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Trace of Positive Integer Power of Adjacency Matrix

Jagdish Kumar Pahade[1*] and Manoj Jha[2]

[1]Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal (M.P.) India.

[2]Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal (M.P.), India.

Abstract

Finding the trace of positive integer power of a matrix is an important problem in matrix theory. In the Graph Theory an important application of the trace of positive integer power of Adjacency matrix is counting the triangles in connected graph. In this paper we present a new efficient formula to find the trace of positive integer power of some special Adjacency matrix of connected simple graph. The key idea of our formula is to multiply the matrix k times, where k *is* positive integer.

Keywords: Adjacency matrix, Complete Graph, Trace, Matrix multiplication, Matrix power.

INTRODUCTION

Traces of powers of matrices arise in several fields of mathematics. More specifically Network Analysis, Number theory, Dynamical systems**,** Matrix theory, and Differential equations [3]. There are many applications in matrix theory and numerical linear algebra. For example, in order to obtain approximations of the smallest and the largest Eigen values of a symmetric matrix A, a procedure based on estimates of the trace of A^n and A^{-n}, n is natural, was proposed in [4]. When analyzing a complex network, an important problem is to compute the total number of triangles of a connected simple graph. This number is equal to $Tr(A^3)/6$ where A is the adjacency matrix of the graph [2].It is possible to solve triangle finding and counting in a graph by adjacency matrix representation of the graph [9]. For any integer matrix M and any prime number p, the entries of the unique Witt vector consisting of p-adic integers are expressed from the traces of powers of the integer matrix M [6].Traces of powers of integer matrices are connected with the Euler congruence [6]

$$Tr\left(M^{p^r}\right) \equiv Tr\left(M^{p^{r-1}}\right) (\text{mod } p^r)$$

Holds for all integer matrices M, all primes p, and all natural *r*. There are many applications of this congruence to dynamical systems, which is natural in studying those invariants of dynamical systems that are described in terms of the traces of powers of integer matrices, for example in studying the Lefschetz numbers [6].Solution of Lyapunov matrix equations can be solved by using matrix polynomials and characteristic polynomials where the computation of the traces of matrix powers are needed [1].There is a question in [11],can trace S^kA(where S^kA is the k-th symmetric power of $n \times n$ matrix A) be written in terms of some coefficient of A^{k+1} and the corresponding coefficient in A ?
The computation of the trace of matrix powers has received much attention. In [1], an algorithm for computing $Tr(A^k)$, k is integer , is proposed, when A is a lower Hessenberg matrix with a unit codiagonal. In [7], a symbolic calculation of the trace of powers of Tridiagonal matrices is presented. Trace of positive integer power of real2×2matrices is presented in [8].A formula for trace of any symmetric power of a $n \times n$ matrix is presented in [10] in terms of the ordinary power of matrix. Let A be a symmetric positive definite matrix, and let $\{\lambda_k\}$ denote its Eigen values. For $q \in R$, A^q is also symmetric positive definite, and it holds [5]

$$TrA^q = \Sigma_k \lambda_k^q$$

But for higher order matrix finding Eigen values λ_k is very difficult and time consuming, therefore other formula to compute trace of matrix power is needed. Now we present a new theorem to compute trace of matrix power for adjacency matrix of a connected simple graph with any number of vertices. Our estimation for the trace of A^k is based on the multiplication of matrix. This formula will depend only on order of the matrix.

Trace of a $n \times n$ matrix $A = [a_{ij}]$, is defined to be the sum of the elements on the main diagonal of A.i.e. $TrA = a_{11} + a_{22} + \ldots + a_{nn}$

A **complete graph** is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. For a given number of vertices, there's a unique complete graph. The adjacency matrix of the complete graph takes the particularly simple form of all 1's with 0's on the diagonal, i.e., the unit matrix minus the identity matrix.

MAIN RESULT

Theorem 1. Let A is an adjacency symmetric matrix of a complete simple graph with *n* vertices, then

$$TrA^k = \sum_{r=1}^{n/2} s(k,r)n(n-1)^r (n-2)^{k-2r}$$, for even positive integer ***k***

and

$$TrA^k = \sum_{r=1}^{n-1/2} s(k,r)n(n-1)^r (n-2)^{k-2r}$$, for odd positive integer ***k.*** where

$S(k,r)$ be a number depending upon ***k*** and ***r,*** and defined by

$$S(k,1)=1, S(k,k/2)=1, S(k,k-1/2)=\frac{k-1}{2}, and, S(k,r)=S(k-1,r)+S(k-2,r-1)$$

Proof: Consider a adjacency symmetric matrix $\boldsymbol{A = (a_{ij}), 1 \le i \le n , 1 \le j \le n}$ where

$$a_{ij} = \begin{cases} 1 & if \quad i \neq j \\ 0 & if \quad i = j \end{cases}$$

Now $\boldsymbol{A^2 = (a_{ij}), 1 \le i \le n , 1 \le j \le n}$ where $a_{ij} = \begin{cases} n-2 & if \quad i \neq j \\ n-1 & if \quad i = j \end{cases}$

