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Optimization of a Weighted Average: Cases of AR(1) and MA(1) Processes

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Abstract

In the time series smoothing literature, we normally use standard smoothing techniques such as moving averages or exponential smoothing based usually on the graphical aspect and the presence or absence of a linear trend and deterministic or stochastic seasonality. This paper focused on the optimal weighted average $Z_t = \sum_{i=1}^{n} \alpha_i Y_{t-(i-1)}$ using Lagrangian approach, where Y_t is a random process that follows $\text{AR}(1)$ or $\text{MA}(1)$ model. We searched for the best linear combination resulted from a minimum variance under the constraint $\sum_{i=1}^{n} \alpha_i = 1$. Here Z_t can be considered as an aggregated process with constraints on the coefficients. It can also be considered as a smoothing process to fill missing data. This paper is a theoretical research which allowed to obtain a weighted average with a minimum variance. We hope that the results of this research will have considerable impact on the practical analysis of time series.

Keywords: AR(1), MA(1) process, optimal weighted average, Lagrangian optimization.

1. INTRODUCTION

In time series analysis, we sometimes end up with daily, monthly, quarterly or annual data, and so on. It is known, for example, that the transition from the daily data Y_t to the weekly data Z_t normally results from a direct aggregation of the five daily observations $Z_t = \sum_{i=5t-4}^{5t} Y_i$. Similarly, the transition from the monthly data Y_t to the quarterly data Z_t is due to an aggregation of three monthly observations $Z_t = \sum_{i=3t-2}^{3t} Y_i$. Obviously the structure of the model associated with the aggregated process depends on the structure of the initial process. Historically, this kind of research has been oriented in two axes, the aggregation of data over time and the aggregation of ARIMA or other models. In this respect, we cite for example Engel (1984), Wei (1990), Mourad (1987), Kadi, Oppenheim, and Viano (1994) and Engle (2008).
To reduce the variability of a random process, analysts use traditional smoothing techniques, such as single or double exponential smoothing. Obviously, the choice of a smoothing technique will take into account the presence of a trend and / or seasonality. In the financial field of stock market analysis, we note a very frequent use of mobile averaging techniques. Indeed, the problem is to apply equal weights (simple equal weighted) on the observations of Yt to obtain the simple rolling average by arbitrarily specifying a reference period. In order to distinguish the importance of the observations according to their occurrence time, the analysts often use a weighted moving average (WMA) according to which the most recent observation has a weight heavier than the most distant observations used in the calculation. As a special case of the WMA technique, there is the exponential moving average (EMA) that suggests exponentially decreasing weights. We end with the so-called End Point Moving Average (EPMA)) proposed by Lafferty (1995). The EPMA has a remarkable efficiency in the follow-up of the evolution of a series especially if the variable in question reveals a regular increase followed by a regular decrease. After this quick look at the most used smoothing techniques in practice, we can introduce our problem. Indeed the construction of a process $Z_t = \sum_{i=1}^{p} \alpha_i Y_{t-i+1}$ as a weighted aggregation of order p of a process Y_t without having any idea of the structure of the evolution of the latter could lead to a wrong decision on the part of the analyst. Given that the Y_t process follows an autoregressive (AR), moving average (MA) or ARMA model, the weights used in the aggregation over time must result after a mathematical justification. If we impose the same weights directly in the construction of Z_t then we make a mistake by neglecting the link between Y_t and its past. In the research on AR(1) models, Andrews (1993) used the exact median unbiased estimators of the AR parameter to test the presence of a unit root. Mourad and Harb (2011) studied the impulse response function and the accumulated impulse response of a random variable generated by an autoregressive process with order of multiplicity of the autoregressive root greater than or equal to unity.

A recent working paper by Mourad, Ghandour, Tawbe and Mourad (2019) focused on the necessity of identifying the model of the time series' past, before the missing data by studying the case of an AR (1) process. The performance of the Z_t smoothing process varies according to the autoregressive parameter.

In the following, we will search for the best weights $\alpha_i,\ i = 1, \cdots, p$ under the constraint ($\sum_{i=1}^{p} \alpha_i = 1$) to form the process Z_t taking into consideration that the initial process Y_t follows a white noise, a centered model AR(1) and a centered model MA(1). We will use a Lagrangian approach to minimize the variance of Z_t under the constraint $\sum_{i=1}^{p} \alpha_i = 1$. To facilitate the progress of the subsequent matrix calculation we assume that ε_t is a white noise of average 0 and variance 1.

2. OPTIMIZATION OF THE WEIGHTED AVERAGE

This is the main section of our research that will be dedicated to the demonstration of four theorems. The first aims at calculating an optimal weighted average of a white noise process. The second treats the weighted average of a first-order centered autoregressive process while the third one proves that the function of the variance of the averaged process is decreasing with respect to the order of the weighted average (p). Finally, Theorem 4 studies the weighted average of a centered moving average process of order 1. The values of the resulting parameters $\alpha_i,\ i = 1, \cdots, p$, the Lagrange multiplier λ and the determinant Δ of the matrix used to solve the Cramer system of first derivatives are calculated for all values of p ranging from 2 to 12.

Theorem 2.1. *Suppose that* $Y_t = \varepsilon_t$ *and* $Z_t = \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i+1}$ *is a weighted average for* $\varepsilon_{t-i+1},\ i = 1, \cdots, p$. *The minimal variance of* Z_t *under the constraint* $\sum_{i=1}^{p} \alpha_i = 1$ *is obtained when* $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \frac{1}{p}$.

Proof. It is enough to minimize the variance $V(Z_t) = \sum_{i=1}^{p} \alpha_i^2$ under the constraint $\sum_{i=1}^{p} \alpha_i = 1$. Using Lagrangian optimization, we obtain

$$L(\alpha_1,\, \alpha_2,\, \cdots,\, \alpha_p,\, \lambda) = \sum_{i=1}^{p} \alpha_i^2 + \lambda(\sum_{i=1}^{p} \alpha_i - 1).$$

Now

$$\begin{cases} \dfrac{dL}{d\alpha_1} & = & 2\alpha_1 + \lambda & = & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \dfrac{dL}{d\alpha_p} & = & 2\alpha_p + \lambda & = & 0 \\ \dfrac{dL}{d\lambda} & = & \displaystyle\sum_{i=1}^{p}(\alpha_i - 1) & = & 0 \end{cases}$$

We obtain a system of $(p+1)$ equations and $(p+1)$ unknowns. In matrix form :

$$\begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & 1 \\ 1 & 1 & \cdots & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \Longleftrightarrow MX = A\,.$$

Using the Pivot Method, The line L^1_{p+1} of the matrix M is replaced by the line $L_{p+1} - \frac{1}{2}L_1$. We write: $L_{p+1} - \frac{1}{2}L_1 \longrightarrow L^1_{p+1}$ We obtain the following matrix :

$$M_1 = \begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & -\frac{1}{2} \end{pmatrix}$$

The line L^2_{p+1} of the matrix M_1 is replaced by the line $L^1_{p+1} - \frac{1}{2}L_2$ then $L^1_{p+1} - \frac{1}{2}L_2 \longrightarrow L^2_{p+1}$ We obtain:

$$M_2 = \begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & \cdots & 1 & -1 \end{pmatrix}$$

The new line L^p_{p+1} of matrice M_p is replaced by the line $L^{p-1}_{p+1} - \frac{1}{2}L_p$ then $L^{p-1}_{p+1} - \frac{1}{2}L_p \longrightarrow L^p_{p+1}$

$$M_p = \begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{p}{2} \end{pmatrix}$$

