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Derivative Free Three-Step Iterative Method with Divided Difference Estimates to Solve Nonlinear Equations

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Abstract

This article discusses an iterative method to solve a nonlinear equation, which is free from derivatives, obtained by approximating a derivative in the method proposed by Rhee et al. [Int. J. Comput. Math., 95 (2018), 2174-2211] by the method of divided difference. We show analytically that the method is of order eighth and for each iteration it requires four evaluation functions. Numerical experiments show that the new method is comparable with other discussed methods.

AMS Subjet classification: 65H05

Keywords: Efficiency index, derivative-free method, divided difference, order of convergence

1. INTRODUCTION

Finding a simple root α of the nonlinear equation $f(x) = 0$ is a prototype for many nonlinear numerical problems. Newton's method is the most widely iterative method used for dealing with such problems, and it is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \dots,$$

which converges quadratically in some neighbohood of α [1, h.58].

Many researchers have modified Newton's method in order to obtain a high order convergence such as [3], [5], [6], [8], [9], [10], [11], and [13]. Sharma and Arora [15] developed the iteration method of the form

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{f(y_n)}{2f[y_n, x_n] - f'(x_n)}, \\
x_{n+1} &= z_n - \frac{f[z_n, y_n]}{f[z_n, x_n]} \frac{f(z_n)}{2f[z_n, y_n] - f[z_n, x_n]},
\end{aligned}
$$

where $f[r,t] = \frac{(f(r)-f(t))}{(r-t)}$, $s = \frac{f(y_n)}{f(x_n)}$ and $u = \frac{f(z_n)}{f(y_n)}$, which has four evaluation functions and an eight order of convergence. Rhee et al. [14] modified Sharma and Arora's method and proposed the iterative method

$$
\left.
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= x_n - L_f(s)\frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= z_n - K_f(s,u)\frac{f(x_n)}{f'(x_n)},
\end{aligned}
\right\} \tag{1}
$$

where $s = \frac{f(y_n)}{f(x_n)}$, $u = \frac{f(z_n)}{f(y_n)}$, $L_f(s) = \frac{(1-s)}{(1-2s)}$ and $K_f(s,u) = \frac{su(1-s)^2(1-u)}{(1-2s)(1-su)(1-2s-2u+3su)}$, which have four evaluation functions and an eight order convergence.

In this article, the authors present a derivative free three-step iterative method, which is a modification of the method of Rhee et al. [14]. The derivation and analysis of convergence of the proposed method are presented in section two. In section three, numerical simulations are carried out on the four test functions using the proposed method and three methods proposed by Rhee et al. (MCX, MAX, and MEX, see [14] for detail) to see the efficiency of the proposed method.

2. DERIVATION AND ITS ANALYSIS OF CONVERGENCE

If the first derivative, $f'(x_n)$, on equation (1) is estimated by using a divided difference of the form

$$
f'(x_n) \approx \frac{f(w_n) - f(x_n)}{w_n - x_n},
$$

where

$$
w_n = x_n + f(x_n)^3, \tag{2}
$$

a new derivative free three-step iteration method (MFTI) is obtained as follows

$$
\left.
\begin{aligned}
y_n &= x_n - \frac{f(x_n)(w_n - x_n)}{f(w_n) - f(x_n)}, \\
z_n &= x_n - L_f(s)\frac{f(x_n)(w_n - x_n)}{f(w_n) - f(x_n)} \\
x_{n+1} &= z_n - K_f(s,u)\frac{f(x_n)(w_n - x_n)}{f(w_n) - f(x_n)},
\end{aligned}
\right\} \tag{3}
$$

where $s = \frac{f(y_n)}{f(x_n)}$, $u = \frac{f(z_n)}{f(y_n)}$, $L_f(s) = \frac{(1-s)}{(1-2s)}$ and $K_f(s,u) = \frac{su(1-s)^2(1-u)}{(1-2s)(1-su)(1-2s-2u+3su)}$. The convergence analysis of (3) is given in Theorem 1.