Hence $\boldsymbol{TrA^2} = n(n-1)$

Or $\boldsymbol{TrA^2} = S(2,1)n(n-1)^1 (n-2)^{2-2.1}$

Or $\boldsymbol{TrA^2} = \sum_{r=1}^{1} s(2,r)n(n-1)^r (n-2)^{2-2.r}$

Again $A^3 = (a_{ij}), 1 \le i \le n,\ 1 \le j \le n$ where $a_{ij} = \begin{cases} (n-1)+(n-2)^2 & if\ \ i \ne j \\ (n-1)(n-2) & if\ \ i = j \end{cases}$

Hence $TrA^3 = n(n-1)(n-2)$

Or $TrA^3 = S(3,1)n(n-1)^1(n-2)^{3-2.1}$

Or $TrA^3 = \sum_{r=1}^{1} S(3,r)n(n-1)^r(n-2)^{3-2.r}$

Again $A^4 = (a_{ij}), 1 \le i \le n,\ 1 \le j \le n$ where

$$a_{ij} = \begin{cases} (n-1)(n-2)+(n-2)[(n-1)+(n-2)^2\ ] & if\ \ i \ne j \\ (n-1)[\ (n-1)+(n-2)^2] & if\ \ i = j \end{cases}$$

$$\text{Or } a_{ij} = \begin{cases} 2(n-1)(n-2)+(n-2)^3 & if\ \ i \ne j \\ (n-1)^2+(n-1)(n-2)^2 & if\ \ i = j \end{cases}$$

Hence $TrA^4 = n(n-1)^2 + n(n-1)(n-2)^2$

Or $TrA^4 = s(4,1)n(n-1)^1(n-2)^{4-2.1} + S(4,2)n(n-1)^2(n-2)^{4-2.2}$

Or $TrA^4 = \sum_{r=1}^{2} S(4,r)n(n-1)^r(n-2)^{4-2.r}$

Again $A^5 = (a_{ij}), 1 \le i \le n,\ 1 \le j \le n$ where

$$a_{ij} = \begin{cases} (n-1)^2+(n-1)(n-2)^2+(n-2)[2(n-1)(n-2)+(n-2)^3\ ] & if\ \ i \ne j \\ (n-1)[2(n-1)(n-2)+(n-2)^3] & if\ \ i = j \end{cases}$$

Or $a_{ij} = \begin{cases} (n-1)^2 + 3(n-1)(n-2)^2 + (n-2)^4 & \text{if } i \neq j \\ 2(n-1)^2(n-2) + (n-1)(n-2)^3 & \text{if } i = j \end{cases}$

Hence $TrA^5 = 2n(n-1)^2(n-2) + n(n-1)(n-2)^3$

Or $TrA^5 = S(5,1)n(n-1)^1(n-2)^{5-2.1} + S(5,2)n(n-1)^2(n-2)^{5-2.2}$

Or

$$Tr\,A^5 = \sum_{r=1}^{2} S(5,r)n(n-1)^r(n-2)^{5-2.r}$$

Again $\boldsymbol{A^6 = (a_{ij}), 1 \le i \le n\ ,\ 1 \le j \le n}$ where

$$a_{ij} = \begin{cases} 2(n-1)^2(n-2) + (n-1)(n-2)^3 + (n-2)\left[(n-1)^2 + 3(n-1)(n-2)^2 + (n-2)^4\ \right] & \text{if } i \neq j \\ (n-1)\left[(n-1)^2 + 3(n-1)(n-2)^2 + (n-2)^4\right] & \text{if } i = j \end{cases}$$

Or $a_{ij} = \begin{cases} 3(n-1)^2(n-2) + 4(n-1)(n-2)^3 + (n-2)^5 & \text{if } i \neq j \\ (n-1)^3 + 3(n-1)^2(n-2)^2 + (n-1)(n-2)^4 & \text{if } i = j \end{cases}$

Hence $TrA^6 = n(n-1)^3 + 3n(n-1)^2(n-2)^2 + n(n-1)(n-2)^4$

Or $TrA^6 = S(6,1)n(n-1)^1(n-2)^{6-2.1} + S(6,2)n(n-1)^2(n-2)^{6-2.2}$

$$+ S(6,3)n(n-1)^3(n-2)^{6-2.3}$$

Or $TrA^6 = \sum_{r=1}^{3} S(6,r)n(n-1)^r(n-2)^{6-2.r}$

Again $\boldsymbol{A^7 = (a_{ij}), 1 \le i \le n\ ,\ 1 \le j \le n}$ where

$$a_{ij} = \begin{cases} (n-1)^3 + 3(n-1)^2(n-2)^2 + (n-1)(n-2)^4 & \\ + (n-2)\left[3(n-1)^2(n-2) + 4(n-1)(n-2)^3 + (n-2)^5\ \right] & \text{if } i \neq j \\ (n-1)\left[3(n-1)^2(n-2) + 4(n-1)(n-2)^3 + (n-2)^5\right] & \text{if } i = j \end{cases}$$