The system becomes as follows :

$$\begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{p}{2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Thus $\lambda = -\frac{2}{p}$, $2\alpha_i + \lambda = 0 \Longleftrightarrow \alpha_i = \frac{1}{p},\ i = 1, \cdots, p$. The Hessian matrix is a square matrix of second-order partial derivatives for the function $L(\alpha_1, \ldots, \alpha_p \lambda)$. In our case, the Hessian matrix is the following:

$$H = \begin{pmatrix} L_{\alpha_1\alpha_1} & L_{\alpha_1\alpha_2} & \cdots & L_{\alpha_1\alpha_p} & L_{\alpha_1\lambda} \\ L_{\alpha_2\alpha_1} & L_{\alpha_1\alpha_2} & \cdots & L_{\alpha_2\alpha_p} & L_{\alpha_2\lambda} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{\lambda\alpha_1} & L_{\lambda\alpha_2} & \cdots & L_{\lambda\alpha_p} & L_{\lambda\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 2 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{p}{2} \end{pmatrix}$$

The determinant of H is negative and thus we are in the presence of a minimum point. Thus the minimum is obtained when $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \frac{1}{p}$. In fact this is a simple moving average of order p. □

Theorem 2.2. *Suppose $Y_t = \varphi Y_{t-1} + \varepsilon_t$; $\varphi \neq 0$ and $|\varphi| < 1$ with ε_t is orthogonal to Y_{t-k} where $k \geq 1$, $(\varepsilon_t \perp Y_{t-1})$. The best weighted average $Z_t = \sum_{i=1}^{p} \alpha_t Y_{t-i+1}$ under the constraint $\sum_{i=1}^{p} \alpha_i = 1$ is obtained if $\alpha_1 = \alpha_p$ and $\alpha_2 = \cdots = \alpha_{p-1}$.*

Proof. Let us first calculate the variance of Z_t:
It is clear that $E(Y_t) = 0$ and $V(Y_t) = \sigma_Y^2 = \frac{1}{1-\varphi^2}$.

$$\begin{aligned} V(Z_t) &= \alpha_1^2 V(Y_t) + \alpha_2^2 V(Y_{t-1}) + \cdots + \alpha_p^2 V\Big(Y_{t-(p-1)}\Big) \\ &+ 2\alpha_1\alpha_2 E(Y_t Y_{t-1}) + \cdots + 2\alpha_1\alpha_p E\Big(Y_t Y_{t-(p-1)}\Big) \\ &+ 2\alpha_2\alpha_3 E(Y_{t-1} Y_{t-2}) + \cdots + 2\alpha_2\alpha_p E\Big(Y_{t-1} Y_{t-(p-1)}\Big) \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ 2\alpha_{p-1}\alpha_p E(Y_{t-(p-2)} Y_{t-(p-1)}) \end{aligned}$$

We can easily verify that $E\Big(Y_{t-(i-1)} Y_{t-(j-1)}\Big) = \frac{1}{1-\varphi^2}\varphi^{j-i}$; $1 \leq i < j \leq p$ and thus the variance of Z_t becomes :

$$
\begin{aligned}
V(Z_t) = \frac{1}{1-\varphi^2}\Big\{ & \Big(\alpha_1^2+\alpha_2^2+\cdots+\alpha_p^2\Big)+\Big(2\alpha_1\alpha_2\varphi+\cdots+2\alpha_1\alpha_p\varphi^{p-1}\Big) \\
& + \Big(2\alpha_2\alpha_3\varphi+\cdots+2\alpha_2\alpha_p\varphi^{p-2}\Big)+\cdots+\Big(2\alpha_{p-1}\alpha_p\varphi\Big)\Big\} \\
& = f(\alpha_1,\alpha_2,\cdots,\alpha_p).
\end{aligned}
$$

The best weighted average Z_t is that obtained when we minimize the variance with respect to the constraint $\sum_{i=1}^{p}\alpha_i = 1$. Similarly to Theorem 2.1, we proceed as follows:

$$
\begin{aligned}
L(\alpha_1,\cdots,\alpha_p\lambda) &= f(\alpha_1,\alpha_2,\cdots,\alpha_p)+\lambda(\sum_{i=1}^{p}\alpha_i-1) \\
\frac{dL}{d\alpha_1} &= 2\alpha_1+2\alpha_2\varphi+\cdots+2\alpha_p\varphi^{p-1}+\lambda=0 \qquad (1) \\
\frac{dL}{d\alpha_2} &= 2\alpha_2+2\alpha_1\varphi+2\alpha_3\varphi+\cdots+2\alpha_p\varphi^{p-2}+\lambda=0 \qquad (2) \\
\vdots & \qquad \vdots \\
\frac{dL}{d\alpha_p} &= 2\alpha_p+2\alpha_1\varphi^{p-1}+\cdots+2\alpha_{p-1}\varphi+\lambda=0 \qquad (p) \\
\frac{dL}{d\lambda} &= \alpha_1+\alpha_2+\cdots+\alpha_p-1=0 \qquad (p+1)
\end{aligned}
$$

In matrix form, we have:

$$
\begin{pmatrix}
2 & 2\varphi & 2\varphi^2 & \cdots & 2\varphi^{p-1} & 1 \\
2\varphi & 2 & 2\varphi & \cdots & 2\varphi^{p-2} & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
2\varphi^{p-1} & \cdots & 2\varphi & 2 & 2\varphi & 1 \\
1 & 1 & \cdots & \cdots & 1 & 0
\end{pmatrix}
\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \\ \lambda \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\Longleftrightarrow MX = A.
$$

Using the Gaussian method, we proceed as in the case of Theorem 2.2

$$
\left|
\begin{array}{cccccc}
2 & 2\varphi & 2\varphi^2 & \dots & 2\varphi^{p-1} & 1 \\
2\varphi & 2 & 2\varphi & \dots & 2\varphi^{p-2} & 1 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
2\phi^{p-1} & 2\varphi^{p-2} & \dots & 2\varphi & 2 & 1 \\
1 & 1 & \dots & \dots & 1 & 0
\end{array}
\right|
\quad
\begin{array}{c}
L_2 = \varphi L_1 \to L_2^1 \\
L_3 = \varphi^2 L_1 \to L_3^1 \\
\vdots \\
L_{p+1} = \ \frac{1}{2}L_1 \to L_{p+1}^1
\end{array}
$$

$$M_1 = \left| \begin{array}{cccccc} 2 & 2\varphi & 2\varphi^2 & \dots & 2\varphi^{p-1} & 1 \\ 0 & 2(1-\varphi^2) & 2(\varphi-\varphi^3) & \dots & 2(\varphi^{p-2}-\varphi^p) & 1-\varphi \\ 0 & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 2(\varphi^{p-2}-\varphi^p) & 2(\varphi^{p-3}-\varphi^{p+1}) & \dots & 2(1-\varphi^{2p-2}) & 1-\varphi^{p-1} \\ 0 & 1-\varphi & 1-\varphi^2 & \dots & 1-\varphi^{p-1} & -\frac{1}{2} \end{array} \right| \begin{array}{c} L_3^1 = \varphi L_2^1 \to L_2^3 \\ L_4^1 = \varphi^2 L_2^1 \to L_4^2 \\ \vdots \\ L_p^1 = \varphi^{p-2} L_2^1 \to L_p^2 \\ L_{p+1}^1 = \frac{1}{2(1+\varphi)} L_2^1 \to L_{p+1}^2 \end{array}$$

$$M_2 = \begin{pmatrix} 2 & 2\varphi & 2\varphi^2 & \dots & 2\varphi^{p-1} & 1 \\ 0 & 2(1-\varphi^2) & 2(\varphi-\varphi^3) & \dots & 2(\varphi^{p-2}-\varphi^p) & 1-\varphi \\ 0 & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 2(\varphi^{p-3}-\varphi^{p-1}) & \dots & 2(1-\varphi^{2p-4}) & 1-\varphi^{p-2} \\ 0 & 0 & 1-\varphi & \dots & 1-\varphi^{p-2} & -\frac{1}{2}-\frac{1}{2}\frac{1-\varphi}{1+\varphi} \end{pmatrix}$$

