3.2), (M, F) must be locally projectively flat, otherwise V is a α- homothety. Which is a contradiction to the assumption that V is non - α-homothety.

Hence $s_{ij} = 0$ and by $e_{ij} = r_{ij} + b_i s_j + b_j s_i = r_{ij=0}$. which is equivalent to $\nabla_i b_j = 0$ and (M, F) is a Berwald space.

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