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NONLINEAR DERIVATIONS ON OPERATOR BANACH-JORDAN PAIRS

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Abstract

The main goal of this paper consists in giving the general form of (nonlinear) derivations on the Banach-Jordan pair $V = (\mathcal{BL}(Y, X), \mathcal{BL}(X, Y))$ *of operators defined between two real or complex Banach spaces* X *and* Y.

Key words: *Associative pair, Operator Banach-Jordan pair, Derivation, Jordan Derivation.*

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1. Introduction

An interesting topic of the theory of derivations centres the question of boundedness of any derivation on a class of Banach algebras. In light of this idea, two questions spring to mind, namely, do unbounded derivations actually exist and, if they do, do there exist, nevertheless, large classes of Banach algebras on which all derivations are automatically bounded. The answer to both questions is affirmative. In this direction, Johnson and Sinclair showed that all derivations on semisimple Banach algebras are automatically continuous [7]. The investigation into the problem of automatic continuity was developed later in different directions and in a general context of Banach triple systems and pairs, providing among other crucial results, that derivations on A are continuous whenever A is a Banach semisimple alternative [14] or Jordan algebra [15] , associative B^*-triple system [16] , a semisimple Banach-Jordan pair [6] . Of particular relevance to our work is the crucial result, proved by P. Šemrl [12] on the general form of nonlinear derivations of Banach algebras.

In this paper, we deal with nonlinear or either additive derivations acting on Banach-Jordan pairs. We give the general form of these mappings defined on the Banach-Jordan pair of operators between two Banach spaces, generalizing by the way the Šemrl's result on nonlinear derivations of Banach algebras [12].

2. Algebraic preliminaries

In this paper we shall deal with associative and Jordan pairs over a ring of scalars $\mathcal{R}$ containing $\frac{1}{2}$. Nevertheless, we shall be mainly interested in the case when $\mathcal{R}$ is the complex or the real field. The definitions relating to associative or Jordan pairs not explicitly stated in the paper can be found in [8] . However, we record in this section some notations and results.

Let $A = (A^+, A^-)$ be a pair of $\mathcal{R}$-modules equipped with trilinear maps

$$\begin{array}{llll} <\ldots>: & A^\sigma \times A^{-\sigma} \times A^\sigma & \longrightarrow & A^\sigma \\ & (x,y,z) & \longmapsto & <xyz>. \end{array}$$

Then A is called an *associative pair* if the identities

$$<uv<xyz>>=<u<yxv>z>=<<uvx>yz>$$

hold for all $u, x, z \in A^\sigma$ and $v, y \in A^{-\sigma}$.

A most typical example of associative pairs is given by taking

$$A^+ = Hom_\Delta(X, Y)\,,\ A^- = Hom_\Delta(Y, X)$$

linear maps between right vector spaces X and Y over a division associative $\mathcal{R}$-algebra Δ, with multiplication $<abc>=abc$, the usual mapping composition.

Given a Jordan pair $V = (V^+, V^-)$, we write

$$Q_\sigma : V^\sigma \to Hom_{\mathcal{R}}\left(V^{-\sigma}, V^\sigma\right)$$

$\sigma \in \{+,-\}$,to denote the quadratic maps of V. The multiplication $Q_x y$ is quadratic in x and linear in y and has linearizations

$$\{x,y,z\} = Q_{(x,z)}y = V_{(x,y)}z = Q_{x+z}y - Q_x y - Q_z y.$$

Note that $\{x,y,x\} = 2Q_x y$, so we only need to consider the triple product in the *linear* case: ($\frac{1}{2} \in \mathcal{R}$).

Any associative pair $A = (A^+, A^-)$ gives rise to a Jordan pair which one usually denotes by A^J with quadratic multiplication $Q_x y =< xyx >$ $[8,\ \text{Theorem 7.1}]$.

A Jordan pair V is said to be *nondegenerate if* $Q_a = 0$ implies $a = 0$. The lower radical of V, denoted by $rad(V)$, is defined as the smallest ideal of V among its ideals I such that V/I is nondegenerate. We also have that V is nondegenerate if and only if $rad(V) = 0$.