So on, finally we get the following upper triangular matrix:

$$D = \begin{pmatrix} 2 & 2\varphi & 2\varphi^2 & \dots & 2\varphi^{p-1} & 1 \\ 0 & 2(1-\varphi^2) & 2(\varphi-\varphi^3) & \dots & 2(\varphi^{p-2}-\varphi^p) & 1-\varphi \\ 0 & 0 & 2(1-\varphi^2) & \dots & 2(\varphi^{p-3}-\varphi^{p-1}) & 1-\varphi \\ 0 & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2(1-\varphi^2) & 1-\varphi \\ 0 & 0 & 0 & \dots & 0 & -\frac{1}{2}(1+(p-1)\frac{1-\varphi}{1+\varphi}) \end{pmatrix}$$

The determinant of this matrix is:

$$|D| = \Delta = -2^{p-1}(1-\varphi^2)^{p-1}\big(1+(p-1)\frac{1-\varphi}{1+\varphi}\big).$$

In fact: $-1 < \varphi < 1$ and $p \geq 2$, by consequence $2 < p + (2-p)\varphi < 2(p-1)$. The matrix D is invertible and consequently the linear system has a unique solution:

$$\lambda = \frac{-2(1+\varphi)}{1+\varphi+(p-1)(1-\varphi)} = \frac{-2(1+\varphi)}{p+\varphi(2-p)}$$

$$\alpha_p = \frac{1}{(1+\varphi)+(p-1)(1-\varphi)} = \frac{1}{p+\varphi(2-p)}$$

$$\alpha_{p-1} = \alpha_{p-2} = \dots = \alpha_2 = \frac{1-\varphi}{(1+\varphi)+(p-1)(1-\varphi)} = \frac{1-\varphi}{p+\varphi(2-p)}$$

$$\alpha_1 = \alpha_p = \frac{1}{(1+\varphi)+(p-1)(1-\varphi)} = \frac{1}{p+\varphi(2-p)}$$

The solutions are tabulated for $p = 2,\ 5,\ 9,\ 12$:

	$p=2$				$p=5$			
ϕ	α_1	α_2	λ	Δ	α_1	α_2	λ	Δ
-0.9	0.5	0.5	-0.1	-7.6	0.130	0.247	-0.026	-1.606
-0.8	0.5	0.5	-0.2	-7.2	0.135	0.243	-0.054	-9.943
-0.7	0.5	0.5	-0.3	-6.8	0.141	0.239	-0.085	-25.618
-0.6	0.5	0.5	-0.4	-6.4	0.147	0.235	-0.118	-45.634
-0.5	0.5	0.5	-0.5	-6.0	0.154	0.231	-0.154	-65.813
-0.4	0.5	0.5	-0.6	-5.6	0.161	0.226	-0.194	-82.315
-0.3	0.5	0.5	-0.7	-5.2	0.169	0.220	-0.237	-92.478
-0.2	0.5	0.5	-0.8	-4.8	0.179	0.214	-0.286	-95.127
-0.1	0.5	0.5	-0.9	-4.4	0.189	0.208	-0.340	-90.509
0.1	0.5	0.5	-1.1	-3.6	0.213	0.191	-0.468	-65.670
0.2	0.5	0.5	-1.2	-3.2	0.227	0.182	-0.545	-49.828
0.3	0.5	0.5	-1.3	-2.8	0.244	0.171	-0.634	-34.604
0.4	0.5	0.5	-1.4	-2.4	0.263	0.158	-0.737	-21.622
0.5	0.5	0.5	-1.5	-2.0	0.286	0.143	-0.857	-11.813
0.6	0.5	0.5	-1.6	-1.6	0.313	0.125	-1.000	-5.369
0.7	0.5	0.5	-1.7	-1.8	0.345	0.103	-1.172	-1.847
0.8	0.5	0.5	-1.8	-0.8	0.385	0.077	-1.385	-0.388
0.9	0.5	0.5	-1.9	-0.4	0.435	0.043	-1.652	-0.025

Table 1: Parameters and determinant in AR(1) model for $p = 2$ and $p = 5$.

	$p=9$				$p=12$			
ϕ	α_1	α_2	λ	Δ	α_1	α_2	λ	Δ
-0.9	0.065	0.124	-0.013	-0.307	0.048	0.090	-0.010	-0.005
-0.8	0.068	0.123	-0.027	-12.839	0.050	0.090	-0.020	-2.696
-0.7	0.072	0.122	-0.043	-93.426	0.053	0.089	-0.032	-78.747
-0.6	0.076	0.121	-0.061	-326.511	0.056	0.089	-0.044	-680.021
-0.5	0.080	0.120	-0.080	-751.781	0.059	0.088	-0.059	-2940.617
-0.4	0.085	0.119	-0.102	-1309.414	0.063	0.088	-0.075	-8023.629
-0.3	0.090	0.117	-0.126	-1852.071	0.067	0.087	-0.093	-15551.722
-0.2	0.096	0.115	-0.154	-2212.261	0.071	0.086	-0.114	-22874.498
-0.1	0.103	0.113	-0.186	-2280.039	0.077	0.085	-0.138	-26486.095
0.1	0.120	0.108	-0.265	-1605.186	0.091	0.082	-0.200	-18336.52
0.2	0.132	0.105	-0.316	-1090.100	0.100	0.080	-0.240	-10892.618
0.3	0.145	0.101	-0.377	-630.925	0.111	0.078	-0.289	-5024.403
0.4	0.161	0.097	-0.452	-302.173	0.125	0.075	-0.350	-1719.349
0.5	0.182	0.091	-0.545	-113.906	0.143	0.071	-0.429	-403.655
0.6	0.208	0.083	-0.667	-30.962	0.167	0.067	-0.533	-56.668
0.7	0.244	0.073	-0.829	-5.135	0.200	0.060	-0.680	-3.657
0.8	0.294	0.059	-1.059	-0.357	0.250	0.050	-0.900	-0.060
0.9	0.370	0.037	-1.407	-0.003	0.333	0.033	-1.267	0.000

Table 2: Parameters and determinant in AR(1) model for $p = 9$ and $p = 12$.

