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Generalized Weighted Ostrowski−Grüss Type Inequality with applications

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Abstract

We obtained generalized weighted Ostrowski−Grüss type inequality by Korkine's identity with weights and parameters for differentiable functions and then we applied these obtain inequalities to probability density functions.Various applications are mentioned of numerical quadrature rules.

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1. INTRODUCTION

In different fields of science inequalities are frequently used. In numerical analysis for error estimation of bounds using inequalities. Inequalities helpful to getting the best bounds in numerical analysis. In the last decay, specially, the Simpson's type rules, mid-point and trapezoid have been analysed with the viewpoint of getting bounds for the quadrature rules. Present article is keen to investigate several refinements of inequalities for Ostrowski−Grüss type inequality with weights and to assume explicit bounds for the numerical quadrature rules in terms of variation in norms by using modern theory of inequalities and weighted Peano kernel approach.

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In 1991, D. S. Mitrinović, J. E. Pečarić and A. M. Fink published their books "Classical and New Inequalities in Analysis" [9] and "Inequalities involving Functions and their Integrals and Derivatives" [10]. With the publication of these books, a relation has been established between the integral of the product of the two functions and the product of the integral of the two functions, which are famous as Grüss inequality [6] and Čebyšev functional (see [9], p.297).

The Grüss inequality is stated as:

Proposition 1.1. *Let ψ, $\psi : [b_0, b_1] \to \mathbb{R}$ be integrable on $[b_0, b_1]$. Further, let the Čebyšev functional be defined as*

$$T(\psi,\psi) = \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\eta)\psi(\eta)d\eta - \left(\frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\eta)d\eta\right)\left(\frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\eta)d\eta\right)$$

$$m \leq \psi(\eta) \leq M, \quad n \leq \psi(\eta) \leq N,$$

for each $\eta \in [b_0, b_1]$, where m, M, n, N are given real constants. Then,

$$|T(\psi,\psi)| \leq \frac{1}{4}(M-m)(N-n),$$

where the constant $\frac{1}{4}$ is the best possible.

S. Wang [4] and S. S. Dragomir verified the following Ostrowski−Grüss type integral inequality with the help of Grüss inequality in 1997:

Proposition 1.2. *Let $\psi : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $b_0, b_1 \in I^0$ with $b_0 < b_1$. If $u \leq \psi'(\eta) \leq \mu$, $\eta \in [b_0, b_1]$ for various constants $u, \mu \in \mathbb{R}$. Then*

$$\left|\psi(\eta) - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger - \frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right| \leq \frac{1}{4}(b_1-b_0)(\mu-u) \tag{1.1}$$

for all $\eta \in [b_0, b_1]$.

(1.1) produce a relation between Gr$\ddot{u}$ss inequality [9] and the Ostrowski inequality [11]. M. Matić, J. E. Pečarić and N. Ujević [8] enhanced the Proposition 1.2 in 2000.

Proposition 1.3. *Let the assumptions of Proposition* 1.2 *be true. Then*

$$\left|\psi(\eta) - \frac{1}{b_1 - b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger - \frac{\psi(b_1)-\psi(b_0)}{b_1 - b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right| \leq \frac{1}{4\sqrt{3}}(\mu - u)(b_1 - b_0) \tag{1.2}$$

for all $\eta \in [b_0, b_1]$.

S. S. Dragomir and A. Sofo [1], N. S. Barnett further enhanced that inequality (1.2) in the similar year, which is stated that:

Proposition 1.4. *Let* $\psi : I \to \mathbb{R}$ *be an absolutely continuous function whose derivative* $\psi' \in L_2[b_0, b_1]$, *if* $u \leq \psi'(\eta) \leq \mu$, $\eta \in [b_0, b_1]$ *for various constants* $u, \mu \in \mathbb{R}$. *Then*

$$\begin{aligned}\left|\frac{\psi(\eta)+\psi(b_1)}{2} - \int_{b_0}^{b_1}\psi(\dagger)d\dagger\right| &\leq \frac{(b_1-b_0)}{2\sqrt{3}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(b_1-b_0)\end{aligned} \tag{1.3}$$

if $\gamma \leq \psi'(\dagger) \leq \Gamma$

In $[13]$, M.A. Shaikh obtained a weighted Ostrowski inequality, which is more generalized.

Proposition 1.5. *Let the assumptions of Proposition* 1.4 *be valid. Then the Ostrowski−Grüss type inequality is*

$$\begin{aligned}&\left|(1-\lambda)\frac{\psi(\eta)+\psi(b_0+b_1-\eta)}{2} + \lambda\frac{(\psi(b_0)+\psi(b_1))}{2} - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(t)dt\right| \\ &\leq \left[\frac{(b_1-b_0)^2}{12}(3\lambda^2-3\lambda+1) + \left(\eta - \frac{b_0+b_1}{2}\right)^2(1-\lambda)\right. \\ &\left.+\frac{(b_1-b_0)(1-\lambda)^2}{2}\left(z - \frac{b_0+b_1}{2}\right)\right]^{\frac{1}{2}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{(b_1-b_0)^2}{12}(3\lambda^2-3\lambda+1) + \left(\eta - \frac{b_0+b_1}{2}\right)^2(1-\lambda)\right. \\ &\left.+\frac{(b_1-b_0)(1-\lambda)^2}{2}\left(\eta - \frac{b_0+b_1}{2}\right)\right]^{\frac{1}{2}}\end{aligned}$$

if $\gamma \leq \psi'(t) \leq \Gamma$, *for all* $\eta \in [b_0, b_1]$ *and* $\lambda \in [0, 1]$.