Write $Rad(V) = (Rad(V^+), Rad(V^-))$ to denote the *Jacobson radical* of a Jordan pair, where $Rad(V^\sigma)$ is the set of properly quasi-invertible elements of V^σ [8]. We say that V is *semisimple* if $Rad(V) = 0$. Note that V stands for an associative or Jordan pair.

A pair of subspaces $I = (I^+, I^-)$ of an associative pair A is said to be ideal of A if

$$< A^\sigma A^{-\sigma} I^\sigma > + < A^\sigma I^{-\sigma} A^\sigma > + < I^\sigma A^{-\sigma} A^\sigma > \subset I^\sigma.$$

A is called *semiprime* if $< I^\sigma A^{-\sigma} I^\sigma >= 0$ implies $I = 0$ for I ideal of A. As in the binary case, semiprimeness can be characterized in terms of elements, that is A is semiprime if and only if $Q_a = 0$ implies $a = 0,\ a \in A^\sigma$ i.e A^J is nondegenerate [8]. Note that semprimeness and nondegeneracy coincide for associative pairs.

3. Jordan derivations on Banach pairs

Let $V = (V^+, V^-)$ be a Jordan pair over an arbitrary ring of scalars $\mathcal{R}$. Following [8], a pair of nonlinear (additive) mappings $D = (D_+, D_-) : D_\sigma : V^\sigma \longrightarrow V^\sigma$, $D_\sigma(x + z) = D_\sigma(x) + D_\sigma(z)$, is called a *derivation* on V if the condition

$$D_\sigma(Q_x y) = \{D_\sigma(x), y, x\} + Q_x D_{-\sigma}(y)$$

holds for every $x \in V^\sigma,\ y \in V^{-\sigma}$ and $\sigma \in \{+, -\}$.

If $\frac{1}{2} \in \mathcal{R}$, a simple computation shows that this is the case if and only if, for every $x, z \in V^\sigma,\ y \in V^{-\sigma}$

$$D_\sigma(\{x, y, z\}) = \{D_\sigma(x), y, z\} + \{x, D_{-\sigma}(y), z\} + \{x, y, D_\sigma(z)\}.$$

Given an associative pair $A = (A^+, A^-)$ over a ring of scalars $\mathcal{R}$ containing $\frac{1}{2}$, we say that a pair of additive maps $D = (D_+, D_-)$ $(D_\sigma : A^\sigma \longrightarrow A^\sigma)$ is a *derivation* on A if the following "Leibnitz rule" condition holds

$$D_\sigma(< xyz >) =< D_\sigma(x)yz > + < xD_{-\sigma}(y)z > + < xyD_\sigma(z) > .$$

D is said to be a *Jordan derivation* on A if it is a derivation on the Jordan pair A^J. An example of Jordan derivations on the associative pair $A = (Hom_{\mathcal{R}}(X, Y), Hom_{\mathcal{R}}(Y, X))$

is given by

$$\begin{aligned} D_+(r) &= ru - vr \\ D_-(s) &= us - sv \end{aligned}$$

for a fixed couple $(u, v) \in End_{\mathcal{R}}(X) \times End_{\mathcal{R}}(Y)$ [13] .

By a *normed associative pair* we mean a (real or complex) associative pair $A = (A^+, A^-)$ where the vector spaces A^+ and A^- are respectively endowed with norms $\|.\|_+$, $\|.\|_-$ satisfying

$$\|< xyz >\|_\sigma \leq \|x\|_\sigma \|y\|_{-\sigma} \|z\|_\sigma \ \forall x, z \in A^\sigma,\ y \in A^{-\sigma}, \sigma = \pm.$$

If these norms are complete, then we shall say that A is a Banach associative pair or merely a Banach pair.