Teorema 1 (Order of Convergence) Suppose that $\alpha \in I$ is a simple zero of a function $f : I \subset \mathbb{R} \to \mathbb{R}$ sufficiently differentiable at open interval I. If x_0 is close enough to α, the iterative method defined in equation (3) has an eight order convergence and satisfies the $error$ equation

$$e_{n+1} = ((\beta_1)c_2^7 + (\beta_2)c_2^5 + (\beta_3)c_2^4 + (\beta_4)c_2^3 - c_3c_4c_2^2 + (c_3^3K_{13} - c_3^3)c_2)e_n^8 + O(e_n^9),$$

where $\beta_1 = 10K_{32}L_3 - L_5L_3 + K_{13}L_3^3 + 225 - K_{70} - 15K_{13}L_3^2 + 75K_{13}L_3 - 2L_4L_3 + 9L_3^2 + 5L_5 - 90L_3 + 10L_4 + K_{51}L_3 - 25K_{32} - K_{32}L_3^2 - 5K_{51} - 125K_{13}$, $\beta_2 = -140c_3 + 3K_{13}L_3^2c_3 + 5f'(\alpha)^3 - 2L_4c_3 - L_5c_3 + 10c_3K_{32} + 75c_3K_{13} + c_3K_{51} - 30K_{13}L_3c_3 - L_3^2c_3 - 2K_{32}L_3c_3 - L_3f'(\alpha)^3 + 33L_3c_3$, $\beta_3 = -L_3c_4 + 5c_4$, and $\beta_4 = -15c_3^2K_{13} + 24c_3^2 + 3K_{13}L_3c_3^2 - c_3^2K_{32} - 2L_3c_3^2 - f'(\alpha)^3c_3$,

$$L_j = \frac{d^j}{ds^j}L_f(0), \qquad 0 \le j \le 7,$$
$$K_{ij} = \frac{1}{(i!j!)}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(0,0), \qquad 0 \le i+j \le 8,$$

and

$$L_0 = 1, L_1 = 1, L_2 = 2,$$
$$K_{00} = K_{10} = K_{01} = K_{20} = K_{02} = K_{30} = K_{03} = K_{40} = K_{50} = K_{60} = 0,$$
$$K_{11} = K_{12} = 1, \quad K_{21} = 2, \quad K_{22} = 4,$$
$$K_{31} = 1 + L_3, \qquad K_{41} = -4 + 2L_3 + L_4.$$

Proof. Suppose that α is the simple root of $f(x) = 0$ then $f(\alpha) = 0$, and $f'(\alpha) \neq 0$. Then by Taylor's expansion [2, h.216] of $f(x)$ about α until the eight order derivative, we get

$$\begin{aligned} f(x) = f(\alpha) + f'(\alpha)\frac{(x-\alpha)}{1!} + f^{(2)}(\alpha)\frac{(x-\alpha)^2}{2!} + f^{(3)}(\alpha)\frac{(x-\alpha)^3}{3!} + f^{(4)}(\alpha)\frac{(x-\alpha)^4}{4!} \\ + f^{(5)}(\alpha)\frac{(x-\alpha)^5}{5!} + f^{(6)}(\alpha)\frac{(x-\alpha)^6}{6!} + f^{(7)}(\alpha)\frac{(x-\alpha)^7}{7!} \\ + f^{(8)}(\alpha)\frac{(x-\alpha)^8}{8!} + O(x-\alpha)^9. \end{aligned} \tag{4}$$

By evaluating equation (4) at $x = x_n$, and recalling $e_n = x_n - \alpha$ and $f(\alpha) = 0$, equation (4) can be written as

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8) + O(e_n^9), \tag{5}$$

with

$$c_n = \frac{f^{(n)}(\alpha)}{n!f'(\alpha)}, \qquad 2 \le n \le 8.$$

Then using equation (5), we obtain

$$\begin{aligned} f(x_n)^3 = f'(\alpha)^3(&e_n^3 + 3c_2e_n^4 + (3c_3 + 3c_2^2)e_n^5 + (3c_4 + 6c_2c_3 + c_2^3)e_n^6 + (3c_3^2 + 3c_5 \\ &+ 3c_2^2c_3 + 6c_2c_4)e_n^7 + (6c_2c_5 + 3c_2^2c_4 + 6c_3c_4 + 3c_2c_3^2 + 3c_6)e_n^8) \\ &+ O(e_n^9). \end{aligned} \tag{6}$$

