Explicit Representation and Some Integral Transforms of Sequence of Functions associated with the Wright type Generalized Hypergeometric Function

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Abstract

The principal aim of the present work is to investigate Explicit Representation of a sequence of functions \( G_{r,n}^{(a,b,c,\delta)}(x;\alpha,k,s) \) [16] associated with the Generalized Hypergeometric function \( _2R_1(a,b;c;\tau;z) \) [6, 24], followed by deduction of several integral transforms of \( G_{r,n}^{(a,b,c,\delta)}(x;\alpha,k,s) \).

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1. INTRODUCTION AND PRELIMINARIES

The Gauss Hypergeometric Function is defined [12] as,

\[
_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad (|z| < 1, c \neq 0, -1, -2, \ldots); \quad \text{and} \quad (1.1)
\]

The Generalized Hypergeometric Function, in a classical sense has been defined [5] by
\[ p \binom{F_q}{a_0, \ldots, a_p; b_0, \ldots, b_q; z} = p \binom{F_q}{a_0, \ldots, a_p; b_0, \ldots, b_q; z} \]
\[ = \sum_{k=0}^{\infty} \frac{(a_0)_k \cdots (a_p)_k}{(b_0)_k \cdots (b_q)_k} \frac{z^k}{k!}, \quad (p = q + 1, |z| < 1); \quad (1.2) \]

with no denominator parameter equals zero or negative integer.

Virchenko et al. [24] defined the Generalized Hypergeometric Function in a different sense as:
\[ _2 \mathcal{R}(z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b + \tau k)}{(c + \tau k)_k} \frac{z^k}{k!}; \quad \tau > 0, \quad |z| < 1. \quad (1.3) \]

On \(|z| = 1\), \(_2 \mathcal{R}(z)\) is meaningful for \(\text{Re}(c - a - b) > 0\).

If \(\tau = 1\), then (1.3) reduces to a Gauss’s hypergeometric function \(_2 \mathcal{F}_1(a, b; c; z)\).

One of the most important special functions is the Gauss hypergeometric function \(_2 \mathcal{F}_1(a, b; c; z)\) as; many special functions of applied mathematics can be expressed in terms of it. This has inspired the study of several generalizations.

It should be noted that many algebraic or transcendental functions that occur in the problems of applied mathematics can be expressed in terms of the hypergeometric functions. The Legendre, Bessel, Whittaker and other special functions, and the classical orthogonal polynomials are particular cases of the hypergeometric functions or their various combinations. Let us note that in a systematic study of the generalized probability density, in solving problems of the theory of special functions, differential and integral equations, integral transforms, diffraction theory etc., these functions and their applications have played a significant role, see for example [1], [2], [3], [5], [7], [8], [9], [18].

Galue [6], Rao et al. [13, 14, 15, 16, 17] have studied several properties of \(_2 \mathcal{R}(a, b; c; z)\) some of them including in the light of Fractional Integral and Differential operators.

Srivastava and Singhal [23] introduced a general class of polynomials in 1971 as
\[ G_n^{(a)}(x, r, p, k) = \frac{x^{-a-kn}}{n!} \exp(px) \left( x^{k+1} D^a \right)^n \left[ x^a \exp(-pxr) \right], \quad (1.4) \]
where Laguerre, Hermite and Konhauser polynomials are the special case of (1.4).
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In 1979 Srivastava and Singh [21] introduced a general sequence of functions

\[ V_n^{(a)}(x; a, k, s) / n = 0, 1, 2, \ldots \] ;

\[ V_n^{(a)}(x; a, k, s) = \frac{x^{-a}}{n!} \exp \{ p_k(x) [x^a(s + xD)]^n [x^a \exp \{-p_k(x)\}] \}. \quad (1.5) \]

where \( p_k(x) \) is a polynomial in \( x \) of degree \( k \) and \( a \) and \( s \) are constants.

Rao et al. [16] introduced a sequence of functions containing Generalized Hypergeometric Function

\[ G_{r, n}^{(a, b, c, \delta)}(x; a, \alpha, k, s) / n = 0, 1, 2, \ldots \] ;

\[ G_{r, n}^{(a, b, c, \delta)}(x; a, \alpha, k, s) = \frac{x^{-\delta - an}}{n!} _2R_1(a, b; c; \tau; p_k(x)) \left[ x^\delta _2R_1(a, b; c; \tau; -p_k(x)) \right], \quad (1.6) \]

where \( p_k(x) \) is a polynomial in \( x \) of degree \( k \) and \( x \in (0, \infty), \alpha, \delta, s \) are constants and \( \tau \in \mathbb{C},(0, \infty); a, b, c \in \mathbb{C}; \text{Re}(a) > 0, \text{Re}(b) > 0, \text{Re}(c) > 0 \); and

\[ \left( \frac{T^s_x}{a} \right)^n \equiv x^{an}(s + xD)(s + \alpha + xD) \ldots (s + (n-1) \alpha + xD) \] with \( D \equiv \frac{d}{dx} \).