It is clear that the Hessian matrix is the matrix D:

$$|H| = \Delta = -2^{p-1}\big(1-\varphi^2\big)^{p-1}(1+(p-1)\frac{1-\varphi}{1+\varphi})$$

Since $|\varphi| < 1$, we have $(1-\varphi^2) > 0$, $\frac{1-\varphi}{1+\varphi} > 0$ leading to $\big(1+(p-1)\frac{1-\varphi}{1+\varphi}\big) > 0$ by consequence $|H| < 0$ and then the unique solution obtained corresponds to a minimum Q.E.D. $\square$

Theorem 2.3. *According to the results obtained in Theorem 2.2, the variance has the form:*

$$\begin{aligned} V(Z_t) &= \frac{1}{1-\varphi^2}(\alpha_1^2+\alpha_2^2+\cdots+\alpha_p^2)+(2\alpha_1\alpha_1\varphi+\cdots+2\alpha_1\alpha_p\varphi^{p-1}) \\ &+(2\alpha_2\alpha_3\varphi+\cdots+2\alpha_2\alpha_p\varphi^{p-2})+\cdots+(2\alpha_{p-1}\alpha_p\varphi) \\ &= f(\alpha_1,\alpha_2,\cdots,\alpha_p) \end{aligned}$$

Proof.

$$\begin{aligned} V(Z_t) &= \Big[2\alpha_1^2(1+\varphi^{p-1})+\alpha_2^2\big((p-2)+2\sum_{k=1}^{p-3}k\varphi^{(p-2)-k}\big)+4\alpha_1\alpha_2\sum_{k=1}^{p-2}\varphi^k\Big]\sigma_Y^2 \\ &= \frac{1}{\Big[(1+\varphi)+(p-1)(1-\varphi)\Big]^2}\Big[2(1+\varphi^{p-1})+(1-\varphi)^2((p-2)+2\sum_{k=1}^{p-3}\varphi^{(p-2)-k}) \\ &+4(1-\varphi)\sum_{k-1}^{p-2}p-2\varphi^k\Big]\sigma_Y^2 \\ &= \frac{1}{\Big[(1+\varphi)+(p-1)(1-\varphi)\Big]^2}\Big[2+4\varphi-2\varphi^{p-1}+(1-\varphi)^2(p-2) \\ &+2(1-\varphi)^2\varphi^{p-3}\sum_{k=1}^{p-3}k(\frac{1}{\varphi})^{k-1}\Big]\sigma_Y^2 \end{aligned}$$

But

$$\sum_{k=1}^{p-3}k\left(\frac{1}{\varphi}\right)^{k-1} = -\varphi^2\left[\sum_{k=1}^{p-3}\left(\frac{1}{\varphi}\right)^k\right]' = \frac{\varphi^2-(p-2)\varphi^{-p+5}+(p-3)\varphi^{-p+4}}{(1-\varphi)^2}$$

Thus $2(1-\varphi)^2\varphi^{p-3}\sum_{k=1}^{p-3}k\left(\frac{1}{\varphi}\right)^{k-1} = 2\varphi^{p-1}-2(p-2)\varphi^2+2(p-3)\varphi$ and hence

$$\begin{aligned} V(Z_t) &= \frac{1}{[p(1-\varphi)+2\varphi]^2}\Big[2+4\varphi+(1-\varphi)^2(p-2)-2\varphi^2+2\varphi(p-3)(1-\varphi)\Big]\sigma_Y^2 \\ &= \frac{1}{[p(1-\varphi)+2\varphi]^2}\Big[(1+\varphi)(p(1-\varphi)+2\varphi)\Big]\sigma_Y^2 = \frac{(1+\varphi)}{(p(1-\varphi)+2\varphi}\sigma_Y^2. \end{aligned}$$

The first partial derivative:

$$\frac{\partial V}{\partial p} = \frac{-(1-\varphi^2)}{(p(1-\varphi)+2\varphi)^2} < 0.$$

Practically, we can use a criteria like AIC (Akaike Information Criterion) proposed by Akaike (1974) and SBC (Schwarz BayesianCriterion)

$$\text{AIC} = -2\ln(L) + 2p\text{ , SBC} = -2\ln(L) + \ln(n)p,$$

where L is the maximized likelihood number of coefficients in the weighted mean and n is the number of observations in the time series Y_t. It is a compromise between the bias represented by the $-2\ln(L)$ component which decreases with the number p of coefficients $(\alpha_i,\ i = 1,\cdots,p)$ and the parsimony represented respectively by the component $2p$ and $\ln(n)p$, which need to describe the data with the smallest possible number of parameters. If the number of coefficients p is large relative to the number of observations n i.e. $\left(\frac{n}{p} < 40\right)$, Hurvich and Tsai(1995) propose the crrected AIC

$$\text{AIC}_c = \text{AIC} + \frac{(2p(p+1))}{n-p-1}.$$

The optimal value of p is the one that minimizes the criteria while p varies from 1 to a certain maximal fixed value $p_{\max}$that depends on the size of the series. □

Theorem 2.4. *Suppose $Y_t = \varepsilon_t + \theta\varepsilon_{t-1}$; $\theta \neq 0$ and $|\theta| < 1$ with ε_t is orthogonal to Y_{t-k}; $k \geq 1$ $(\varepsilon_t \perp Y_{t-1})$. The best weighed average $Z_t = \sum_{i=1}^{p} \alpha_i Y_{t-i+1}$ under the constraint $\sum_{i=1}^{p} \alpha_i = 1$ is obtained as follows:*

1. *If p is even then there are $\frac{p}{2}$ equal couples: $\alpha_1 = \alpha_p,\ \alpha_2 = \alpha_{p-1}, \ldots, \alpha_{\frac{p}{2}} = \alpha_{\frac{p}{2}+1}$.*
2. *If p is odd then there are $\frac{p-1}{2}$ equal couples: $\alpha_1 = \alpha_p,\ \alpha_2 = \alpha_{p-1}, \ldots, \alpha_{\frac{p-1}{2}} = \alpha_{\frac{p+1}{2}+1}$ and the coefficient $\alpha_{\frac{p+1}{2}}$ has its value distinct from the other coefficients.*

Proof.

$$Z_t = \alpha_1 Y_t + \alpha_2 Y_{t-1} + \cdots + \alpha_p Y_{t-(p-1)} = \sum_{i=1}^{p} \alpha_i(\epsilon_{t-i+1} + \theta\epsilon_{t-i})$$

$$V(Z_t) = \alpha_1^2 + \sum_{i=1}^{p-1}(\alpha_i\theta + \alpha_{i+1})^2 + \alpha_p^2\theta^2 = f(\alpha_1, \alpha_2, \ldots, \alpha_p).$$

Nullifying the partial derivative of a function $L(\alpha_1, \ldots, \alpha_p, \lambda) = V(Z_t) + \lambda(\sum_{i=1}^{p} \alpha_i - 1)$:

$$\frac{dL}{d\alpha_1} = 2(1+\theta^2)\alpha_1 + 2\theta\alpha_2 + \lambda = 0 \qquad (1)$$
$$\frac{dL}{d\alpha_2} = 2\theta^2\alpha_1 + 2(1+\theta^2)\alpha_2 + 2\theta\alpha_3 + \lambda = 0 \qquad (2)$$
$$\frac{dL}{d\alpha_3} = 2\theta^2\alpha_2 + 2(1+\theta^2)\alpha_3 + 2\theta\alpha_4 + \lambda = 0 \qquad (3)$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$\frac{dL}{d\alpha_{p-1}} = 2\theta^2\alpha_{p-2} + 2(1+\theta^2)\alpha_{p-1} + 2\theta\alpha_p + \lambda = 0 \qquad (p-1)$$
$$\frac{dL}{d\alpha_p} = 2\theta^2\alpha_{p-1} + 2(1+\theta^2)\alpha_p + \lambda = 0 \qquad (p)$$
$$\frac{dL}{d\lambda} = \alpha_1 + \alpha_2 + \cdots + \alpha_p - 1 = 0 \qquad (p+1)$$

In matrix form: $MX = A$ with M is a square matrix of order $(p+1)$ given by

$$M = \begin{bmatrix} 2(1+\theta^2) & 2\theta & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 2\theta & 2(1+\theta^2) & 2\theta & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 2\theta & 2(1+\theta^2) & 2\theta & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 2\theta & 2(1+\theta^2) & 2\theta & \cdots & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & & \cdots & 2\theta & 2(1+\theta^2) & 2\theta & 1 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 2(1+\theta^2) & 2\theta & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

In fact, M is not a positive definite matrix. According to Gilbert (1991), we can just use the Sylvester's Criterion finding negative the principal minor. Per example, for $p = 2$, the minors are:

$$\Delta_{11} = 2(1+\theta^2) > 0$$
$$\Delta_{22} = \begin{vmatrix} 2(1+\theta^2) & 2\theta \\ 2\theta & 2(1+\theta^2) \end{vmatrix} = 4(1+\theta^2) - 4\theta^2 = 4(1+\theta^2+\theta^4) > 0$$
$$\Delta_{33} = |M| = 4(-\theta^2 + \theta - 1) < 0$$

We limit ourselves to cases $p = 2, \ldots, 12$ that we find in practice. In the following, for each value of p, we give the algebraic expressions of the coefficients and the determinant Δ. □

The tables below show the symbolic solutions and their numerical values for $p = 2, \ldots 12$ and for $-0.9 \leq \theta \leq 0.9$. The variables a and b are functions of θ : $a = 2\left(1 + \theta^2\right)$ and $b = 2\theta$, and Δ is the determinant of each matrix.