Using equation (6) and recalling $x_n = e_n + \alpha$ then equation (2) becomes

$$\begin{aligned} w_n = \alpha + e_n + f'(\alpha)^3(&e_n^3 + 3c_2e_n^4 + (3c_3 + 3c_2^2)e_n^5 + (3c_4 + 6c_2c_3 + c_2^3)e_n^6 \\ &+ (3c_3^2 + 3c_5 + 3c_2^2c_3 + 6c_2c_4)e_n^7 + (6c_2c_5 + 3c_2^2c_4 + 6c_3c_4 + 3c_2c_3^2 + 3c_6)e_n^8 \\ &+ O(e_n^9)). \end{aligned} \tag{7}$$

Substituting equation (7) into equation (4) we get

$$f(w_n) = f'(\alpha)e_n + c_2f'(\alpha)e_n^2 + (c_3f'(\alpha) + f'(\alpha)^4)e_n^3 + \cdots + O(e_n^9),$$

and it follows that

$$\begin{aligned} f(w_n) - f(x_n) = {} & f'(\alpha)^4e_n^3 + 5f'(\alpha)^4c_2e_n^4 + (9f'(\alpha)^4c_2^2 + 6f'(\alpha)^4c_3)e_n^5 \\ &+ (7f'(\alpha)^4c_4 + 21f'(\alpha)^4c_2c_3 + 7f'(\alpha)^4c_2^3 + c_2f'(\alpha)^7)e_n^6 \\ &+ (24f'(\alpha)^4c_2^2c_3 + 8f'(\alpha)^4c_5 + 24f'(\alpha)^4c_2c_4 + 12f'(\alpha)^4 \\ &+ 3c_3f'(\alpha)^7 + 2c_2^4f'(\alpha)^4 + 6c_2^2f'(\alpha)^7)e_n^7 + (9f'(\alpha)^4c_6 \\ &+ 15c_2^3f'(\alpha)^7 + 27f'(\alpha)^4c_2c_3^2 + 27f'(\alpha)^4c_2^2c_4 \\ &+ 27f'(\alpha)^4c_2c_5 + 24c_2f'(\alpha)^7c_3 + 9c_2^3f'(\alpha)^4c_3 \\ &+ 6c_4f'(\alpha)^7 + 27f'(\alpha)^4c_3c_4)e_n^8 + O(e_n^9). \end{aligned} \tag{8}$$

From equation (7), we obtain

$$\begin{aligned} w_n - x_n = {} & f'(\alpha)^3e_n^3 + c_23f'(\alpha)^3e_n^4 + (3c_3 + 3c_2^2)f'(\alpha)^3e_n^5 + (3c_4 + 6c_2c_3 + c_2^3)f'(\alpha)^3e_n^6 \\ &+ (3c_5 + 3c_3^2 + 3c_2^2 + 3c_2^2c_3 + 6c_2c_4)f'(\alpha)^3e_n^7 \\ &+ (3c_2^2c_4 + 3c_6 + 6c_3c_4 + 6c_2c_5 + 3c_2c_3^2)f'(\alpha)^3e_n^8 + O(e_n^9). \end{aligned} \tag{9}$$

Moreover from equation (5) and (9) we get

$$\begin{aligned} f(x_n)(w_n - x_n) = f'(\alpha)^4(&e_n^4 + 4c_2e_n^5 + (4c_3 + 6c_2^2)e_n^6 + (12c_2c_3 + 4c_2^3 + 4c_4)e_n^7 \\ &+ (12c_2^2c_3 + 6c_3^2 + 4c_5 + 12c_2c_4 + c_2^4)e_n^8 + O(e_n^9)). \end{aligned} \tag{10}$$