A typical example of Banach associative pairs is given by taking

$$A^+ = \mathcal{BL}(X, Y),\ A^- = \mathcal{BL}(Y, X),$$

the pair of linear operators between two Banach spaces X and Y with the multiplication $< abc >= abc$, the usual mapping composition.

An example of Jordan derivations on the associative pair $A = (\mathcal{BL}(X, Y), \mathcal{BL}(Y, X))$ is given by

$$\begin{aligned} D_+(x) &= xa + bx \\ D_-(y) &= -ay - yb \end{aligned}$$

for a suitable $(a, b) \in \mathcal{BL}(X) \times \mathcal{BL}(Y)$.

3.1. Theorem. [13] . *Every linear derivation on a semisimple complex Banach associative pair consists of continuous operators.*

3.2. Theorem. *Let $A = (A^+, A^-)$ be a semisimple Banach associative pair. Then any linear Jordan derivation $D = (D_+, D_-)$ on A is continuous.*

Proof. Assume that the ground field is $\mathbb{C}$. Since semisimple pairs are semiprime, a tedious computation enables to see that $D = (D_+, D_-)$ is a derivation on A. Now Theorem (3.1) applies to have the continuity of D.

If the ground field is the real field $\mathbb{R}$. we may consider the algebraic complexification $A \otimes \mathbb{C} = A_{\mathbb{C}} = (A^+_{\mathbb{C}}, A^-_{\mathbb{C}})$ of A. It is a complex Banach pair for the standard norms

$$\|x + iy\|_\sigma = 2^{-\frac{1}{2}} \max_\theta(\|xcos\theta - ysin\theta\| + \|xsin\theta - ycos\theta\|),$$

which extend those of A (See [2] for the complexification of a Banach algebra). The pair $A_{\mathbb{C}}$, is also endowed with the conjugate linear involutive automorphism $\tau = (\tau_+, \tau_+)$ defined by

$$\tau_\sigma(x + iy) = x - iy, \text{ for all } x, y \in A^\sigma.$$

We proceed as in [5], to show that $A_{\mathbb{C}}$ is also semisimple. Indeed, $A \cap Rad(A_{\mathbb{C}})$ is a quasi-invertible ideal of A and hence, by [8, 4.2], $A \cap Rad(A_{\mathbb{C}}) \subset Rad(A)$. Since the Jacobson radical is invariant under automorphisms, in particular τ, for all $x + iy \in Rad(A^\sigma_{\mathbb{C}})$,

$$x = \frac{1}{2}[(x + iy) + (x - iy)],\ y = \frac{1}{2i}[(x + iy) - (x - iy)] \in A^\sigma \cap Rad(A^\sigma_{\mathbb{C}}) = 0$$

and then $Rad(A_{\mathbb{C}}) = (0)$. On the other hand, the pair Jordan derivation $D = (D_+, D_-)$ extends to a pair Jordan derivation $\delta = (\delta_+, \delta_-)$ on $A_{\mathbb{C}}$ defined by

$$\delta_\sigma(x + iy) = D_\sigma(x) + iD_\sigma(y),$$

for all $x, y \in A^\sigma$ and $\sigma \in \{+, -\}$. By means of what it is just proved, δ_σ is continuous and then so is D_σ. ■

Now the well-known result by Johnson and Sinclair [7] can be derived from Theorem (3.2) by considering, for any linear derivation D on a semisimple real or complex Banach algebra A, the derivation $\delta = (D, D)$ on the semisimple real or complex Banach associative pair (A, A) whose triple products are defined by $< xyz >= xyz$.