For $p = 2$:

$$\alpha_1 = \alpha_2 = \frac{1}{2}$$

$$\lambda = -\frac{a}{2} - \frac{b}{2}$$

$$\Delta = 2b - 2a$$

θ	α_1	α_2	λ	Δ
-0.9	0.500	0.500	-0.910	-10.840
-0.8	0.500	0.500	-0.840	-9.760
-0.7	0.500	0.500	-0.790	-8.760
-0.6	0.500	0.500	-0.760	-7.840
-0.5	0.500	0.500	-0.750	-7.000
-0.4	0.500	0.500	-0.760	-6.240
-0.3	0.500	0.500	-0.790	-5.560
-0.2	0.500	0.500	-0.840	-4.960
-0.1	0.500	0.500	-0.910	-4.440
0.1	0.500	0.500	-1.110	-3.640
0.2	0.500	0.500	-1.240	-3.360
0.3	0.500	0.500	-1.390	-3.160
0.4	0.500	0.500	-1.560	-3.040
0.5	0.500	0.500	-1.750	-3.000
0.6	0.500	0.500	-1.960	-3.040
0.7	0.500	0.500	-2.190	-3.160
0.8	0.500	0.500	-2.440	-3.360
0.9	0.500	0.500	-2.710	-3.640

Table 3: Parameters and determinant in MA(1) model for $p = 2$.

For $p = 3$:

$$\alpha_1 = \alpha_3 = \frac{a-b}{3a-4b}$$
$$\alpha_2 = \frac{a-2b}{3a-4b}$$
$$\lambda = -\frac{a^2-2b^2}{3a-4b}$$
$$\Delta = 4ab - 3a^2$$

θ	$\alpha_1 = \alpha_3$	α_2	λ	Δ
-0.9	0.300	0.400	-0.367	-65.377
-0.8	0.300	0.399	-0.347	-53.267
-0.7	0.301	0.398	-0.341	-43.329
-0.6	0.302	0.395	-0.349	-35.251
-0.5	0.304	0.391	-0.370	-28.750
-0.4	0.307	0.386	-0.404	-23.571
-0.3	0.311	0.378	-0.451	-19.489
-0.2	0.316	0.367	-0.511	-16.307
-0.1	0.324	0.353	-0.583	-13.857
0.1	0.346	0.308	-0.761	-10.625
0.2	0.362	0.276	-0.863	-9.651
0.3	0.382	0.237	-0.974	-9.025
0.4	0.404	0.191	-1.091	-8.723
0.5	0.429	0.143	-1.214	-8.750
0.6	0.452	0.095	-1.345	-9.139
0.7	0.473	0.054	-1.485	-9.953
0.8	0.488	0.023	-1.639	-11.283
0.9	0.497	0.005	-1.810	-13.249

Table 4: Parameters and determinant in MA(1) model for $p = 3$.

For $p = 4$:

$$\alpha_1 = \alpha_4 = \frac{a}{4a-2b}$$
$$\alpha_2 = \alpha_3 = \frac{a-b}{4a-2b}$$
$$\lambda = -\frac{ab+a^2-b^2}{4a-2b}$$
$$\Delta = 2ab^2 + 6a^2b - 4a^3 - 2b^3$$

θ	$\alpha_1 = \alpha_4$	$\alpha_2 = \alpha_3$	λ	Δ
-0.9	0.200	0.300	-0.185	-296.158
-0.8	0.201	0.299	-0.181	-219.445
-0.7	0.202	0.298	-0.187	-163.280
-0.6	0.205	0.295	-0.203	-122.473
-0.5	0.208	0.292	-0.229	-93.000
-0.4	0.213	0.287	-0.265	-71.791
-0.3	0.220	0.280	-0.311	-56.548
-0.2	0.228	0.272	-0.366	-45.585
-0.1	0.238	0.262	-0.429	-37.689
0.1	0.263	0.237	-0.579	-27.928
0.2	0.277	0.223	-0.665	-25.075
0.3	0.290	0.210	-0.758	-23.195
0.4	0.302	0.198	-0.859	-22.168
0.5	0.312	0.188	-0.969	-22.000
0.6	0.321	0.179	-1.088	-22.849
0.7	0.327	0.173	-1.216	-25.065
0.8	0.331	0.169	-1.355	-29.268
0.9	0.333	0.167	-1.506	-36.431

Table 5: Parameters and determinant in MA(1) model for $p = 4$.

For $p = 5$:

$$\alpha_1 = \alpha_5 = \frac{ab - a^2 + b^2}{8ab - 5a^2 + b^2}$$
$$\alpha_2 = \alpha_4 = \frac{2ab - a^2}{8ab - 5a^2 + b^2}$$
$$\alpha_3 = \frac{-\left(a^2 - 2ab + b^2\right)}{8ab - 5a^2 + b^2}$$
$$\lambda = -\frac{3ab^2 - a^3}{8ab - 5a^2 + b^2}$$
$$\Delta = 8a^3b - 8ab^3 - 5a^4$$
$$-b^4 + 6a^2b^2$$

θ	$\alpha_1 = \alpha_5$	$\alpha_2 = \alpha_4$	α_3	λ	Δ
-0.9	0.143	0.228	0.257	-0.107	-1128.586
-0.8	0.144	0.228	0.255	-0.108	-764.222
-0.7	0.146	0.227	0.253	-0.118	-524.691
-0.6	0.150	0.226	0.249	-0.136	-367.419
-0.5	0.154	0.224	0.244	-0.162	-263.812
-0.4	0.160	0.221	0.237	-0.195	-195.007
-0.3	0.168	0.218	0.228	-0.236	-148.753
-0.2	0.178	0.213	0.219	-0.284	-117.192
-0.1	0.188	0.207	0.209	-0.339	-95.329
0.1	0.212	0.191	0.193	-0.467	-69.212
0.2	0.225	0.180	0.190	-0.540	-61.729
0.3	0.238	0.165	0.193	-0.619	-56.829
0.4	0.253	0.146	0.202	-0.703	-54.177
0.5	0.268	0.122	0.220	-0.793	-53.812
0.6	0.285	0.092	0.245	-0.887	-56.247
0.7	0.303	0.059	0.275	-0.986	-62.740
0.8	0.319	0.028	0.305	-1.092	-75.819
0.9	0.330	0.007	0.326	-1.207	-100.163

Table 6: Parameters and determinant in MA(1) model for $p = 5$.