Some important results are listed below for our further study as:

(i) We are using the operational formula based on [10,11]:

\[ \left( \frac{T^s_x}{a} \right)^n (x^a) = \alpha^n \left( \frac{a + s}{\alpha} \right)_n x^{a+an}. \quad (1.7) \]

(ii) Srivastava and Manocha [22], gave following result:

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} A(k, n-k) \quad (1.8) \]

(iii) Some well-known integral transforms [04], [12], [19], [20] are listed as under.

**Beta (Eular) Transform:**

\[ B\{ f(x); a, b \} = \int_0^1 x^{a-1}(1-x)^{b-1} f(z) \, dz; \quad \text{Re}(a), \text{Re}(b) > 0. \quad (1.9) \]

**Finite Laplace Transform:**
\( L_{T} \{ f(x) \} = \int_{0}^{T} e^{-st} f(t) \, dt \); \( \text{Re}(s) > 0 \),

\( T \) is a finite positive number. \hfill (1.10)

**Laplace Transform:**

\( L \{ f(x) \} = \int_{0}^{\infty} e^{-st} f(t) \, dt \); \( \text{Re}(s) > 0 \). \hfill (1.11)

**Laguerre Transform:**

\( L \{ f(x) \} = \int_{0}^{\infty} e^{-\alpha x} L_{n}^{\mu}(x) f(x) \, dx \), \hfill (1.12)

where \( L_{n}^{\mu}(x) \) is the Laguerre polynomial of degree \( n \geq 0 \) and order \( \mu > -1 \);

defined by \( L_{n}^{\mu}(x) = \sum_{r=0}^{n} \left( -1 \right)^{r} \binom{n+\mu}{n-r} \frac{x^{r}}{r!} \). \hfill (1.13)

\[ 2. \text{ EXPLICIT REPRESENTATION OF } G_{\tau, a, b, c, \delta}^{(a, b, c, \delta)}(x; \alpha, k, s). \]

As in (1.6), Rao et al. [17] introduced a sequence \( G_{\tau, a, b, c, \delta}^{(a, b, c, \delta)}(x; \alpha, k, s) \) of functions containing Generalized Hypergeometric Function.

In this work, we introduce the Explicit Representation of \( G_{\tau, a, b, c, \delta}^{(a, b, c, \delta)}(x; \alpha, k, s) \), in terms of following result.

**Theorem 2.1**

\[ G_{\tau, a, b, c, \delta}^{(a, b, c, \delta)}(x; \alpha, k, s) \]

\[ = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{(a)_{n-l}}{n!} \frac{\Gamma(c) \Gamma(b + \tau (m-l))}{\Gamma(c + \tau (m-l)) \Gamma(b)} \frac{x^m}{m!} \frac{(a)_{l} \Gamma(c) \Gamma(b + \tau l)}{\Gamma(c + \tau l) \Gamma(b)} \left( \frac{\delta + s + kl}{\alpha} \right)^{n}, \]

where \( p_{k}(x) \) is a polynomial in \( x \) of degree \( k \) and \( x \in (0, \infty) \), \( \alpha, \delta, s \) are constants and \( \tau \in \overline{0, \infty} \); \( a, b, c \in \overline{0}; \text{Re}(a) > 0, \text{Re}(b) > 0, \text{Re}(c) > 0 \) ; and

\[ \left( \frac{T}{s} \right) = x^{\alpha n} (s + xD)(s + \alpha + xD) \ldots (s + (n-1)\alpha + xD) \text{ with } D = \frac{d}{dx}. \]
Proof: For simplicity purpose, putting $p_k(x) = x^k$ in (1.6), we get

$$G'_{r,a}^{(a,b,c,\delta)}(x; \alpha, k, s) = \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n x^{\delta+kl+an}$$

$$= \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} x^{\delta+kl}$$

$$= \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n$$

$$= \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n$$

$$= \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n$$

Using, (1.7) yields

$$= \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n$$

Applying, (1.8) gives us

$$G_{r,a}^{(a,b,c,\delta)}(x; \alpha, k, s)$$

$$= \frac{x^{-\delta-an}}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n$$

Equation (2.1) gives the required result.