In this article, the generalized Grüss inequality has recognized by introducing Peano kernel with weights. In the first section we focused on Introduction and Propositions. In the second section, with the help of weighted Korkine's identity, we mentioned weighted Ostrowski−Grüss inequality. In the third section, we then put on to probability density functions. Our last section is focused on various applications of numerical quadrature rules.

2. MAIN RESULT

Two lemmas is needed here, with the help of Korkine's identity (2.1) and Grüss inequality (2.2) with weights as mentioned in [2], our main result is proved.

Lemma 2.1. *Let* $p, \psi, \psi : [b_0, b_1] \to \mathbb{R}$ *be the measurable mapping for which the integrals involved in the following identity exist and finite. Then*

$$\int_{b_0}^{b_1} p(\dagger)d\dagger \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)\psi(\dagger)d\dagger - \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger$$
$$= \frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\psi(\dagger)-\psi(t)\right)\left(\psi(\dagger)-\psi(t)\right)d\dagger dt. \quad (2.1)$$

Lemma 2.2. *Let the assumptions of lemma* 2.1 *be valid. Then we have the following Ostrowski−Grüss inequality*

$$0 \le \int_{b_0}^{b_1} p(\dagger)\psi^2(\dagger)d\dagger - \left(\int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger\right)^2 \le \frac{1}{4}(M-m)^2 \quad (2.2)$$

where $m \le \psi(\dagger) \le M$ *a.e on* $[b_0, b_1]$.

Throughout the paper $\alpha = b_0 + \lambda\frac{b_1 - b_0}{2}$ and $\beta = b_1 - \lambda\frac{b_1 - b_0}{2}$ where $\lambda \in [0, 1]$.

Theorem 2.3. *Let the assumptions of Proposition* 1.4 *be valid. Then we get the* *inequality*

$$\Bigg|\psi(b_1)\int_{\beta}^{b_1} p(\dagger)d\dagger - \psi(b_0)\int_{\alpha}^{b_0} p(\dagger)d\dagger + \psi(\eta)\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(\dagger)d\dagger + \psi(b_0+b_1-\eta)$$
$$\times\int_{\frac{\alpha+\beta}{2}}^{\beta} p(\dagger)d\dagger + b_1\int_{\beta}^{b_1} p(u)du - \Bigg(\eta\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(u)du + (b_0+b_1-\eta)$$
$$\int_{\frac{\alpha+\beta}{2}}^{\beta} p(u)du + b_0\int_{b_0}^{\alpha} p(u)du + b_1\int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\Bigg)$$
$$\times\left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right) - \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger\Bigg|$$
$$\leq \left[\int_{b_0}^{b_1}\frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1} K_p(\eta,\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\int_{b_0}^{b_1} p(\dagger)[\psi'(\dagger)]^2 d\dagger - \left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}$$
$$= \frac{1}{2}(\mu-u)H_p(\eta,\dagger) \tag{2.3}$$

where

$$H_p(\eta,\dagger) = \int_{b_0}^{b_1}\frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1} K_p(\eta,\dagger)d\dagger\right)^2$$

and $p : [b_0, b_1] \to [0,\infty)$ *various probability density function is satisfied*

$$\int_{b_0}^{b_1} p(\dagger)d\dagger = 1$$

for all $\eta \in [b_0, b_1]$ *and* $\lambda \in [0,1]$.

Proof. The kernel as defined in [13], $K_p(\eta,\dagger) : [b_0, b_1]^2 \to \mathbb{R}$

$$K_p(\eta,\dagger) = \begin{cases} \int_{\alpha}^{\dagger} p(u)du, & \text{if } \dagger \in [\alpha,\eta], \\ \int_{\frac{\alpha+\beta}{2}}^{\dagger} p(u)du, & \text{if } \dagger \in (\eta, b_0+b_1-\eta], \\ \int_{\beta}^{\dagger} p(u)du, & \text{if } \dagger \in (b_0+b_1-\eta, b_1], \end{cases}$$

From (2.1), the Korkine's identity in the form of

$$\int_{b_0}^{b_1} K_p(\eta,\dagger)\psi'(\dagger)d\dagger - \int_{b_0}^{b_1} K_p(\eta,\dagger)d\dagger \int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger$$
$$= \int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\frac{K_p(\eta,\dagger)}{p(\dagger)} - \frac{K_p(\eta,t)}{p(t)}\right)(\psi'(\dagger)-\psi'(t))\,d\dagger dt. \quad (2.4)$$

From [13], we have

$$\int_{b_0}^{b_1} K_p(\eta,\dagger)\psi'(\dagger)d\dagger = \psi(b_1)\int_{\beta}^{b_1} p(\dagger)d\dagger - \psi(b_0)\int_{\alpha}^{b_0} p(\dagger)d\dagger$$
$$+\psi(\eta)\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(\dagger)d\dagger + \psi(b_0+b_1-\eta)\int_{\frac{\alpha+\beta}{2}}^{\beta} p(\dagger)d\dagger - \int_{b_0}^{b_1}\psi(\dagger)p(\dagger)d\dagger. \quad (2.5)$$

and

$$\int_{b_0}^{b_1} K_p(\eta,\dagger)d\dagger = \eta\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(u)du + (b_0+b_1-\eta)\int_{\frac{\alpha+\beta}{2}}^{\beta} p(u)du$$
$$+b_0\int_{b_0}^{\alpha} p(u)du + b_1\int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(\dagger)\dagger d\dagger. \quad (2.6)$$