For $p = 6$:

$$\alpha_1 = \alpha_6 = \frac{-a^2 + b^2}{4ab - 6a^2 + 4b^2}$$
$$\alpha_2 = \alpha_5 = \frac{ab - a^2 + b^2}{4ab - 6a^2 + 4b^2}$$
$$\alpha_3 = \alpha_4 = \frac{ab - a^2}{4ab - 6a^2 + 4b^2}$$
$$\lambda = \frac{-2ab^2 + a^2b + a^3 - b^3}{4ab - 6a^2 + 4b^2}$$
$$\Delta = 10a^4b - 4ab^4 - 6a^5 + 4b^5$$

θ	$\alpha_1 = \alpha_6$	$\alpha_2 = \alpha_5$	$\alpha_3 = \alpha_4$	λ	Δ
-0.9	0.108	0.179	0.214	-0.068	-3828.483
-0.8	0.109	0.179	0.213	-0.071	-2380.414
-0.7	0.111	0.179	0.210	-0.082	-1520.365
-0.6	0.115	0.178	0.206	-0.099	-1004.790
-0.5	0.121	0.178	0.201	-0.124	-690.562
-0.4	0.128	0.178	0.195	-0.154	-494.634
-0.3	0.136	0.176	0.188	-0.191	-369.135
-0.2	0.145	0.174	0.180	-0.233	-286.460
-0.1	0.156	0.171	0.173	-0.280	-230.564
0.1	0.178	0.160	0.162	-0.391	-165.137
0.2	0.189	0.152	0.159	-0.455	-146.605
0.3	0.201	0.141	0.158	-0.523	-134.444
0.4	0.213	0.129	0.158	-0.597	-127.697
0.5	0.223	0.117	0.160	-0.676	-126.312
0.6	0.233	0.105	0.162	-0.760	-131.452
0.7	0.241	0.096	0.164	-0.851	-146.459
0.8	0.246	0.089	0.165	-0.949	-179.137
0.9	0.249	0.085	0.166	-1.054	-246.501

Table 7: Parameters and determinant in MA(1) model for $p = 6$.

For $p = 7$:

$$\alpha_1 = \alpha_7 = \frac{2ab^2 + a^2b - a^3 - b^3}{6ab^2 + 12a^2b - 7a^3 - 8b^3}$$
$$\alpha_2 = \alpha_6 = \frac{ab^2 + 2a^2b - a^3 - 2b^3}{6ab^2 + 12a^2b - 7a^3 - 8b^3}$$
$$\alpha_3 = \alpha_5 = -\frac{a^3 - 2a^2b + b^3}{6ab^2 + 12a^2b - 7a^3 - 8b^3}$$
$$\alpha_4 = \frac{2a^2b - a^3}{6ab^2 + 12a^2b - 7a^3 - 8b^3}$$
$$\lambda = \frac{a^4 + 2b^4 - 4a^2b^2}{6ab^2 + 12a^2b - 7a^3 - 8b^3}$$
$$\Delta = 16ab^5 + 12a^5b - 7a^6 - 12a^2b^4 - 32a^3b^3 + 20a^4b^2$$

θ	$\alpha_1 = \alpha_7$	$\alpha_2 = \alpha_6$	$\alpha_3 = \alpha_5$	α_4	λ	Δ
-0.9	0.084	0.143	0.178	0.190	-0.046	-11944.403
-0.8	0.085	0.143	0.177	0.188	-0.050	-6850.629
-0.7	0.088	0.144	0.175	0.185	-0.061	-4101.102
-0.6	0.093	0.145	0.172	0.180	-0.077	-2581.865
-0.5	0.098	0.146	0.168	0.174	-0.100	-1714.609
-0.4	0.106	0.147	0.163	0.167	-0.127	-1199.941
-0.3	0.114	0.148	0.158	0.160	-0.160	-881.719
-0.2	0.123	0.148	0.153	0.153	-0.197	-677.084
-0.1	0.133	0.146	0.147	0.148	-0.239	-540.937
0.1	0.153	0.138	0.139	0.139	-0.337	-383.702
0.2	0.164	0.131	0.137	0.136	-0.393	-339.506
0.3	0.174	0.122	0.138	0.132	-0.453	-310.516
0.4	0.185	0.110	0.142	0.125	-0.518	-294.344
0.5	0.196	0.096	0.151	0.114	-0.587	-290.859
0.6	0.209	0.077	0.166	0.096	-0.660	-302.972
0.7	0.222	0.054	0.189	0.070	-0.737	-339.419
0.8	0.235	0.029	0.217	0.038	-0.818	-422.379
0.9	0.246	0.008	0.241	0.010	-0.905	-606.550

Table 8: Parameters and determinant in MA(1) model for $p = 7$.

For $p = 8$:

$$\alpha_1 = \alpha_8 = \frac{2ab^2 - a^3}{2(6ab^2 + 3a^2b - 4a^3 - 2b^3)}$$
$$\alpha_2 = \alpha_7 = \frac{2ab^2 + a^2b - a^3 - b^3}{2(6ab^2 + 3a^2b - 4a^3 - 2b^3)}$$
$$\alpha_3 = \alpha_6 = \frac{ab^2 + a^2b - a^3 - b^3}{2(6ab^2 + 3a^2b - 4a^3 - 2b^3)}$$
$$\alpha_4 = \alpha_5 = \frac{ab^2 + a^2b - a^3}{2(6ab^2 + 3a^2b - 4a^3 - 2b^3)}$$
$$\lambda = \frac{a^3b - 2ab^3 + a^4 + b^4 - 3a^2b^2}{2(6ab^2 + 3a^2b - 4a^3 - 2b^3)}$$
$$\Delta = 4ab^6 + 14a^6b - 8a^7 - 4b^7 + 42a^2b^5 - 28a^3b^4 - 50a^4b^3 + 30a^5b^2$$

θ	$\alpha_1 = \alpha_8$	$\alpha_2 = \alpha_7$	$\alpha_3 = \alpha_6$	$\alpha_4 = \alpha_5$	λ	Δ
-0.9	0.067	0.117	0.150	0.166	-0.033	-34986.361
-0.8	0.069	0.118	0.149	0.164	-0.038	-18592.533
-0.7	0.072	0.119	0.148	0.161	-0.048	-10504.340
-0.6	0.077	0.121	0.146	0.156	-0.063	-6349.127
-0.5	0.083	0.124	0.143	0.151	-0.083	-4103.969
-0.4	0.090	0.126	0.140	0.145	-0.108	-2822.387
-0.3	0.098	0.127	0.136	0.139	-0.137	-2050.449
-0.2	0.107	0.128	0.132	0.133	-0.171	-1562.500
-0.1	0.116	0.127	0.128	0.129	-0.208	-1241.496
0.1	0.134	0.121	0.122	0.122	-0.296	-874.257
0.2	0.144	0.115	0.121	0.120	-0.346	-771.605
0.3	0.154	0.108	0.121	0.118	-0.399	-704.255
0.4	0.163	0.098	0.124	0.115	-0.457	-666.371
0.5	0.173	0.087	0.128	0.112	-0.519	-657.281
0.6	0.182	0.076	0.134	0.108	-0.585	-683.127
0.7	0.190	0.065	0.140	0.105	-0.656	-763.740
0.8	0.195	0.057	0.146	0.102	-0.732	-954.176
0.9	0.199	0.052	0.149	0.101	-0.813	-1408.548

Table 9: Parameters and determinant in MA(1) model for $p = 8$.