3.3. Corollary. *Any linear (Jordan) derivation of a real or complex semisimple Banach algebra is continuous.*

3.4. Remarks.

1. Theorem (3.2) can be immediately obtained from the main result in [6, Theorem (3.9)].

2. The linear Jordan derivation $D = (D_+, D_-)$ defined by

$$\begin{aligned} D_+(x) &= xa + bx \\ D_-(y) &= -ay - yb \end{aligned}$$

on the Banach associative pair $A = (\mathcal{BL}(X, Y), \mathcal{BL}(Y, X))$, for a suitable $(a, b) \in \mathcal{BL}(X) \times \mathcal{BL}(Y)$, is continuous. To prove this assertion, it suffices to show that A is semisimple. Indeed, It is known that the Banach algebras $\mathcal{BL}(X)$ and $\mathcal{BL}(Y)$ are semisimple. It is easily seen that $u \in Rad(A^+)$ if and only if $uv \in Rad(\mathcal{BL}(Y))\ \forall v \in A^-$. Assume that $u \neq 0$, that is $u(x_0) = y_0 \neq 0$ for some $0 \neq x_0 \in X$. Consider the linear operator $v \in A^-$ defined by $v(y_0) = x_0$ and v restricted to the complement of

$\mathbb{C}y_0$. We then have $uv(y_0) = y_0 \neq 0$, which is a contradiction. By a similar argument we prove that $Rad(A^-) = 0$.

3. Theorem (3.2) is not valid when D_σ is not linear. It suffices to consider the following counterexample. Recall that every additive derivation $d : K \longrightarrow K$ $(K = \mathbb{R}$ or $\mathbb{C}.)$ vanishes at all algebraic numbers. Moreover, if $\lambda \in K$ is transcendental then there exists an additive derivation which does not vanish at λ $[11]$. It follows that any nontrivial additive derivation $d : K \longrightarrow K$ is discontinuous. Consider now the complex associative pair $A = (A^+, A^-)$ with

$$A^+ = M_{p,q}(\mathbb{C}),\ A^- = M_{q,p}(\mathbb{C}),\ (p, q \in \mathbb{N}^*)$$

the pair of rectangular matrices with entries in the complex field $\mathbb{C}$. It is easily seen that A is a simple Banach pair. Let's show that it is also semisimple. Indeed, A has descending chain condition dcc on principal inner ideals since it is finite-dimensional, then Theorem (10.8) in $[8]$ applies to see that A is semisimple because $rad(A) = 0$ by simplicity of A. Moreover, d induces a Jordan nonlinear derivation $D = (D_+, D_-)$ defined by

$$D_\sigma((\alpha_{ij})_{1\leq i\leq p, 1\leq j\leq q}) = (d(\alpha_{ij}))_{1\leq i\leq p, 1\leq j\leq q}).$$

But D_σ is clearly discontinuous because so is d.

4. Nonlinear derivations on operator Banach-Jordan pairs

This section is devoted to the determination of all nonlinear (additive) derivations of a well-known class of Banach-Jordan pairs, namely, the standard Jordan pairs.

By a Banach-Jordan pair we mean a real or complex Jordan algebra $V = (V^+, V^-)$, where the vector spaces V^+ and V^- are equipped with complete norms making continuous the triple products of V.

Let (X, Y) be a pair of real or complex Banach spaces. A Banach subpair (U^+, U^-) of the Banach-Jordan pair $V = (\mathcal{BL}(Y, X), \mathcal{BL}(X, Y))$ is said to be *standard* if $\mathcal{F}(Y, X) \subset U^+$ and $\mathcal{F}(X, Y) \subset U^-$, where $(\mathcal{F}(Y, X), \mathcal{F}(X, Y))$ is the pair of finite rank operators.

Let V be a Jordan pair over a field *K*. By the centroid of V we mean the set $\Gamma(V)$ of all additive endomorphisms $T = (T^+, T^-)$ satisfying

$$T^\sigma \{x, y, z\} = \{T^\sigma(x), y, z\} = \left\{x, T^{-\sigma}(y), z\right\},$$

for all $x, z \in V^\sigma$, $y \in V^{-\sigma}$ (see $[4]$ for more details). Clearly, every element of *K* lies in the centroid. We call V *central* if the elements of *K* form all of the centroid.

Before giving the following result, we recall that the *socle* of a Jordan pair $V = (V^+, V^-)$ is the ideal $Soc(V) = (Soc(V^+), Soc(V^-))$, where $Soc(V^\sigma)$ is the sum of all simple inner or trivial ideals of V^σ [9] .