By putting (2.5) and (2.6) in (2.4), we get

$$\psi(b_1)\int_{\beta}^{b_1} p(\dagger)d\dagger - \psi(b_0)\int_{\alpha}^{b_0} p(\dagger)d\dagger + \psi(\eta)\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(\dagger)d\dagger$$
$$+\psi(b_0+b_1-\eta)\int_{\frac{\alpha+\beta}{2}}^{\beta} p(\dagger)d\dagger + b_1\int_{\beta}^{b_1} p(u)du$$
$$-\left(\eta\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(u)du + (b_0+b_1-\eta)\int_{\frac{\alpha+\beta}{2}}^{\beta} p(u)du + b_0\int_{b_0}^{\alpha} p(u)du\right.$$
$$\left.+b_1\int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\right)\left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right) - \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger$$
$$= \int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\frac{K_p(\eta,\dagger)}{p(\dagger)} - \frac{K_p(\eta,t)}{p(t)}\right)(\psi'(\dagger)-\psi'(t))\,d\dagger dt \quad (2.7)$$

$\forall\, \eta \in [b_0, b_1]$.

Using Cauchy$-$Schwartz inequality for double integrals, we get

$$\left|\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\frac{K_p(\eta,\dagger)}{p(\dagger)} - \frac{K_p(\eta,t)}{p(t)}\right)(\psi'(\dagger)-\psi'(t))\,d\dagger dt\right|$$
$$\leq \left(\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\frac{K_p(\eta,\dagger)}{p(\dagger)} - \frac{K_p(\eta,t)}{p(t)}\right)^2 d\dagger dt\right)^{\frac{1}{2}}$$
$$\times\left(\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\psi'(\dagger)-\psi'(t)\right)^2 d\dagger dt\right)^{\frac{1}{2}}. \quad (2.8)$$

From[2], we have the following identity

$$\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\frac{K_p(\eta,\dagger)}{p(\dagger)}-\frac{K_p(\eta,t)}{p(t)}\right)^2 d\dagger dt$$
$$=\int_{b_0}^{b_1}\frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)}-\left(\int_{b_0}^{b_1}K_p(\eta,\dagger)d\dagger\right)^2 \quad (2.9)$$

and from Korkine(2.1), we have

$$\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(\dagger)p(t)\left(\psi'(\dagger)-\psi'(t)\right)^2 d\dagger dt$$
$$=\int_{b_0}^{b_1} p(\dagger)[\psi'(\dagger)]^2 d\dagger-\Big(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\Big)^2. \quad (2.10)$$

Using weighted Ostrowski−Grüss inequality (2.2), if $u \leq \psi'(\dagger) \leq \mu$ and $\dagger \in (b_0, b_1)$, we get

$$0 \leq \int_{b_0}^{b_1} p(\dagger)\big(\psi'(\dagger)\big)^2 d\dagger-\Big(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\Big)^2 \leq \frac{1}{4}(\mu-u)^2. \quad (2.11)$$

Using (2.7)-(2.11), we obtain

$$\Bigg|\psi(b_1)\int_{\beta}^{b_1} p(\dagger)d\dagger-\psi(b_0)\int_{\alpha}^{b_0} p(\dagger)d\dagger+\psi(\eta)\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(\dagger)d\dagger+\psi(b_0+b_1-\eta)$$
$$\times\int_{\frac{\alpha+\beta}{2}}^{\beta} p(\dagger)d\dagger-\Bigg(\eta\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(u)du+(b_0+b_1-\eta)\int_{\frac{\alpha+\beta}{2}}^{\beta} p(u)du$$
$$+b_0\int_{b_0}^{\alpha} p(u)du+b_1\int_{\beta}^{b_1} p(u)du-\int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\Bigg)\Bigg(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\Bigg)$$
$$-\int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger\Bigg| \leq \left[\int_{b_0}^{b_1}\frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)}-\left(\int_{b_0}^{b_1}K_p(\eta,\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\int_{b_0}^{b_1} p(\dagger)[\psi'(\dagger)]^2 d\dagger-\left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{2}(\mu-u)\left[\int_{b_0}^{b_1}\frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)}-\left(\int_{b_0}^{b_1}K_p(\eta,\dagger)d\dagger\right)^2\right]$$
$$= \frac{1}{2}(\mu-u)H_p(\eta,\dagger)$$

which proves our result (2.3). □

We can define various special cases of (2.3).

Remark 2.4. If we put $p(\dagger) \equiv \dfrac{1}{b_1 - b_0}$ in (2.3), then we get the special result (1.4) of [14].

The following remarks contains various special cases of (2.1).