For $p = 9$:

$$\alpha_1 = \alpha_9 = \frac{2ab^3 - a^3b + a^4 + b^4 - 3a^2b^2}{24ab^3 - 16a^3b + 9a^4 + b^4 - 15a^2b^2}$$
$$\alpha_2 = \alpha_8 = \frac{4ab^3 - 2a^3b + a^4 - 2a^2b^2}{24ab^3 - 16a^3b + 9a^4 + b^4 - 15a^2b^2}$$
$$\alpha_3 = \alpha_7 = -\frac{2a^3b - 3ab^3 - a^4 + b^4 + a^2b^2}{24ab^3 - 16a^3b + 9a^4 + b^4 - 15a^2b^2}$$
$$\alpha_4 = \alpha_6 = -\frac{(a-b)(2ab^2 + a^2b - a^3)}{24ab^3 - 16a^3b + 9a^4 + b^4 - 15a^2b^2}$$
$$\alpha_5 = \frac{2ab^3 - 2a^3b + a^4 + b^4 - a^2b^2}{24ab^3 - 16a^3b + 9a^4 + b^4 - 15a^2b^2}$$
$$\lambda = -\frac{5ab^4 + a^5 - 5a^3b^2}{24ab^3 - 16a^3b + 9a^4 + b^4 - 15a^2b^2}$$
$$\Delta = 16a^7b - 24ab^7 - 9a^8 - b^8 + 18a^2b^6 + 88a^3b^5 - 55a^4b^4 - 72a^5b^3 + 42a^6b^2$$

θ	$\alpha_1 = \alpha_9$	$\alpha_2 = \alpha_8$	$\alpha_3 = \alpha_7$	$\alpha_4 = \alpha_6$	α_5	λ	Δ
-0.9	0.055	0.097	0.127	0.145	0.151	-0.024	-97557.253
-0.8	0.057	0.099	0.127	0.143	0.149	-0.029	-48243.204
-0.7	0.060	0.101	0.126	0.140	0.145	-0.039	-25884.284
-0.6	0.065	0.103	0.125	0.136	0.140	-0.053	-15119.653
-0.5	0.071	0.107	0.124	0.132	0.134	-0.071	-9565.316
-0.4	0.078	0.109	0.122	0.127	0.128	-0.094	-6490.714
-0.3	0.086	0.112	0.119	0.122	0.122	-0.120	-4674.988
-0.2	0.094	0.113	0.117	0.117	0.118	-0.151	-3541.667
-0.1	0.103	0.113	0.114	0.114	0.114	-0.185	-2802.234
0.1	0.120	0.108	0.109	0.109	0.109	-0.264	-1962.222
0.2	0.129	0.103	0.108	0.107	0.107	-0.309	-1728.395
0.3	0.137	0.096	0.109	0.105	0.106	-0.357	-1574.976
0.4	0.146	0.088	0.111	0.101	0.107	-0.410	-1488.283
0.5	0.155	0.077	0.117	0.095	0.110	-0.465	-1466.566
0.6	0.165	0.064	0.127	0.085	0.118	-0.525	-1523.851
0.7	0.175	0.048	0.144	0.066	0.135	-0.588	-1706.559
0.8	0.186	0.027	0.167	0.040	0.161	-0.654	-2150.052
0.9	0.196	0.008	0.190	0.012	0.188	-0.724	-3270.226

Table 10: Parameters and determinant in MA(1) model for $p = 9$.

For $p = 10$:

$$\alpha_1 = \alpha_{10} = \frac{a^4 + b^4 - 3a^2b^2}{2(6ab^3 - 4a^3b + 5a^4 + 3b^4 - 12a^2b^2)}$$
$$\alpha_2 = \alpha_9 = \frac{2ab^3 - a^3b + a^4 + b^4 - 3a^2b^2}{2(6ab^3 - 4a^3b + 5a^4 + 3b^4 - 12a^2b^2)}$$
$$\alpha_3 = \alpha_8 = \frac{2ab^3 - a^3b + a^4 - 2a^2b^2}{2(6ab^3 - 4a^3b + 5a^4 + 3b^4 - 12a^2b^2)}$$
$$\alpha_4 = \alpha_7 = \frac{ab^3 - a^3b + a^4 - 2a^2b^2}{2(6ab^3 - 4a^3b + 5a^4 + 3b^4 - 12a^2b^2)}$$
$$\alpha_5 = \alpha_6 = \frac{ab^3 - a^3b + a^4 + b^4 - 2a^2b^2}{2(6ab^3 - 4a^3b + 5a^4 + 3b^4 - 12a^2b^2)}$$
$$\lambda = -\frac{3ab^4 + a^4b + a^5 + b^5 - 3a^2b^3 - 4a^3b^2}{2(6ab^3 - 4a^3b + 5a^4 + 3b^4 - 12a^2b^2)}$$
$$\Delta = 18a^8b - 6ab^8 - 10a^9 + 6b^9 - 78a^2b^7 + 52a^3b^6 + 160a^4b^5 - 96a^5b^4 - 98a^6b^3 + 56a^7b^2$$

θ	$\alpha_1 = \alpha_{10}$	$\alpha_2 = \alpha_9$	$\alpha_3 = \alpha_8$	$\alpha_4 = \alpha_7$	$\alpha_5 = \alpha_6$	λ	Δ
-0.9	0.046	0.082	0.109	0.127	0.136	-0.019	-261521.501
-0.8	0.048	0.084	0.109	0.125	0.133	-0.023	-120839.619
-0.7	0.051	0.086	0.109	0.123	0.130	-0.032	-61918.692
-0.6	0.056	0.090	0.109	0.120	0.125	-0.046	-35144.132
-0.5	0.062	0.093	0.109	0.116	0.119	-0.062	-21853.324
-0.4	0.069	0.097	0.108	0.112	0.114	-0.083	-14674.053
-0.3	0.077	0.100	0.106	0.108	0.109	-0.107	-10498.197
-0.2	0.084	0.101	0.104	0.105	0.105	-0.135	-7916.667
-0.1	0.092	0.101	0.102	0.102	0.102	-0.166	-6242.951
0.1	0.108	0.097	0.098	0.098	0.098	-0.238	-4351.858
0.2	0.116	0.093	0.098	0.097	0.097	-0.279	-3827.160
0.3	0.124	0.087	0.098	0.095	0.096	-0.323	-3482.871
0.4	0.132	0.079	0.101	0.092	0.095	-0.371	-3287.508
0.5	0.141	0.070	0.105	0.089	0.095	-0.422	-3236.246
0.6	0.149	0.060	0.112	0.084	0.096	-0.476	-3358.762
0.7	0.156	0.050	0.119	0.078	0.097	-0.534	-3755.911
0.8	0.162	0.041	0.127	0.072	0.099	-0.596	-4735.392
0.9	0.165	0.035	0.132	0.068	0.100	-0.662	-7334.689

Table 11: Parameters and determinant in MA(1) model for $p = 10$.

For $p = 11$:

$$\alpha_1 = \alpha_{11} = \frac{3ab^4 - a^4b + a^5 - b^5 + 3a^2b^3 - 4a^3b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\alpha_2 = \alpha_{10} = \frac{ab^4 - 2a^4b + a^5 - 2b^5 + 6a^2b^3 - 3a^3b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\alpha_3 = \alpha_9 = -\frac{ab^4 + 2a^4b - a^5 + b^5 - 5a^2b^3 + 2a^3b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\alpha_4 = \alpha_8 = -\frac{2a^4b - a^5 - 4a^2b^3 + 2a^3b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\alpha_5 = \alpha_7 = \frac{ab^4 - 2a^4b + a^5 - b^5 + 4a^2b^3 - 2a^3b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\alpha_6 = \frac{ab^4 - 2a^4b + a^5 - 2b^5 + 4a^2b^3 - 2a^3b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\lambda = -\frac{a^6 - 2b^6 + 9a^2b^4 - 6a^4b^2}{9ab^4 - 20a^4b + 11a^5 - 12b^5 + 48a^2b^3 - 28a^3b^2}$$
$$\Delta = 36ab^9 + 20a^9b - 11a^10 - 27a^2b^8 - 192a^3b^7 + 120a^4b^6 + 264a^5b^5$$
$$-154a^6b^4 - 128a^7b^3 + 72a^8b^2$$

