Fixed Point Theorems for \((\varepsilon, \lambda)-\)Uniformly Locally Generalized Contractions

G. Sudhaamsh Mohan Reddy

Faculty of Science and Technology, ICFAI Foundation for Higher Education
Dontanapalli, Shankarpalli Road, Hyderabad-501203, India

Abstract

In this paper, we define a class called \((\varepsilon, \lambda)-\)uniformly locally generalized contractions and establish a fixed point theorem for such contractions.

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Introduction:

1. Definition: A selfmap \(f\) of a \(D^*-\)metric space \((X, D^*)\) is called a \((\varepsilon, \lambda)-\)uniformly locally generalized contraction, if there is a number \(q\) with \(0 \leq q < 1\) and a positive constant \(\varepsilon\), such that

\[
D^*(fx, fy, fz) \leq q \cdot D^*(x, y, y) + r \cdot D^*(x, fx, fx) + s \cdot D^*(y, fy, fy) + t \cdot \{D^*(x, fx, fy) + D^*(y, fx, fy)\}
\]

for all \(x, y \in X\) with \(D^*(x, y, y) < \varepsilon\), where \(\sup_{x, y \in X} \{q + r + s + 2t\} = \lambda < 1\).

We now prove

Main Theorem:

2. Theorem: Suppose \(f\) is a \((\varepsilon, \lambda)-\)uniformly locally generalized contraction of a \(D^*-\)metric space \((X, D^*)\) and \(X\) is \(f\)-orbitally complete. Then for every \(x \in X\), either
(2.1) \( D^s(f^s x, f^{s+1} x, f^{s+1} x) \geq \varepsilon \) for all integers \( s \geq 0 \)

or

(2.2) the sequence \( \{f^n x\} \) converges to \( u \), which is a fixed point of \( f \). Also there is no other fixed point \( v \in X \) with \( D^s(u,v,v) < \varepsilon \).

**Proof:** For any \( x \in X \), consider \( \{D^s(f^s x, f^{s+1} x, f^{s+1} x)\}_{s=0}^{\infty} \). Then we have either each of the term in this sequence is greater than or equal to \( \varepsilon \) or for some term in it is less than \( \varepsilon \).

In the first case, the alternative of (2.1) of the hypothesis holds.

Let for some integer \( s = s_0 \), \( D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x) < \varepsilon \). Since \( f \) is a \((\varepsilon, \lambda)\)-uniformly locally generalized contraction and \( D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x) < \varepsilon \), we get numbers \( q, r, s, \) and \( t \) (all depending on \( x \) and \( y \)) such that

\[
D^s(f^{s_0+1} x, f^{s_0+2} x, f^{s_0+2} x) = D^s(ff^{s_0} x, ff^{s_0+1} x, ff^{s_0+1} x)
\]

\[
\leq q.D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x) + r.D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x)
\]

\[
+ s.D^s(f^{s_0+1} x, f^{s_0+2} x, f^{s_0+2} x)
\]

\[
+ t\left\{D^s(f^{s_0} x, f^{s_0+2} x, f^{s_0+2} x) + D^s(f^{s_0+1} x, f^{s_0+1} x, f^{s_0+1} x)\right\}
\]

\[
\leq q.D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x) + r.D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x)
\]

\[
+ s.D^s(f^{s_0+1} x, f^{s_0+2} x, f^{s_0+2} x)
\]

\[
+ t\left\{D^s(f^{s_0} x, f^{s_0+2} x, f^{s_0+2} x) + D^s(f^{s_0+1} x, f^{s_0+2} x, f^{s_0+2} x)\right\}
\]

\[
\leq (q + r + t).D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x)
\]

\[
+ (s + t).D^s(f^{s_0+1} x, f^{s_0+2} x, f^{s_0+2} x)
\]

Therefore

\[
(1-s-t)D^s(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}) \leq (q + r + t)D^s(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x)
\]
This implies that

\[ D^*(f_{s_0+1}, f_{s_0+2}, f_{s_0+3}) \leq \left(\frac{q+r+t}{1-s-t}\right) D^*(f_{s_0}x, f_{s_0+1}x, f_{s_0+2}x) \]

\[ \leq \lambda \cdot D^*(f_{s_0}x, f_{s_0+1}x, f_{s_0+2}x) \]

Also we get by repeated use of the above inequality that

\[ D^*(f_{s_0+p}, f_{s_0+p+1}, f_{s_0+p+2}) \leq \lambda \cdot D^*(f_{s_0+p-1}x, f_{s_0+p}x, f_{s_0+p+1}x) \]

\[ \leq \lambda^2 \cdot D^*(f_{s_0+p-2}x, f_{s_0+p-1}x, f_{s_0+p}x) \]

\[ \ldots \]

\[ \ldots \]

\[ \leq \lambda^p \cdot D^*(f_{s_0}x, f_{s_0+1}x, f_{s_0+2}x) \]