Remark 2.5. If we put $\lambda = 0$ in (2.3), then $\alpha = b_0$ and $\beta = b_1$, then following inequality holds

$$\begin{aligned}
&\left|\frac{\psi(m)+\psi(n)}{2} - \int_m^n \psi(t)dt\right| \leq \left[\int_{b_0}^{b_1} \frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1} K_p(\eta,\dagger)d\dagger\right)^2\right]^{\frac{1}{2}} \\
&\times \left[\int_{b_0}^{b_1} p(\dagger)[\psi'(\dagger)]^2 d\dagger - \left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}} \\
\leq& \frac{1}{2}(\mu - u)H_p(\eta,\dagger). \qquad (2.12)
\end{aligned}$$

The above inequality is the special result of our paper [7].

Remark 2.6. If we put $p(\dagger) \equiv \dfrac{1}{b_1 - b_0}$ in (2.12), then we get the inequality (1.3) of [1].

Remark 2.7. If we put $\lambda = 1$ in (2.3), then $\alpha = \beta = \dfrac{b_0 + b_1}{2}$, then following inequality holds

$$\begin{aligned}
&\left|\psi(b_1)\int_{\frac{b_0+b_1}{2}}^{b_1} p(\dagger)d\dagger - \psi(b_0)\int_{\frac{b_0+b_1}{2}}^{b_0} p(\dagger)d\dagger + b_1\int_{\frac{b_0+b_1}{2}}^{b_1} p(u)du\right. \\
&- \left(b_0\int_{b_0}^{\frac{b_0+b_1}{2}} p(u)du + b_1\int_{\frac{b_0+b_1}{2}}^{b_1} p(u)du - \int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\right)\left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right) \\
&\left.- \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger\right| \leq \left[\int_{b_0}^{b_1} \frac{K_p^2(\eta,\dagger)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1} K_p(\eta,\dagger)d\dagger\right)^2\right]^{\frac{1}{2}} \\
&\times \left[\int_{b_0}^{b_1} p(\dagger)[\psi'(\dagger)]^2 d\dagger - \left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}} \\
\leq& \frac{1}{2}(\mu - u)H_p(\eta,\dagger). \qquad (2.13)
\end{aligned}$$

Remark 2.8. If we put $p(\dagger) \equiv \dfrac{b_1 - b_0}{2}$ in (2.13), then trapezoidal inequality holds

$$\begin{aligned}
&\left|\frac{\psi(b_0)+\psi(b_1)}{2} - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger\right| \\
\leq\ & \frac{b_1-b_0}{2\sqrt{3}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{4\sqrt{3}}(\mu-u)(b_1-b_0). \qquad (2.14)
\end{aligned}$$

The above inequality is the Remark of [14].

Corollary 2.9. *If we put* $\eta = \dfrac{b_0+b_1}{2}$ *in* (2.3)*, then following inequality holds*

$$\begin{aligned}
&\Bigg|\psi(b_1)\int_{\beta}^{b_1}p(\dagger)d\dagger - \psi(b_0)\int_{\alpha}^{b_0}p(\dagger)d\dagger + \psi\left(\frac{b_0+b_1}{2}\right)\int_{\alpha}^{\beta}p(\dagger)d\dagger \\
&+b_1\int_{\beta}^{b_1}p(u)du - \left(\left(\frac{b_0+b_1}{2}\right)\int_{\alpha}^{\beta}p(u)du + b_0\int_{b_0}^{\alpha}p(u)du\right. \\
&\left.+b_1\int_{\beta}^{b_1}p(u)du - \int_{b_0}^{b_1}p(\dagger)\dagger d\dagger\right)\left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right) \\
&-\int_{b_0}^{b_1}p(\dagger)\psi(\dagger)d\dagger\Bigg| \\
&\leq\left[\int_{b_0}^{b_1}\frac{K_p^2\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1}K_p\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger\right)^2\right]^{\frac{1}{2}} \\
&\times\left[\int_{b_0}^{b_1}p(\dagger)[\psi'(\dagger)]^2d\dagger - \left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}} \\
&\leq\frac{1}{2}(\mu-u)H_p\left(\frac{b_0+b_1}{2},\dagger\right). \qquad (2.15)
\end{aligned}$$

Remark 2.10. If we put $p(\dagger) \equiv \dfrac{1}{b_1-b_0}$ in (2.15), then the bound of average midpoint and trapezoidal inequality holds

$$\begin{aligned}
&\left|(1-\lambda)\psi\left(\frac{b_0+b_1}{2}\right) + \lambda\frac{\psi(b_0)+\psi(b_1)}{2} - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger\right| \\
\leq\ & \frac{b_1-b_0}{2\sqrt{3}}\sqrt{3\lambda^2-3\lambda+1}\left[\frac{1}{b_1-b_0}\|\psi'\|_2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{(\mu-u)(b_1-b_0)}{4\sqrt{3}}\sqrt{3\lambda^2-3\lambda+1}.
\end{aligned}$$