From equation (8), equation (10) and the aid of geometric series [16] we obtain

$$\frac{f(x_n)(w_n - x_n)}{f(w_n) - f(x_n)} = e_n - c_2 e_n^2 + (-2c_3 + 2c_2^2)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4 - f'(\alpha)^3 c_2)e_n^4 + (8c_2^4 + 10c_2c_4 + 6c_3^2 - 3f'(\alpha)^3 c_3 - 4c_5 - 20c_2^2c_3)e_n^5 + \cdots + O(e_n^9). \quad (11)$$

Next equation (11) is substituted to equation (3) and simplifying the resulting equation we get

$$y_n = \alpha + c_2e_n^2 - (-2c_3 + 2c_2^2)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4 - f'(\alpha)^3c_2)e_n^4 + \cdots + O(e_n^9). \quad (12)$$

Then by evaluating equation (4) at $x = y_n$ as in equation (12) we get

$$f(y_n) = c_2 f'(\alpha)e_n^2 + (2c_3 - 2c_2^2)f'(\alpha)e_n^3 + ((3c_4 + 5c_2^3 - 7c_2c_3)f'(\alpha) + f'(\alpha)^4c_2)e_n^4 + \cdots + O(e_n^9). \quad (13)$$

To calculate $s = \frac{f(y_n)}{f(x_n)}$ we use equation (5), equation (13) and the aid of geometric series [16, h.730] we end up with

$$s = c_2e_n + (-3c_2^2 + 2c_3)e_n^2 + (f'(\alpha)^3c_2 - 10c_2c_3 + 8c_2^3 + 3c_4)e_n^3 + \cdots + O(e_n^9). \quad (14)$$

Since $L_f(s) = \frac{(1-s)}{(1-2s)}$, then by Taylor's expansion of $L_f(s)$ around $s = 0$, and by using equation (14) we obtain

$$L_f(s) = L_0 + L_1c_2e_n + (2L_1c_3 - 3L_1c_2^2 + L_2c_2^2)e_n^2 + (L_1c_2f'(\alpha)^3 + 4L_2c_2c_3 + L_3c_2^3 - 6L_2c_2^3 - 10L_1c_2c_3 + 3L_1c_4 + 8L_1c_2^3)e_n^3 + \cdots + O(e_n^9), \quad (15)$$

where

$$L_j = \frac{d^j}{ds^j}L_f(s), \qquad 0 \leq j \leq 7.$$

Next equation (11) an equation (15) are substituted into equation (3) and simplifying the resulting equation, we end up with

$$z_n = \alpha + (1 - L_0)e_n + (L_0c_2 - L_1c_2)e_n^2 + (2L_0c_3 - 2L_0c_2^2 - 2L_1c_3 + 4L_1c_2^2 - L_2c_2^2)e_n^3 + \cdots + O(e_n^9). \quad (16)$$

Then by evaluating equation (4) at $x = z_n$ as in equation (16), and setting $L_0 = 1, L_1 =$

$1, L_2 = 2$ then simplifying we obtain

$$f(z_n) = (- c_2c_3 + 5c_2^3 - L_3c_2^3)f'(\alpha)e_n^4 + ((10L_3c_2^4 - 6L_3c_2^2c_3 - 2c_2c_4 - 36c_2^4 - L_4c_2^4 + 32c_2^2c_3 - 2c_3^2)f'(\alpha) - f'(\alpha)^4c_2^2)e_n^5 + \cdots + O(e_n^9). \quad (17)$$

Then using equation (17), equation (13) and the aid of geometric series [16, h.730] we have

$$u = \frac{f(z_n)}{f(x_n)} = (- c_2^2L_3 + 5c_2^2 - c_3)e_n^2 + (-2c_4 + 20c_2c_3 - 4c_2L_3c_3 - 26c_2^3 - c_2^3L_4 - f'(\alpha)^3c_2 + 8L_3c_2^3)e_n^3 + \cdots + O(e_n^9). \quad (18)$$