Or $a_{ij} = \begin{cases} (n-1)^3 + 6(n-1)^2(n-2)^2 + 5(n-1)(n-2)^4 + (n-2)^6 & if \ i \neq j \\ 3(n-1)^3(n-2) + 4(n-1)^2(n-2)^3 + (n-1)(n-2)^5 & if \ i = j \end{cases}$

Hence **TrA^7 =** $3n(n-1)^3\,(n-2) + 4n(n-1)^2\,(n-2)^3$

$$+ \ n(n-1)(n-2)^5$$

Or **TrA^7 =** $S(7,1)n(n-1)^1(n-2)^{7-2.1} + S(7,2)n(n-1)^2(n-2)^{7-2.2}$

$$+ S(7,3)n(n-1)^3(n-2)^{7-2.3}$$

Or **TrA^7 =** $\sum_{r=1}^{3} S(7,r)n(n-1)^r(n-2)^{7-2.r}$

Continuing this process of multiplication of matrix, we conclude that,

$TrA^k = \sum_{r=1}^{n/2} s(k,r)n(n-1)^r(n-2)^{k-2r}$, for even positive integer ***k*** and

$TrA^k = \sum_{r=1}^{n-1/2} S(k,r)n(n-1)^r(n-2)^{k-2r}$, for odd positive integer ***k.***

where $S(k,r)$ be a number depending upon ***k*** and ***r,*** and defined by

$$S(k,1) = 1, S(k,k/2) = 1, S(k,k-1/2) = \frac{k-1}{2}, and, S(k,r) = S(k-1,r) + S(k-2,r-1)$$

This completes the proof.

Example 1: Consider a matrix A = $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ and let we are to find $\boldsymbol{TrA^5}$.

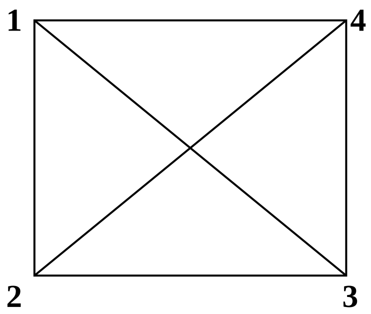
Here $\boldsymbol{n = 6}$ and $\boldsymbol{k = 5}$ .then by our theorem, we have

$$TrA^5 = \sum_{r=1}^{2} S(5,r)n(n-1)^r (n-2)^{5-2.r}$$

Or $\quad TrA^5 = S(5,1)6(6-1)^1(6-2)^{5-2.1} + S(5,2)6(6-1)^2(6-2)^{5-2.2}$

Or $\quad \mathbf{TrA^5} = 6\times5\times4^3 + 2\times6\times5^2\times4 = 3120$

Example 2: How many triangles are there in the following graph?



We know that the number of triangles in the connected graph is $\mathbf{Tr(A^3)}/6$, where A is the adjacency matrix of the graph.

The adjacency matrix of this graph is A = $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

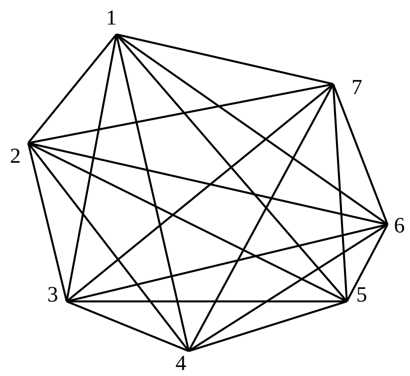
By our theorem $\boldsymbol{TrA^3} = \sum_{r=1}^{1} S(3,r)n(n-1)^r (n-2)^{3-2.r}$

Here $n = 4$, $\quad \boldsymbol{TrA^3} = S(3,1)4(4-1)^1 (4-2)^{3-2.1}$

Or $TrA^3 = 4\times3\times2 = 24$

Number of triangles in the graph = **Tr**(**A**^{3})/6 = 24/6 = 4,namely 123,124,134,234.

Example 3: How many triangles in the following graph?



Here adjacency matrix of the graph is $A = (a_{ij}), 1 \le i \le 7\ ,\ 1 \le j \le 7$ where

$$a_{ij} = \begin{cases} 1 \text{ if } i \neq j \\ 0 \text{ if } i = j \end{cases}$$

By our theorem $TrA^3 = \sum_{r=1}^{1} S(3,r)n(n-1)^r(n-2)^{3-2.r}$

Here$n = 7$, $TrA^3 = S(3,1)7(7-1)^1(7-2)^{3-2.1}$

Or $TrA^3 = 7\times6\times5 = 210$

Number of triangles in the graph = $Tr(A^3)/6$ = 210/6 = 35.

CONCLUSION AND FUTURE WORK

In this paper ,we extended the order of matrix presented in [8] for the trace of positive integer power of 2×2 real matrix to the trace of positive integer power of some adjacency matrix of any order.as explained in the paper such estimates have applications in various branches of mathematics. The ideas presented in this paper could be extended to any matrix.

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