Remark 2.11. If we put, $\lambda = 1$, then $\alpha = \beta = \dfrac{b_0+b_1}{2}$ in (2.15), then following inequality holds

$$\begin{aligned}
&\Bigg|\psi\left(\frac{b_0+b_1}{2}\right)\int_{\alpha}^{\beta}p(u)du+\psi(b_0)\int_{b_0}^{\alpha}p(u)du+\psi(b_1)\int_{\beta}^{b_1}p(u)du\\
&-\left(\frac{b_0+b_1}{2}\int_{\alpha}^{\beta}p(u)du+b_0\int_{b_0}^{\alpha}p(u)du+b_1\int_{\beta}^{b_1}p(u)du-\int_{b_0}^{b_1}p(\dagger)\dagger d\dagger\right)\\
&\times\left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right)-\int_{b_0}^{b_1}p(\dagger)\psi(\dagger)d\dagger\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq\left[\int_{b_0}^{b_1}\frac{K_p^2\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger}{p(\dagger)}-\left(\int_{b_0}^{b_1}K_p\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger\right)^2\right]^{\frac{1}{2}}\\
&\quad\times\left[\int_{b_0}^{b_1}p(\dagger)[\psi'(\dagger)]^2d\dagger-\left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}\\
&\leq\frac{1}{2}(\mu-u)H_p\left(\frac{b_0+b_1}{2},\dagger\right).
\end{aligned}\tag{2.16}$$

Remark 2.12. If we put $p(\dagger)\equiv\dfrac{1}{b_1-b_0}$ in (2.16), then trapezoidal inequality holds as we achieve in (2.14).

Remark 2.13. If we put, $\lambda = 0$, then $\alpha = b_0$ and $\beta = b_1$ in (2.15), then weighted midpoint inequality holds

$$\begin{aligned}
&\Bigg|\psi\left(\frac{b_0+b_1}{2}\right)-\left(\frac{b_0+b_1}{2}-\int_{b_0}^{b_1}p(\dagger)\dagger d\dagger\right)\left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right)\\
&-\int_{b_0}^{b_1}p(\dagger)\psi(\dagger)d\dagger\Bigg|\\
&\leq\left[\int_{b_0}^{b_1}\frac{K_p^2\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger}{p(\dagger)}-\left(\int_{b_0}^{b_1}K_p\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger\right)^2\right]^{\frac{1}{2}}\\
&\quad\times\left[\int_{b_0}^{b_1}p(\dagger)[\psi'(\dagger)]^2d\dagger-\left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}\\
&\leq\frac{1}{2}(\mu-u)H_p\left(\frac{b_0+b_1}{2},\dagger\right).
\end{aligned}\tag{2.17}$$

Remark 2.14. If we put $p(\dagger) = \dfrac{1}{b_1 - b_0}$ in (2.17), then midpoint inequality holds

$$\begin{aligned}
&\left|\psi\left(\frac{b_0+b_1}{2}\right) - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger\right| \\
\leq\ & \frac{(b_1-b_0)}{2\sqrt{3}}\left[\frac{1}{b_1-b_0}\left\|\psi'\right\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{4\sqrt{3}}(\mu-u)(b_1-b_0).
\end{aligned}$$

The above inequality is the Remark of [14].

Remark 2.15. If we put $\lambda = \frac{1}{2}$, then $\alpha = \dfrac{3b_0+b_1}{4}$ and $\beta = \dfrac{b_0+3b_1}{4}$ in (2.15), then following inequality holds

$$\begin{aligned}
&\Bigg|\psi\left(\frac{b_0+b_1}{2}\right)\int_{\frac{3b_0+b_1}{4}}^{\frac{b_0+3b_1}{4}} p(u)du + \psi(b_0)\int_{b_0}^{\frac{3b_0+b_1}{4}} p(u)du + \psi(b_1)\int_{\frac{b_0+3b_1}{4}}^{b_1} p(u)du \\
&-\left(\frac{b_0+b_1}{2}\int_{\frac{3b_0+b_1}{4}}^{\frac{b_0+3b_1}{4}} p(u)du + b_0\int_{b_0}^{\frac{3b_0+b_1}{4}} p(u)du + b_1\int_{\frac{b_0+3b_1}{4}}^{b_1} p(u)du\right. \\
&\left.-\int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\right)\left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right) - \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger\Bigg| \\
\leq&\left[\int_{b_0}^{b_1}\frac{K_p^2\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1}K_p\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger\right)^2\right]^{\frac{1}{2}} \\
&\times\left[\int_{b_0}^{b_1}p(\dagger)[\psi'(\dagger)]^2d\dagger - \left(\int_{b_0}^{b_1}p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}} \\
\leq&\frac{1}{2}(\mu-u)H_p\left(\frac{b_0+b_1}{2},\dagger\right).
\end{aligned} \tag{2.18}$$

Remark 2.16. If we put $p(\dagger) = \dfrac{1}{b_1 - b_0}$ in (2.18), then the bound of average midpoint and trapezoidal inequality holds

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\psi(b_0)+\psi(b_1)}{2} + \psi\left(\frac{b_0+b_1}{2}\right)\right] - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger\right| \\
\leq\ & \frac{(b_1-b_0)}{4\sqrt{3}}\left[\frac{1}{b_1-b_0}\left\|\psi'\right\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{8\sqrt{3}}(\mu-u)(b_1-b_0).
\end{aligned}$$

The above inequality is the Remark of [14].