Lemma 4.1. *The Banach-Jordan pairs* $V = (\mathcal{BL}(Y,X), \mathcal{BL}(X,Y)) \supset U \supset W = (\mathcal{F}(Y,X), (X,Y))$ *are central.*

Proof. V is clearly strongly prime and $Soc(V) = (\mathcal{F}(Y,X), \mathcal{F}(X,Y))$ is obviously nonzero. Moreover, $Soc(Soc(V)) = Soc(V)$. Hence, by [3] , V, W and U are central. ∎

A simple computation enables to check the following result.

Lemma 4.2. *Let V be a Jordan pair and let D be an additive derivation on V. Then D induces an additive derivation d on the centroid $\Gamma(V)$ of V defined by:*

$$d(T) = DT - TD,$$

for all T in $\Gamma(V)$.

To determine all additive derivations on $V = (\mathcal{BL}(Y,X), \mathcal{BL}(X,Y))$, we distinguish separately finite and infinite dimensional cases.

Suppose that $dim_K(X) = p < \infty$ and $dim_K(Y) = q < \infty$ ($K = \mathbb{R}$ or $\mathbb{C}$.). The Banach spaces $\mathcal{BL}(X,Y)$ and $\mathcal{BL}(Y,X)$ has dimension pq and then are respectively isomorphic to the Banach spaces $M_{q,p}(K)$, $M_{p,q}(K)$ of $q \times p$ and $p \times q$ matrices. Under these conditions, we have the following result.

Theorem 4.3. *Let $D = (D_+, D_-)$ be a pair of additive maps on the Banach-Jordan pair of rectangular matrices $V = (M_{p,q}(K), M_{q,p}(K))$. Then $D = (D_+, D_-)$ is an additive derivation if and only if there exist a nontrivial additive derivation $d : K \longrightarrow K$ and a $p \times p$ matrix (a_{mn}) and a $q \times q$ matrix (b_{hk}) such that*

$$\begin{aligned} D_+(x_{ij}) &= (x_{ij})(b_{hk}) - (a_{mn})(x_{ij}) + (dx_{ij}) \\ D_-(y_{ij}) &= (y_{ij})(a_{mn}) - (b_{hk})(y_{ij}) + (dy_{ij}), \end{aligned}$$

for all (x_{ij}) in $M_{p,q}(K)$ and (y_{ij}) in $M_{q,p}(K)$.

Proof. By Lemma (4.2), D induces an additive derivation d on the centroid $\Gamma(V)$ of V. But Lemma (4.1) implies that $\Gamma(V)$ is isomorphic to K. An easy computation enables to check that the pair $\delta = (\delta_+, \delta_-)$, defined on V by

$$\begin{aligned} \delta_+(x_{ij}) &= D_+(x_{ij}) - (dx_{ij}) \\ \delta_-(y_{ij}) &= D_-(y_{ij}) - (dy_{ij}), \end{aligned}$$

is a linear derivation on V. Therefore, in virtue of $[13,\ \text{Theorem } 2.3.1]$, there exist square matrices $(a_{mn}) \in M_{p,p}(K)$ and $(b_{hk}) \in M_{q,q}(K)$ such that

$$\begin{aligned} \delta_+(x_{ij}) &= (x_{ij})(b_{hk}) - (a_{mn})(x_{ij}) \\ \delta_-(y_{ij}) &= (y_{ij})(a_{mn}) - (b_{hk})(y_{ij}). \end{aligned}$$

It follows that

$$\begin{aligned} D_+(x_{ij}) &= (x_{ij})(b_{hk}) - (a_{mn})(x_{ij}) + (dx_{ij}) \\ D_-(y_{ij}) &= (y_{ij})(a_{mn}) - (b_{hk})(y_{ij}) + (dy_{ij}) \end{aligned}$$

as required, which completes the proof. ∎

Suppose actually that either X or Y (or both) is infinite dimensional. We shall show that D is linear and then continuous.