That is, \( D^*(f_{s_0+p}, f_{s_0+p+1}, f_{s_0+p+2}) < \varepsilon \) for every integer \( p = 0, 1, 2, 3, \ldots \)

and hence for \( n \geq s_0 \), we have

\[ D^*(f^n x, f^{n+p} x, f^{n+p} x) \leq D^*(f^n x, f^{n+1} x, f^{n+1} x) + D^*(f^{n+1} x, f^{n+2} x, f^{n+2} x) \]

\[ + \ldots + D^*(f^{n+p-1} x, f^{n+p} x, f^{n+p} x) \]

\[ \leq (\lambda^{n-s_0} + \lambda^{n-s_0+1} + \ldots + \lambda^{n-s_0+p-1}) D^*(f_{s_0}x, f_{s_0+1}x, f_{s_0+2}x) \]

\[ \leq (\lambda^{n-s_0} + \lambda^{n-s_0+1} + \ldots + \lambda^{n-s_0+p-1} + \ldots) \cdot D^*(f_{s_0}x, f_{s_0+1}x, f_{s_0+2}x) \]

\[ \leq \lambda^{n-s_0} \cdot D^*(f_{s_0}x, f_{s_0+1}x, f_{s_0+2}x) \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \]

Thus the sequence \( \{f^n x\} \) is a Cauchy sequence in a \( f \)-orbitally complete \( D^*- \)metric space \((X, D^*)\) and hence there exists \( u \in X \) such that

\[ u = \lim_{n \to \infty} f^n x = \lim_{n \to \infty} f^{s_0+p} \]
Therefore there is an integer \( n_0 > s_0 \) such that
\[
D^*(f^n x, u, u) < \varepsilon \quad \text{for all} \quad n \geq n_0
\]
Now
\[
D^*(f u, f^n x, f^n x) \leq q D^*(u, f^n x, f^n x) + r D^*(u, f u, f u) + s D^*(f^n x, f^{n+1} x, f^{n+1} x)
\]
\[
+ t \left\{ D^*(u, f^{n+1} x, f^{n+1} x) + D^*(f^n x, f^{n+1} x, f^{n+1} x) \right\}
\]
\[
D^*(f u, f^{n+1} x, f^{n+1} x) \leq q D^*(u, f^n x, f^n x) + r D^*(u, f^{n+1} x, f^{n+1} x)
\]
\[
+ r D^*(f^{n+1} x, f u, f u) + s D^*(f^n x, f^{n+1} x, f^{n+1} x)
\]
\[
+ t D^*(u, f^{n+1} x, f^{n+1} x) + t D^*(f^n x, f^{n+1} x, f^{n+1} x)
\]
\[
+ t D^*(f^{n+1} x, f u, f u)
\]
\[
\leq q D^*(u, f^n x, f^n x) + (r + t) D^*(u, f^{n+1} x, f^{n+1} x)
\]
\[
+ (s + t) D^*(f^n x, f^{n+1} x, f^{n+1} x)
\]
\[
+ (r + t) D^*(f u, f^{n+1} x, f^{n+1} x)
\]
\[
\leq \lambda D^*(u, f^n x, f^n x) + \lambda D^*(u, f^{n+1} x, f^{n+1} x)
\]
\[
+ \lambda D^*(f^n x, f^{n+1} x, f^{n+1} x) + \lambda D^*(f u, f^{n+1} x, f^{n+1} x)
\]
which gives
\[
(1 - \lambda) D^*(f u, f^{n+1} x, f^{n+1} x) \leq \lambda \left\{ D^*(u, f^n x, f^n x) + D^*(u, f^{n+1} x, f^{n+1} x)
\right\}
\]
\[
+ D^*(f^n x, f^{n+1} x, f^{n+1} x) \right\}
\]
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Therefore,

\[
D^*(fu, f^{n+1}x, f^{n+1}x) \leq \frac{\lambda}{(1-\lambda)} \left[ D^*(u, f^n x, f^n x) + D^*(u, f^{n+1}x, f^{n+1}x) \right] + D^*(f^n x, f^{n+1}x, f^{n+1}x)
\]

Now letting \(n \to \infty\), it follows that \(D^*(fu, u, u) = 0\) which implies that \(fu = u\), showing that the sequence \(\{f^nx\}\) converges to some point of \(X\).

To prove the uniqueness of fixed point of \(f\), suppose that \(fv = v\) for some \(v \in X\) and \(D^*(u,v,v) < \varepsilon\). Then

\[
D^*(u,v,v) = D^*(fu, fv, fv)
\]

\[
\leq qD^*(u,v,v) + rD^*(u, fu, fu) + sD^*(v, fv, fv)
\]

\[
+ t[D^*(u, fv, fv) + D^*(v, fu, fu)]
\]

\[
= qD^*(u,v,v) + rD^*(u,u,u) + sD^*(v,v,v)
\]

\[
+ t[D^*(u,v,v) + D^*(v,u,u)]
\]

\[
= (q + 2t)D^*(u,v,v) = \lambda D^*(u,v,v)
\]

which implies that \(D^*(u,v,v) = 0\), since \(\lambda < 1\) and hence \(u = v\), proving the second part of (2.2).

2.2 **Corollary:** Suppose \(f\) is a \((\varepsilon, \lambda)\)-uniformly locally generalized contraction of a \(D^*\)-metric space \((X, D^*)\) and \(X\) is \(f\)-orbitally complete. If for every \(x \in X\), there is an integer \(n(x)\) such that

\[
(2.2.1) \quad D^*(f^{n(x)}x, f^{n(x)+1}x, f^{n(x)+1}x) < \varepsilon
\]

Then \(f\) has a unique fixed point, provided any two fixed points \(u, v\) of \(f\) are such that \(D^*(u,v,v) < \varepsilon\).

Also the sequence \(\{f^nx\}\) for any \(x \in X\) converges to the unique fixed point of \(f\).

**Proof:** Follows from Theorem 1.1.
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REFERENCES


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