Remark 2.17. If we put, $\lambda = \frac{1}{3}$, then $\alpha = \frac{5b_0+b_1}{6}$ and $\beta = \frac{b_0+5b_1}{6}$ in (2.15), then following inequality holds

$$\Bigg|\psi\left(\frac{b_0+b_1}{2}\right)\int_{\frac{5b_0+b_1}{6}}^{\frac{b_0+5b_1}{6}} p(u)du + \psi(b_0)\int_{b_0}^{\frac{5b_0+b_1}{6}} p(u)du + \psi(b_1)\int_{\frac{b_0+5b_1}{6}}^{b_1} p(u)du$$
$$-\left(\frac{b_0+b_1}{2}\int_{\frac{5b_0+b_1}{6}}^{\frac{b_0+5b_1}{6}} p(u)du + b_0\int_{b_0}^{\frac{5b_0+b_1}{6}} p(u)du + b_1\int_{\frac{b_0+5b_1}{6}}^{b_1} p(u)du\right.$$
$$\left.-\int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\right)\left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right) - \int_{b_0}^{b_1} p(\dagger)\psi(\dagger)d\dagger\Bigg|$$
$$\leq\left[\int_{b_0}^{b_1}\frac{K_p^2\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger}{p(\dagger)} - \left(\int_{b_0}^{b_1} K_p\left(\frac{b_0+b_1}{2},\dagger\right)d\dagger\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\int_{b_0}^{b_1} p(\dagger)[\psi'(\dagger)]^2 d\dagger - \left(\int_{b_0}^{b_1} p(\dagger)\psi'(\dagger)d\dagger\right)^2\right]^{\frac{1}{2}}$$
$$\leq\frac{1}{2}(\mu-u)H_p\left(\frac{b_0+b_1}{2},\dagger\right). \tag{2.19}$$

Remark 2.18. If we put $p(\dagger) = \dfrac{1}{b_1-b_0}$ in (2.19), then bound of $\frac{1}{3}$ Simpson's rule holds

$$\left|\frac{1}{3}\left[\frac{\psi(b_0)+\psi(b_1)}{2} + 2\psi\left(\frac{b_0+b_1}{2}\right)\right] - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger\right|$$
$$\leq \frac{(b_1-b_0)}{6}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{12\sqrt{3}}(\mu-u)(b_1-b_0).$$

The above inequality is the Remark of [14].

Remark 2.19. If we put $\eta = b_0$ or $\eta = b_1$ and $p(\dagger) \equiv \dfrac{1}{b_1-b_0}$ in (2.3), then trapezoidal inequality holds which is independent the value of h

$$\left|\frac{\psi(b_0)+\psi(b_1)}{2} - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\dagger)d\dagger\right|$$
$$\leq \frac{b_1-b_0}{2\sqrt{3}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{(b_1-b_0)(\mu-u)}{4\sqrt{3}}. \tag{2.20}$$

In Section 3 and 4, we will present applications of weighted Ostrowski$-$Grüss for numerical quadrature rules and probability density function respectively.

3. APPLICATION FOR PROBABILITY DENSITY FUNCTIONS

Let X be a continuous random variable having the probability density function $\psi : [b_0, b_1] \to \mathbb{R}_+$ and the cumulative distribution function $\Psi : [b_0, b_1] \to [0, 1]$, i.e.,

$$\Psi(\eta) = \int_{b_0}^{\eta} \psi(\dagger)d\dagger, \quad \eta \in [\alpha, \beta] \subset [b_0, b_1],$$

$$E(X) = \int_{b_0}^{b_1} \dagger\psi(\dagger)d\dagger,$$

and from [2] weighted expectation would be

$$E_p(X) = \int_{b_0}^{b_1} p(\dagger)\dagger\psi(\dagger)d\dagger$$

on the interval $[b_0, b_1]$. Then we have the following theorem.

Theorem 3.1. *Let the assumptions of Theorem* 2.3 *be valid and if probability density function belongs to* $L_2[b_0, b_1]$ *space, then following inequality holds*

$$\begin{aligned}
&\Bigg|\Psi(b_1)\int_{\beta}^{b_1} p(\dagger)d\dagger - \Psi(b_0)\int_{\alpha}^{b_0} p(\dagger)d\dagger + \Psi(\eta)\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(\dagger)d\dagger + \Psi(b_0 + b_1 - \eta) \\
&\times \int_{\frac{\alpha+\beta}{2}}^{\beta} p(\dagger)d\dagger + b_1\int_{\beta}^{b_1} p(u)du - \left(\eta\int_{\alpha}^{\frac{\alpha+\beta}{2}} p(u)du + (b_0 + b_1 - \eta)\right. \\
&\left.\int_{\frac{\alpha+\beta}{2}}^{\beta} p(u)du + b_0\int_{b_0}^{\alpha} p(u)du + b_1\int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(\dagger)\dagger d\dagger\right) \\
&\times \left(p(b_1) - \int_{b_0}^{b_1} p'(\dagger)\Psi(\dagger)d\dagger\right) - b_1 p(b_1) - E_p(X) - \int_{b_0}^{b_1} p'(\dagger)\dagger\Psi(\dagger)d\dagger\Bigg| \\
&\leq \frac{1}{2}(\mu - u)H_p(\eta, \dagger) \qquad (3.1)
\end{aligned}$$

for all $\eta \in [\alpha, \beta]$.