Theorem 4.4. *Let $X,\ Y$ be Banach spaces with either X or Y is infinite dimensional. Then every additive derivation $D = (D_+, D_-)$, $D_\sigma : U^\sigma \longrightarrow V^\sigma$, on a standard operator pair (U^+, U^-) on the couple (X, Y), is continuous and is of the form*

$$\begin{aligned} D_+(x) &= ax - xb \\ D_-(y) &= ya - by \end{aligned}$$

for some $(a, b) \in \mathcal{BL}(X) \times \mathcal{BL}(Y)$.

Proof. Let us note that V^+ and V^- can be considered as left and right R-modules with respect to the multiplications

$$\begin{array}{llll} \mu_+ : & R \times V^+ & \longrightarrow & V^+ \text{ and } \quad \mu_- : \quad R \times V^- \longrightarrow \quad V^- \\ & (r, x) & \longmapsto & rx \qquad\qquad\qquad\ (r, y) \ \longmapsto \quad yr \end{array}$$

where the products $rx,\ yr$ are the usual mapping composition and $R = \mathcal{BL}(X)$. It is clear that μ_+ and μ_- are continuous. Moreover, the map

$$\begin{array}{llll} \psi : & V^+ \times V^- & \longrightarrow & R \\ & (x, y) & \longmapsto & xy \end{array}$$

is a nondegenerate bilinear map in the sense that $\psi(ax, yb) = a\psi(x, y)b$. Note that ψ is also continuous. The products of V may be expressed via ψ as follows

$$\begin{aligned} Q_x y &= \psi(x, y)x, Q_y x = y\psi(x, y), \\ \{x, y; z\} &= \psi(x, y)z + \psi(z, y)x \ \forall x, z \in V^+,\ y \in V^-. \end{aligned}$$

By means of Lemmas (4.1) and (4.2), D induces an additive derivation d on the complex field and we have, for every $x \in V^\sigma$, and $\alpha \in \mathbb{C}$,

$$(*) \quad D_\sigma(\alpha x) = (d\alpha)x + \alpha D_\sigma(x).$$

We shall show that d is continuous. Suppose on the contrary that d is discontinuous. Then d is unbounded on every open ball of the complex field $\mathbb{C}$ $[1]$. Since ψ is nondegenerate, as in $[4]$, we can find an infinite system $\{(e_n^+, e_n^-)\}_n$ of $V = (V^+, V^-)$ such that

$$\psi(e_p^+, e_q^-) = \delta_{pq} I.$$

For every $n \in \mathbb{N}$ there exist $x_n \in B(0, \frac{1}{2^n \lambda_n})$ such that

$$2^n \lambda_n < |d(x_n)| \text{ and } \lambda_n = \max(\|D_+ e_n^+\|, \|e_n^+\|).$$

Thus the series $\sum\limits_k x_k e_k^+$ and $\sum\limits_n \frac{D_+(e_n^+)}{2^n\lambda_n}$ converge. Now, using the continuity of μ_σ ($\sigma = \pm$) and that of ψ we obtain

$$\begin{aligned}
&\sum_n \tfrac{1}{2^n\lambda_n} D_+(x_n \sum_k x_k e_k^+) &=\\
&\sum_n \tfrac{1}{2^n\lambda_n} D_+[\psi(\sum_k x_k e_k^+, e_n^-)\sum_k x_k e_k^+] &=\\
&\sum_n \tfrac{1}{2^n\lambda_n} D_+(Q_{\sum_k x_k e_k^+} e_n^-) &=\\
&\sum_n \tfrac{1}{2^n\lambda_n} [\{D_+(\sum_k x_k e_k^+), e_n^-, \sum_k x_k e_k^+\} + Q_{\sum_k x_k e_k^+} D_- e_n^-] &=
\end{aligned}$$