Some Particular cases:

(i) For $a = b = c = \tau = 1$

$$G_{r,a}^{(1,1,\delta)}(x; \alpha, k, s) = \frac{\alpha^n}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j(a)_j \Gamma(c) \Gamma(b+\tau l)}{\Gamma(c+\tau l) \Gamma(b) l!} \left( \frac{\alpha}{\alpha} \right)^n$$
\[ \frac{\Gamma(c)\Gamma(b+\tau(m-l))}{\Gamma(c+\tau(m-l))\Gamma(b)} \left( \frac{\delta+kl+s}{\alpha} \right)_n^{(m)} \]
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\[ B\left(G_{\alpha,\beta,\gamma}(x;\alpha,\beta,\gamma):d,e\right) \]

\[ = \frac{\alpha^n}{n!} \beta(d,e) \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{(a)m-\Gamma(c)\Gamma(b+\gamma(m-l))}{\Gamma(c+\gamma(m-l))}\Gamma(b) \right) \left( -1 \right)^l \left( \frac{\Gamma(c)\Gamma(b+\gamma)}{\Gamma(c+\gamma)} \right) \left( \frac{\delta+s+kl}{\alpha} \right) \int_0^{\infty} e^{-\alpha t} t^m dt \]

\[ \text{(II) The Finite Laplace Transform:} \]

Using (1.10), we have

\[ L_T \left\{ G_{\alpha,\beta,\gamma}(x;\alpha,\beta,\gamma) \right\} = \int_0^{\infty} e^{-\alpha t} \Gamma\left(\alpha,\beta,\gamma;\delta+s+kl\right) \frac{e^{-\alpha t}}{\alpha} \int_0^{\infty} e^{-\alpha t} t^m dt \]

\[ = \frac{\alpha^n}{n!} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{(a)m-\Gamma(c)\Gamma(b+\gamma(m-l))}{\Gamma(c+\gamma(m-l))}\Gamma(b) \right) \left( -1 \right)^l \left( \frac{\Gamma(c)\Gamma(b+\gamma)}{\Gamma(c+\gamma)} \right) \left( \frac{\delta+s+kl}{\alpha} \right) \int_0^{\infty} e^{-\alpha t} t^m dt \]

where, \( \gamma(\alpha,x) \) is an incomplete Gamma function [12].

Thus the finite Laplace transform of \( G_{\alpha,\beta,\gamma}(x;\alpha,\beta,\gamma) \) is given by (3.3).

**Particular case**: Setting, \( a=b=c=\gamma=1, \delta=s=0 \) and \( \alpha=k \)

\[ L_T \left\{ G_{1,1,0}(x;\alpha,\beta,\gamma) \right\} \]

\[ = \frac{k^n}{n!} \sum_{m=0}^{\infty} \frac{1}{s^{km+1}} \gamma(km+1,\alpha) \sum_{l=0}^{m} \left( -1 \right)^l \left( \begin{array}{c} m \\ l \end{array} \right) \]

\[ \text{(III) Laplace Transform:} \]

Using (1.11), we have

\[ L \left\{ G_{\alpha,\beta,\gamma}(x;\alpha,\beta,\gamma) \right\} = \int_0^{\infty} e^{-\alpha t} \Gamma\left(\alpha,\beta,\gamma;\delta+s+kl\right) \frac{e^{-\alpha t}}{\alpha} \int_0^{\infty} e^{-\alpha t} t^m dt \]

\[ = \frac{\alpha^n}{n!} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{(a)m-\Gamma(c)\Gamma(b+\gamma(m-l))}{\Gamma(c+\gamma(m-l))}\Gamma(b) \right) \left( -1 \right)^l \left( \frac{\Gamma(c)\Gamma(b+\gamma)}{\Gamma(c+\gamma)} \right) \left( \frac{\delta+s+kl}{\alpha} \right) \int_0^{\infty} e^{-\alpha t} t^m dt \]
\[
\frac{\alpha^n}{n!} \sum_{m=0}^{\infty} \sum_{l=0}^{m} \left( \frac{(a)_{m-l}}{\gamma(c)\Gamma(b+\tau(m-l))} \frac{(-1)^{l} (a) \Gamma(c)\Gamma(b+\tau l)}{\Gamma(c+\tau(m-l))! \Gamma(c+\tau l)} \frac{(\delta + s + kl)}{\alpha} \right) \frac{\Gamma(km+1)}{s^{km+1}} \right). 
\]

Thus the Laplace transform of \(G_{\tau,a}^{(a,b,c,d)}(x;\alpha,k,s)\) is given by (3.4).

**References**


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