Proof. Put $\psi = \Psi$ in (2.3) and by using these two identities mention below, we get (3.1),

$$\int_{b_0}^{b_1} p(\dagger)\Psi(\dagger)d\dagger = b_1 p(b_1) - E_p(X) - \int_{b_0}^{b_1} p'(\dagger)s\Psi(\dagger)d\dagger$$

and

$$\int_{b_0}^{b_1} p(\dagger)\Psi'(\dagger)d\dagger = p(b_1) - \int_{b_0}^{b_1} p'(\dagger)\Psi(\dagger)d\dagger.$$

□

Remark 3.2. Let the assumptions of Theorem 3.1 be valid, if we substitute $p(\dagger) \equiv \frac{1}{b_1 - b_0}$ in (3.1), then following inequality holds

$$\begin{aligned}
&\left|(1-\lambda)\frac{\psi(\eta)+\psi(b_0+b_1-\eta)}{2}+\frac{\lambda}{2}-\frac{b_1-E(X)}{b_1-b_0}\right| \\
&\leq \frac{1}{b_1-b_0}\left[\frac{(b_1-b_0)^2}{12}(3\lambda^2-3\lambda+1)+\left(\eta-\frac{b_0+b_1}{2}\right)^2(1-\lambda)\right. \\
&\left.+\frac{(b_1-b_0)(1-\lambda)^2}{2}\left(\eta-\frac{b_0+b_1}{2}\right)\right]^{\frac{1}{2}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2-1\right]^{\frac{1}{2}} \\
&\leq \frac{(M-b_0)}{2(b_1-b_0)}\left[\frac{(b_1-b_0)^2}{12}(3\lambda^2-3\lambda+1)+\left(\eta-\frac{b_0+b_1}{2}\right)^2\right. \\
&\left.(1-\lambda)+\frac{(b_1-b_0)(1-\lambda)^2}{2}\left(\eta-\frac{b_0+b_1}{2}\right)\right]^{\frac{1}{2}}
\end{aligned}$$

if $b_0 \leq \psi'(t) \leq$ M

4. APPLICATIONS TO NUMERICAL QUADRATURE RULES

Let $I_n : b_0 = z_0 < z_1 < \ldots < z_n = b_1$ be a partition of the interval $[b_0, b_1]$ and let $\Delta z_k = z_{k+1} - z_k, k \in \{0, 1, 2, \ldots, n-1\}$. Then

$$\int_{z_k}^{z_{k+1}} p(\dagger)\psi(\dagger)d\dagger = Q_n(I_n, \psi, p) + R_n(I_n, \psi, p) \tag{4.1}$$

where $Q_n(I_n, \psi, p)$ is a quadrature formula, define as

$$\begin{aligned}
&Q_n(I_n,\psi,p) = \\
&\sum_{k=0}^{n-1}\left[\psi(z_{k+1})\int_{\beta_k}^{z_{k+1}}p(\dagger)d\dagger-\psi(z_k)\int_{\alpha_k}^{z_k}p(\dagger)d\dagger+\psi(\eta)\int_{\alpha}^{\frac{\alpha_k+\beta_k}{2}}p(\dagger)d\dagger\right. \\
&+\psi(z_k+z_{k+1}-\eta)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k}p(\dagger)d\dagger+z_{k+1}\int_{\beta_k}^{z_{k+1}}p(u)du-\left(\eta_k\int_{\alpha_k}^{\frac{\alpha_k+\beta_k}{2}}p(u)du\right. \\
&+(z_k+z_{k+1}-\eta_k)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k}p(u)du+z_k\int_{z_k}^{\alpha_k}p(u)du+z_{k+1}\int_{\beta_k}^{z_{k+1}}p(u)du \\
&\left.\left.-\int_{z_k}^{z_{k+1}}p(\dagger)\dagger d\dagger\right)\left(\int_{z_k}^{z_{k+1}}p(\dagger)\psi'(\dagger)d\dagger\right)\right]
\end{aligned} \tag{4.2}$$

for all $\eta_k \in [z_k, z_{k+1}]$.

Theorem 4.1. *Let ψ as be defined in Theorem 2.3. Then (2.3) holds where $Q_n(I_n,\psi,p)$ is given by formula (4.2) and the remainder $R_n(I_n,\psi,p)$ satisfies the estimates*

$$|R_n(I_n,\psi,p)| \leq \sum_{k=0}^{n-1} \frac{(\mu-u)}{2} H_p(\eta_k,\dagger) \tag{4.3}$$

for all $\eta_k \in [z_k, z_{k+1}]$.

Proof. Using (2.3) on $[z_k, z_{k+1}]$,

$$\begin{aligned}
&R_k(I_k,\psi,p) \\
&= \int_{z_k}^{z_{k+1}} p(\dagger)\psi(\dagger)d\dagger - \psi(z_{k+1})\int_{\beta_k}^{z_{k+1}} p(\dagger)d\dagger + \psi(z_k)\int_{\alpha_k}^{z_k} p(\dagger)d\dagger \\
&-\psi(\eta)\int_{\alpha}^{\frac{\alpha_k+\beta_k}{2}} p(\dagger)d\dagger - \psi(z_k+z_{k+1}-\eta)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k} p(\dagger)d\dagger - z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du \\
&+\left(\eta_k\int_{\alpha_k}^{\frac{\alpha_k+\beta_k}{2}} p(u)du + (z_k+z_{k+1}-\eta_k)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k} p(u)du + z_k\int_{z_k}^{\alpha_k} p(u)du\right. \\
&\left.+z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(\dagger)\dagger d\dagger\right)\left(\int_{z_k}^{z_{k+1}} p(\dagger)\psi'(\dagger)d\dagger\right).
\end{aligned}$$