$$\begin{aligned}
&\sum_n \tfrac{1}{2^n\lambda_n} \psi(D_+(\sum_k x_k e_k^+), e_n^-)\sum_k x_k e_k^+ + \sum_n \tfrac{1}{2^n\lambda_n}\psi(\sum_k x_k e_k^+, D_- e_n^+)\sum_k x_k e_k^+ &\\
&\qquad + \sum_n \tfrac{1}{2^n\lambda_n}\psi(\sum_k x_k e_k^+, e_n^-) D_+(\sum_k x_k e_k^+) &=\\
&\sum_n \tfrac{1}{2^n\lambda_n} \psi(D_+(\sum_k x_k e_k^+), e_n^-)\sum_k x_k e_k^+ + \sum_n \tfrac{1}{2^n\lambda_n}\psi(\sum_k x_k e_k^+, D_- e_n^-)\sum_k x_k e_k^+ &\\
&\qquad + \sum_n \tfrac{x_n}{2^n\lambda_n} D_+(\sum_k x_k e_k^+). &
\end{aligned}$$

On the other hand, by making use of $(*)$, we obtain

$$\begin{aligned}
&[\sum_n \tfrac{1}{2^n\lambda_n} d(x_n)]\sum_k x_k e_k^+) &=\\
&\sum_n \tfrac{1}{2^n\lambda_n} D_+(x_n \sum_k x_k e_k^+) - \sum_n \tfrac{x_n}{2^n\lambda_n} D_+(\sum_k x_k e_k^+) &=\\
&\sum_n \tfrac{1}{2^n\lambda_n} \psi(D_+(\sum_k x_k e_k^+), e_n^-)\sum_k x_k e_k^+ + \sum_n \tfrac{1}{2^n\lambda_n}\psi(\sum_k x_k e_k^+, D_- e_n^-)\sum_k x_k e_k^+ &=\\
&\psi(D_+(\sum_k x_k e_k^+), \sum_n \tfrac{1}{2^n\lambda_n} e_n^-)\sum_k x_k e_k^+ + \psi(\sum_k x_k e_k^+, \sum_n \tfrac{1}{2^n\lambda_n} D_- e_n^-)\sum_k x_k e_k^+ &
\end{aligned}$$

But this is a contradiction since the series

$$[\sum_n \frac{1}{2^n\lambda_n} d(x_n)]\sum_k x_k e_k^+$$

diverges. Therefore d is trivial and then D_+ is linear. Therefore, since V is semisimple, D_+ is continuous by [6]. By symmetry of the argument, D_- is continuous. Finally, in virtue of $[13,\ \text{Theorem 2.3.1}]$, there exist $a \in \mathcal{BL}(X)$ and $b \in \mathcal{BL}(Y)$ such that, for all $x \in V^+$ and $y \in V^-$,

$$\begin{aligned} D_+(x) &= ax - xb \\ D_-(y) &= ya - by, \end{aligned}$$

which completes the proof. ■

Actually, the main result in $[12,\ \text{Theorem 2.3}]$ is an immediate consequence of Theorem (4.4). Let X be a Banach space. We denote by $\mathcal{FL}(X)$ the subalgebra of bounded finite rank operators on X.

Corollary 4.5. *Let X be an infinite dimensional Banach space (not necessarily Hilbert). Then every additive derivation $D : \mathcal{FL}(X) \longrightarrow \mathcal{BL}(X)$ is of the form*

$$D(x) = xa - ax$$

for some operator $a \in \mathcal{BL}(X)$.

Proof. Consider the Banach-Jordan pairs $V = (\mathcal{BL}(X), \mathcal{BL}(X))$ and $U = (\mathcal{FL}(X), \mathcal{FL}(X))$ associated to the Banach algebras $\mathcal{BL}(X)$ and $\mathcal{FL}(X)$. D defines an additive derivation $\delta : U \longrightarrow V$ with $\delta_+ = \delta_- = D$. Hence Theorem (4.4) applies to obtain $a, b \in \mathcal{BL}(X)$ such that, for all $x \in \mathcal{FL}(X)$,

$$D(x) = xb - ax.$$

Substitute the unit element I of $\mathcal{FL}(X)$ in the previous equality, we obtain $a = b$. ■

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