Next by Taylor's expansion of $K_f(s, u)$ about $(s = 0, u = 0)$, up to the order of seven in s and three in u, using the equation (14) and equation (18), after simplifying we obtain

$$\begin{aligned} K_f(s,u) = K_{00} &+ c_2K_{10}e_n + (5K_{01}c_2^2 + c_2^2K_{20} + 2c_3K_{10} - 3c_2^2K_{10} - K_{01}c_3 \\ &- K_{01}c_2^2L_3)en^2 + (c_2^3K_{30} + c_2f'(\alpha)^3K_{10} - 6c_2^3K_{20} - K_{11}L_3c_2^3 \\ &- K_{01}c_2f'(\alpha)^3 - K_{11}c_2c_3 - 2K_{01}c_4 - 4K_{01}c_2L_3c_3 - 10c_2c_3K_{10} \\ &+ 8c_2^3K_{10} + 5K_{11}c_2^3 + 3c_4K_{10} - 26K_{01}c_2^3 + 4c_2c_3K_{20} + 8K_{01}L_3c_2^3 \\ &+ 20K_{01}c_2c_3 - K_{01}c_2^3L_4)e_n^3 + \cdots + O(e_n^9), \end{aligned} \quad (19)$$

where

$$K_{ij} = \frac{1}{(i!j!)}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(s,u), \qquad 0 \le i + j \le 8.$$

Substituting the equation (11), equation (16) and equation (19) into the equation (3), we obtain

$$x_{n+1} = \alpha - K_{00}e_n + (K_{00}c_2 - c_2K_{10})e_n^2 + (-2c_3K_{10} - c_2^2K_{20} + K_{01}c_2^2L_3 - 5K_{01}c_2^2 - 2K_{00}c_2^2 + 4c_2^2K_{10} + K_{01}c_3 + 2K_{00}c_3)e_n^3 + \sum_{l=4}^{8}\gamma_l e_n^l + O(e_n^9), \quad (20)$$

where $\gamma_l = \gamma_l(c_2, c_3, \cdots, c_6, L_3, \cdots, L_7, K_{ij})$, for $4 \le l \le 8, 0 \le i \le 7$ and $0 \le j \le 3$. Setting

$$K_{00} = K_{10} = K_{01} = K_{20} = 0,$$

from equation (20) along with $\gamma_4 = 0$, we immediately obtain

$$K_{11} = 1, \quad K_{30} = 0.$$

Continuing in this manner at the l-th stage with $4 \le l \le 7, \gamma_l = 0$ and solve $\gamma_l = 0$ for remaining K_{ij} to find

$$K_{02} = 0, \quad K_{21} = 2, \quad K_{40} = 0, \quad K_{12} = 1, \quad K_{31} = 1 + L_3, \quad K_{50} = 0,$$
$$K_{03} = 0, \quad K_{22} = 4, \quad K_{41} = -4 + 2L_3 + L_4, \quad K_{60} = 0.$$

On substituting these values of K_{ij} into γ_8, recalling $e_{n+1} = x_{n+1} - \alpha$, we find

$$\begin{aligned} e_{n+1} = ((&- c_3^3 + K_{13}c_3^3)c_2 + (24c_3^2 + 3K_{13}L_3c_3^2 - f'(\alpha)^3c_3 - 15K_{13}c_3^2 - 2L_3c_3^2 \\ &- K_{32}c_3^2)c_2^3 - c_2^2c_4c_3 + (5c_4 - L_3c_4)c_2^4 + (-2K_{32}L_3c_3 - 140c_3 + 3K_{13}L_3^2c_3 \\ &- 30K_{13}L_3c_3 + 75K_{13}c_3 + 33L_3c_3 - L_3^2c_3 + K_{51}c_3 + 5f'(\alpha)^3 - 2L_4c_3 \\ &- L_3f'(\alpha)^3 - L_5c_3 + 10K_{32}c_3)c_2^5 + (225 - 25K_{32} - 125K_{13} + 10K_{32}L_3 \\ &+ 9L_3^2 - 5K_{51} - K_{70} + 10L_4 + 5L_5 + K_{13}L_3^3 - 90L_3 + 75K_{13}L_3 - 2L_4L_3 \\ &- L_5L_3 - 15K_{13}L_3^2 - K_{32}L_3^2 + K_{51}L_3)c_2^7)e_n^8 + O(e_n^9). \end{aligned}$$