Summing over k from 0 to $n-1$. This yields

$$\begin{aligned}
&R_n(I_k,\psi,p) \\
&= \sum_{k=0}^{n-1}\int_{z_k}^{z_{k+1}} p(\dagger)\psi(\dagger)d\dagger - \sum_{k=0}^{n-1}\left[\psi(z_{k+1})\int_{\beta_k}^{z_{k+1}} p(\dagger)d\dagger - \psi(z_k)\int_{\alpha_k}^{z_k} p(\dagger)d\dagger\right. \\
&+\psi(\eta)\int_{\alpha}^{\frac{\alpha_k+\beta_k}{2}} p(\dagger)d\dagger + \psi(z_k+z_{k+1}-\eta)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k} p(\dagger)d\dagger + z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du \\
&-\left(\eta_k\int_{\alpha_k}^{\frac{\alpha_k+\beta_k}{2}} p(u)du + (z_k+z_{k+1}-\eta_k)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k} p(u)du - z_k\int_{z_k}^{\alpha_k} p(u)du\right. \\
&\left.\left.-z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(\dagger)\dagger d\dagger\right)\left(\int_{z_k}^{z_{k+1}} p(\dagger)\psi'(\dagger)d\dagger\right)\right].
\end{aligned}$$

Applying absolute property on the above identity, we get

$$\begin{aligned}
&|R_n(I_k,\psi,p)| \\
&= \Bigg|\sum_{k=0}^{n-1}\int_{z_k}^{z_{k+1}} p(\dagger)\psi(\dagger)d\dagger - \sum_{k=0}^{n-1}\Bigg[\psi(z_{k+1})\int_{\beta_k}^{z_{k+1}} p(\dagger)d\dagger + \psi(z_k)\int_{\alpha_k}^{z_k} p(\dagger)d\dagger \\
&-\psi(\eta)\int_{\alpha}^{\frac{\alpha_k+\beta_k}{2}} p(\dagger)d\dagger - \psi(z_k+z_{k+1}-\eta)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k} p(\dagger)d\dagger - z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du \\
&-\Bigg(\eta_k\int_{\alpha_k}^{\frac{\alpha_k+\beta_k}{2}} p(u)du + (z_k+z_{k+1}-\eta_k)\int_{\frac{\alpha+\beta_k}{2}}^{\beta_k} p(u)du - z_k\int_{z_k}^{\alpha_k} p(u)du \\
&-z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(\dagger)\dagger d\dagger\Bigg)\left(\int_{z_k}^{z_{k+1}} p(\dagger)\psi'(\dagger)d\dagger\right)\Bigg]\Bigg| \\
\leq\ & \frac{1}{2}(\mu-u)H_p(\eta_k,\dagger).
\end{aligned}$$

□

If we put $p(\dagger)\equiv\dfrac{1}{\Delta z_k}$ in (4.1) and (4.2), then we get the identity

$$\frac{1}{\Delta z_k}\int_{z_k}^{z_{k+1}}\psi(\dagger)d\dagger = Q_n(I_n,\psi)+R_n(I_n,\psi) \tag{4.4}$$

and the quadrature formula becomes

$$\begin{aligned}
&Q_n(I_n,\psi) \\
&=\sum_{k=0}^{n-1}\left[(1-\lambda)\left[\psi(\eta_k)-\frac{\psi(z_{k+1})-\psi(z_k)}{\Delta z_k}\left(\eta_k-\frac{z_k+z_{k+1}}{2}\right)\right.\right. \\
&\left.+\lambda\frac{\psi(z_k)+\psi(z_{k+1})}{2}\right].
\end{aligned} \tag{4.5}$$

Remark 4.2. If we put $p(\dagger)\equiv\dfrac{1}{\Delta z_k}$ in (4.3), we achieve the identity (4.4) and the quadrature formula $Q_n(I_n,\psi)$ (4.5), then remainder $R_n(I_n,\psi)$ satisfies the estimates

$$\begin{aligned}
&|R_n(I_n,\psi)| \\
&\leq\sum_{k=0}^{n-1}\frac{(\mu-u)}{2}\left[\frac{(\Delta z_k)^2}{12}(3\lambda^2-3\lambda+1)+\lambda(1-\lambda)\left(\eta_k-\frac{z_k+z_{k+1}}{2}\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

where

$$
\begin{aligned}
&R_n(I_n, \psi) \\
&= \sum_{k=0}^{n-1} \left[\frac{1}{\Delta z_k} \int_{z_k}^{z_{k+1}} \psi(\dagger) d\dagger - \lambda \frac{\psi(z_k) + \psi(z_{k+1})}{2} \right. \\
&\left. -(1-\lambda) \left\{ \psi(\eta_k) - \frac{\psi(z_{k+1}) - \psi(z_k)}{\Delta z_k} \left(\eta_k - \frac{z_k + z_{k+1}}{2} \right) \right\} \right].
\end{aligned}
$$

5. CONCLUSION

Aim of this article is to generalization of the result of [3]. with the help of weighted kernel as defined in [13], we have obtained generalized weighted Ostrowski-Grüss integral inequality for first differentiable functions. We get different previous published results by using suitable replacement. We have also discussed various applications for probability density functions and numerical quadrature rules in the end.

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