θ	$\alpha_1 = \alpha_{11}$	$\alpha_2 = \alpha_{10}$	$\alpha_3 = \alpha_9$	$\alpha_4 = \alpha_8$	$\alpha_5 = \alpha_7$	α_6	λ	Δ
-0.9	0.039	0.071	0.095	0.112	0.122	0.125	-0.015	-678837.976
-0.8	0.041	0.072	0.095	0.111	0.120	0.123	-0.019	-294242.196
-0.7	0.045	0.075	0.096	0.109	0.116	0.118	-0.028	-144716.781
-0.6	0.050	0.079	0.097	0.107	0.112	0.113	-0.040	-80173.384
-0.5	0.056	0.083	0.097	0.104	0.107	0.108	-0.056	-49159.995
-0.4	0.062	0.087	0.097	0.101	0.102	0.103	-0.074	-32733.951
-0.3	0.069	0.090	0.096	0.098	0.098	0.098	-0.097	-23292.863
-0.2	0.076	0.091	0.094	0.095	0.095	0.095	-0.122	-17500.000
-0.1	0.084	0.092	0.093	0.093	0.093	0.093	-0.150	-13762.869
0.1	0.098	0.089	0.090	0.089	0.089	0.089	-0.216	-9558.546
0.2	0.106	0.085	0.089	0.088	0.088	0.088	-0.254	-8395.062
0.3	0.113	0.079	0.090	0.087	0.087	0.087	-0.295	-7631.586
0.4	0.121	0.073	0.092	0.084	0.087	0.086	-0.339	-7197.010
0.5	0.129	0.064	0.097	0.080	0.089	0.083	-0.386	-7079.604
0.6	0.136	0.054	0.104	0.073	0.094	0.077	-0.436	-7344.394
0.7	0.145	0.041	0.117	0.060	0.105	0.065	-0.489	-8216.234
0.8	0.154	0.025	0.135	0.039	0.126	0.043	-0.544	-10400.629
0.9	0.163	0.008	0.156	0.013	0.153	0.014	-0.603	-16445.661

Table 12: Parameters and determinant in MA(1) model for $p = 11$.

For $p = 12$:

$$\alpha_1 = \alpha_{12} = \frac{3ab^4 + a^5 - 4a^3b^2}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\alpha_2 = \alpha_{11} = \frac{3ab^4 - a^4b + a^5 - b^5 + 3a^2b^3 - 4a^3b^2}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\alpha_3 = \alpha_{10} = \frac{ab^4 - a^4b + a^5 - b^5 + 3a^2b^3 - 3a^3b^2}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\alpha_4 = \alpha_9 = \frac{\big(-a+b\big)\big(ab^3 - a^4 + 3a^2b^2\big)}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\alpha_5 = \alpha_8 = \frac{a\big(2ab^3 - a^3b + a^4 + 2b^4 - 3a^2b^2\big)}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\alpha_6 = \alpha_7 = \frac{2ab^4 - a^4b + a^5 - b^5 + 2a^2b^3 - 3a^3b^2}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\lambda = \frac{-3ab^5 - a^5b - a^6 + b^6 - 6a^2b^4 + 4a^3b^3 + 5a^4b^2}{24ab^4 - 10a^4b + 12a^5 - 6b^5 + 24a^2b^3 - 40a^3b^2}$$
$$\Delta = 6ab^{10} + 22a^{10}b - 12a^{11} - 6b^{11} + 132a^2b^9 - 88a^3b^8 - 400a^4b^7 + 240a^5b^6 \\ +406a^6b^5 - 232a^7b^4 - 162a^8b^3 + 90a^9b^2$$

θ	$\alpha_1 = \alpha_{12}$	$\alpha_2 = \alpha_{11}$	$\alpha_3 = \alpha_{10}$	$\alpha_4 = \alpha_9$	$\alpha_5 = \alpha_8$	$\alpha_6 = \alpha_7$	λ	Δ
-0.9	0.034	0.061	0.083	0.099	0.109	0.115	-0.012	-1715516.840
-0.8	0.036	0.063	0.084	0.098	0.107	0.112	-0.016	-700169.891
-0.7	0.039	0.067	0.085	0.097	0.104	0.108	-0.024	-332025.354
-0.6	0.044	0.071	0.086	0.096	0.101	0.103	-0.036	-180208.391
-0.5	0.050	0.075	0.087	0.094	0.096	0.098	-0.050	-109234.664
-0.4	0.056	0.079	0.088	0.091	0.093	0.093	-0.067	-72240.071
-0.3	0.063	0.082	0.087	0.089	0.090	0.090	-0.088	-51178.678
-0.2	0.070	0.083	0.086	0.087	0.087	0.087	-0.111	-38333.333
-0.1	0.076	0.084	0.085	0.085	0.085	0.085	-0.138	-30079.672
0.1	0.090	0.081	0.082	0.082	0.082	0.082	-0.199	-20826.750
0.2	0.097	0.078	0.082	0.081	0.081	0.081	-0.234	-18271.605
0.3	0.104	0.073	0.082	0.080	0.080	0.080	-0.271	-16594.857
0.4	0.111	0.067	0.085	0.078	0.080	0.079	-0.312	-15637.921
0.5	0.118	0.059	0.089	0.074	0.081	0.078	-0.355	-15372.543
0.6	0.125	0.050	0.095	0.069	0.083	0.077	-0.402	-15936.655
0.7	0.132	0.041	0.103	0.062	0.087	0.075	-0.451	-17811.963
0.8	0.138	0.032	0.111	0.055	0.091	0.073	-0.503	-22539.670
0.9	0.142	0.026	0.117	0.050	0.094	0.072	-0.559	-36061.015

Table 13: Parameters and determinant in MA(1) model for $p = 12$.

Investigating the determinants Δ of M matrices we observe a negative value and by consequence the Lagrangian optimization occurs a minimum.

3. CONCLUSION

At the end of this research, we can present to the specialists in the field of aggregated process ARMA model, especially when there is missing data in the economic and financial series, some theoretical points that can clarify the structure of a process resulting from an optimal choice of a smooth linear combination of the basic process. The coefficients change depending wether we have a white noise, $\mathrm{AR}(1)$ or $\mathrm{MA}(1)$ process. In the case of a white noise process, the specialists can consider a weighted average of equal coefficients. This requires researchers to test whether the basic phenomenon follows a white noise or not. On the other hand, if the basic process follows a stationary $\mathrm{AR}(1)$ process then there will be two types of coefficients: the extreme coefficients will be equal and the coefficients in the middle are governed by equal points. If the basic process follows a stationary $\mathrm{MA}(1)$ process then we have two types of coefficients according to the value of p i.e. if p is even then there are $\frac{p}{2}$ equal couples of coefficients $\alpha_i, i = 1, \ldots, p$ but if p is odd then there are $\frac{p-1}{2}$ equal couples of coefficients α_i and the (median) coefficient $\alpha_{\frac{p+1}{2}}$ has a value different from the other coefficients. Also the choice of the optimal value p can be done using one of the criteria like AIC or AIC_c for example.

Recommendation:

Our future proposal will be destined to find a general solution for the matrix M in the case MA(1) using 3 methods:

- The matrix M without the last column and the last row is a positive definite matrix (proof is available). The strategy leads to prove the invertibility of the matrix M using the inductive principle.
- We think that we must distinguish 4 cases:
  - p is divided by 4.
  - p is even but not divided by 4.
  - p is odd and divided by 3.
  - p is odd and not divided by 3.
- We will focus our attention to elaborate a general formula taking into account the 4 possible cases of p.

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