From the definition of the convergence order [12, h.75], we see that the iterative method (3) is of order eight and Theorem 1 is proven. □

3. NUMERICAL SIMULATION

In this section numerical simulations are performed which aim to compare the proposed method (MFTI) and the known methods as the MCX, the MAX, and the MEX as defined in [14]. The following nonlinear equations used to perform the comparison:

(i) $f_1(x) = \log(x) - \sqrt{x} + x^3$.

(ii) $f_2(x) = x^5 + \log[1 + \sin(x)]$.

(iii) $f_3(x) = \sin(x)\exp(x^2 - 3x) + \ln(1 + x^2)$.

In computation, we use tolerance of $1.0 \times 1E - 200$ and addition the maximum number of iterations allowed is 100. The computational order of convergence (COC) is estimated using a formula [4]

$$\text{COC} \approx \frac{\ln|x_{k+1} - x_k|/|x_k - x_{k-1}|}{\ln|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|}.$$

In Table 1, $f_n(x)$ states the nonlinear function, n states the number of iterations, COC is the computational order of convergence, $|f(x_n)|$ is the absolute value of the function and $|x_n - x_{n-1}|$ is the absolute value of difference between two consecutive approximation.

Based on Table 1, it can be seen that for all the functions with different initial guesses the method MFTI requires the same number of iterations as MCX, MAX, and MEX. In addition, the COC of the MFTI obtained is in agreement with the analytic result that is the method is eight order of convergence. Since the method needs four evaluation functions in each iteration, based on the efficiency index definition [7, h.261], the MFTI

Table 1: Comparison of several iteration methods

$f_n(x)$	x_0	Method	n	COC	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$
f_1	0.9	MFTI	3	8.00	$9.61e-402$	$2.745286e-402$
		MCX	3	8.00	$2.13e-631$	$6.084325e-632$
		MAX	3	8.00	$7.41e-613$	$2.118174e-613$
		MEX	3	8.00	$8.18e-600$	$2.336771e-600$
f_1	0.7	MFTI	3	8.00	$2.03e-205$	$5.798626e-206$
		MCX	3	8.00	$4.37e-488$	$1.249273e-488$
		MAX	3	8.00	$6.49e-475$	$1.855657e-475$
		MEX	3	8.00	$1.85e-461$	$5.284096e-462$
f_2	0.8	MFTI	4	8.00	$7.53e-305$	$7.530220e-305$
		MCX	4	8.00	$4.33e-645$	$4.334704e-645$
		MAX	4	8.00	$8.98e-266$	$8.981312e-266$
		MEX	4	8.00	$5.48e-236$	$5.481051e-236$
f_2	0.35	MFTI	3	8.00	$5.13e-330$	$5.130751e-330$
		MCX	3	8.00	$1.94e-353$	$1.940880e-353$
		MAX	3	8.00	$6.38e-363$	$6.379317e-363$
		MEX	3	8.00	$1.25e-365$	$1.248019e-365$
f_3	0.35	MFTI	3	8.00	$2.13e-210$	$2.125722e-210$
		MCX	3	8.00	$1.96e-317$	$1.959292e-317$
		MAX	3	8.00	$2.61e-219$	$2.607581e-219$
		MEX	3	8.00	$1.63e-202$	$1.625053e-202$
f_3	0.2	MFTI	3	8.00	$1.22e-262$	$1.221857e-262$
		MCX	3	8.00	$4.17e-292$	$4.166594e-292$
		MAX	3	8.00	$5.43e-300$	$5.430683e-300$
		MEX	3	8.00	$6.58e-242$	$6.575072e-242$

has an efficiency index $8^{\frac{1}{4}} \approx 1.68179$. The advantages of this method is that it does not use the first derivative of the function.

Over all based on Table 1, the MFTI can compete the existing eight-order method and it can be used as an alternative method to solve the nonlinear equation.

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